G-BUNDLES ON THE FARGUES-FONTAINE CURVE

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1. p-divisible groups

From now on fix an algebraically closed field k of characteristic p. Let $(\mathsf{Sch}_k)_{\text{fppf}}$ denote the big fppf site over $\operatorname{Spec} k$.

Definition 1.1. A **p-divisible group of height** h over k (for $h \geq 0$) is a sheaf $G : (\mathsf{Sch}_k)_{\text{fppf}} \to \mathsf{Ab}$ such that

(1) (*p-divisiblity*) the multiplication by p map $G \xrightarrow{[p]} G$ is an epimorphism of fppf sheaves,

(2) (*p-power torsion*) $G = \varinjlim_n G[p^n]$, where $G[p^n] := \ker(G \xrightarrow{[p^n]} G)$, and

(3) $G[p]$ is (represented by) a finite flat group scheme of order p^h.

Remark 1.2. There is an equivalent formulation as a system $(G_n)_{n\geq 1}$ of finite flat group schemes and maps $G_n \hookrightarrow G_{n+1}$ identifying the image with the p^n-torsion of the target for each n, subject to some additional conditions. In fact this is equivalent to the definition we've given: in one direction define

$$G(T) := \varinjlim_n \operatorname{Hom}_{\mathsf{Sch}_k}(T, G_n)$$

and in the other direction define

$$G_n := \ker G \xrightarrow{[p^n]} G$$

Example 1.3.

(1) Let $\underline{\mathbf{Q}_p/\mathbf{Z}_p}$ denote the constant sheaf associated to the abelian group $\mathbf{Q}_p/\mathbf{Z}_p$: this is a p-divisible group of height 1.

(2) Set

$$\mu_{p^\infty}(T) = \left\{ x \in \mathscr{O}_T(T)^\times \mid x^{p^n} = 1 \text{ for some } n \right\}$$

In the other definition, μ_{p^∞} is $(\mu_{p^n})_{n\geq 0}$, which is defined analogously. This also has height 1.

(3) If A/k is an abelian variety of dimension g, then $A[p^\infty]$ is a p-divisible group of height $2g$. If A is ordinary, then

$$A[p^\infty] = \mu_{p^\infty}^g \times \underline{\mathbf{Q}_p/\mathbf{Z}_p}^g$$

Remark 1.4.

(1) Since we are working over a perfect field k, we have a decomposition

$$G = G_c \times G_e$$

where G_c is connected and G_e is étale.

Date: June 19, 2019.

(2) Given a p-divisible group we can define its **Cartier dual** $G^\vee$ by

$$G^\vee(T) = \varinjlim_n \mathrm{Hom}_T(G[p^n]_T, \mathbf{G}_{m,T})$$

which is again a p-divisible group over k, and induces a natural isomorphism $(G^\vee)^\vee \cong G$.

(3) Recall we have a relative Frobenius map $F : G \to \varphi^* G$, and a Verschiebung $V : \varphi^* G \to G$, which is dual to the Frobenius on $G^\vee$.

2. Dieudonné modules

The goal of this section is to reduce the study of p-divisible groups to (semi)-linear algebra. Let $\varphi : W(k) \to W(k)$ denote the Frobenius map.

Definition 2.1. A **Dieudonné module** is a triple (M, F, V), where

(1) M is a finite free $W(k)$-module,

(2) $F : M \to M$ is a φ-semilinear endomorphism,

(3) $V : M \to M$ is a φ^{-1}-semilinear endomorphism, and

(4) $FV = VF = p$.

Definition 2.2. We define the dual Dieudonné module $(M^\vee, F^\vee, V^\vee)$ by taking

$$M^\vee = \mathrm{Hom}_{W(k)}(M, W(k))$$

and

$$(F^\vee \ell)(m) = \varphi(\ell(V(m)))$$
$$(V^\vee \ell)(m) = \varphi^{-1}(\ell(F(m)))$$

Theorem 2.3. *There exists an equivalence of categories*

$$\{p\text{-divisible groups over } k\} \xrightarrow{\sim} \{\text{Dieudonné modules}\}$$
$$G \mapsto M(G)$$

such that

(1) $\mathrm{rank}(M(G))$ *is the height of* G.

(2) $M(G^\vee) \cong M(G)^\vee$ *and this is natural in* G.

Sketch of Construction. We will sketch two possible constructions, which give the same thing.

(1) The first construction is the more classical one, and roughly goes as follows. Let $\widehat{\mathbf{W}}$ denote the formal completion of the Witt-vector scheme over k, whose representing functor sends a k-algebra R to $W(R)$. For H a connected p-divisible group we define

$$M(H) = \mathrm{Hom}_{\mathsf{FmlGrp}}(\widehat{\mathbf{W}}, H)$$

and one shows that $M(\varphi^* H) \cong \varphi^* M(H)$ which allows us to define $F = M(V)$ and $V = M(F)$. In general we define

$$M(G) = M(G_c) \times M(G_e^\vee)^\vee$$

(2) The second construction is more abstract and roughly goes as follows. First lift G to a p-divisible group G_0 over $W(k)$. Then there exists a universal vector extension of G_0, i.e. an exact sequence of sheaves
$$0 \to V \to EG_0 \to G_0 \to 0$$
such that $V \cong \mathbf{G}_{a,W(k)}^m$ which is initial in the category of such objects, where we require morphisms to be $W(k)$-linear on the V-part. Then $\mathrm{Lie}(EG_0)$ can be made into a crystal on the crystalline site over k, and we evaluate this crystal at the PD-thickening $W(k) \to k$ to obtain $M(G)$, and one can show that this doesn't depend on the lift that you started with. See **?** for more details.

□

Example 2.4.

(1) If we start with the height 1 p-divisible group μ_{p^∞}, then $M(G)$ is the rank 1 $W(k)$-module with basis vector e, where $F(e) = e$ and $V(e) = pe$: to see this, note that $V : \varphi^*\mu_{p^\infty} \to \mu_{p^\infty}$ is an isomorphism, so F becomes an isomorphism on $M(G)$, and $F : \mu_{p^n} \to \mu_{p^n}$ is nilpotent, so it becomes topologically nilpotent on $M(G)$.

(2) Note $\mathbf{Q}_p/\mathbf{Z}_p$ is Cartier dual to μ_{p^∞}, so $M(G)$ is the same but we now have $F(e) = pe$ and $V(e) = e$.

Let $K = W(k)[1/p]$.

Definition 2.5. An **isocrystal** or **φ-module** is a pair (N, F) where N is finite dimensional K-vector space and $F : N \to N$ is a φ-semilinear map whose linearization $\varphi^*N \to N$ is an isomorphism.

Theorem 2.6 (Dieudonné-Manin)**.** *The category of isocrystals is abelian semisimple, with simple objects given by $N_{r/s}$ for $(r, s) = 1$ and $s > 0$, which has a basis*
$$e, Fe, \ldots, F^{s-1}e$$
and $F^s e = p^r e$.

Definition 2.7. We call r/s the **slope** of the simple isocrystal, and it determines the simple isocrystal uniquely.

There is a natural functor
$$\{\text{Dieudonné modules}\} \to \{\text{isocrystals}\}$$
$$(M, F, V) \mapsto (M[1/p], F)$$
Note that V is determined because $FV = p$ and now p is invertible.

Remark 2.8. An isogeny of p-divisible groups is an epimorphism with finite kernel, and one can show that a map is an isogeny if and only if the associated map on isocrystals is an isomorphism. In fact, one can show that
$$\{p\text{-divisible groups over } k\} / \text{isogeny} \xrightarrow{\sim} \{\text{isocrystals with slopes } [0,1]\}$$
$$G \mapsto M(G)[1/p]$$
is an equivalence of categories.

Example 2.9. The isocrystal attached to μ_{p^∞} has Frobenius acting by p^0, so the slope is 0, and the isocrystal attached to $\mathbf{Q}_p/\mathbf{Z}_p$ has Frobenius acting by p^1, so the slope is 1. By the above remark, these are all of the height 1 p-divisible groups over k, up to isogeny.

3. Fargues-Fontaine curve

Now let F be an algebraically closed perfectoid field of characteristic p. Recall that this means that F is perfect and complete with respect to a nondiscrete non-archimedean norm. We let $A_{\text{inf}} := W(\mathscr{O}_F)$. Fix a pseudo-uniformizer $\varpi \in \mathscr{O}_F$, and denote by $[\varpi]$ its Teichmüller lift to A_{inf}.

Definition 3.1. Let A_{inf} carry the $(p, [\varpi])$-adic topology, and set

$$\mathcal{Y}_F = \mathrm{Spa}(A_{\text{inf}}, A_{\text{inf}}) \setminus V(p[\varpi]) := \left\{ v : A_{\text{inf}} \xrightarrow{\text{cts}} \Gamma \sqcup \{0\} \mid v(p[\varpi]) \neq 0 \right\}_{/\sim}$$

(note this does not depend on the choice of ϖ).

One caveat is that we haven't shown that this is sheafy, so we don't yet know if this gives an adic space, or just a "pre-adic space". In fact the first thing we will do is to cover $\mathcal{Y}_F$ with adic spaces.

The definition may seem mysterious at first, but we can get a better handle on it by noticing that there are effectively two things going on: there are characteristic p phenomena coming from the fact that F is a perfectoid field of characteristic p, and we have characteristic 0 phenomena coming from the fact that we took Witt vectors, and now have a nonzero $p \in A_{\text{inf}}$. In particular, since we chose the $(p, [\varpi])$-adic topology on A_{inf}, we are really incorporating both phenomena.

Remark 3.2. Let's find some familiar ideals in A_{inf}, to get a better idea of what kind of points there are on the curve.

(1) Obviously $(A_{\text{inf}}/p)[1/[\varpi]] = F$.

(2) Also $(A_{\text{inf}}/[\varpi])[1/p]$ is a field of characteristic 0. For example, if we start with the perfectoid field $\overline{\mathbf{F}_p((t))}^{\wedge_t}$, then we get $W(\overline{\mathbf{F}_p})[1/p] = \breve{\mathbf{Q}}_p$.

(3) Finally let $\mathscr{O}_{E_\varpi} := (A_{\text{inf}}/(p - [\varpi]))$. Then $E_\varpi := \mathscr{O}_{E_\varpi}[1/p]$ is a perfectoid field of characteristic 0: note we have a surjection

$$\widetilde{\theta} : \mathscr{O}_F \xrightarrow{\sim} (\mathscr{O}_F/\varpi)^\flat \xrightarrow{\sim} (W(\mathscr{O}_F)/(p - [\varpi], p))^\flat = (\mathscr{O}_{E_\varpi}/p)^\flat \xrightarrow{\sim} \mathscr{O}_{E_\varpi}^\flat \twoheadrightarrow \mathscr{O}_{E_\varpi}$$

so we get a valuation on $\mathscr{O}_{E_\varpi}$ by taking $v(\widetilde{\theta}(x)) = v_F(x)$. This is well-defined, and shows that E_ϖ is a characteristic 0 algebraically closed perfectoid field with $E_\varpi^\flat \cong F$.

In fact, there is a nice moduli description of $\mathcal{Y}_F$ by varying ϖ in part (3) of the above remark.

Fact 3.3. *The map from part (3) induces a bijection*

$$\{\textit{ideals } (p - [\varpi]) \subseteq A_{\text{inf}} \textit{ for } \varpi \in \mathscr{O}_F \textit{ with } 0 < |\varpi| < 1\} \xrightarrow{\sim} \left\{\textit{perfectoid fields } E \textit{ with } \mathit{char}(E) = 0 \textit{ and } E^\flat \cong F\right\}_{/\cong}$$

and these live inside the rank one points of $\mathcal{Y}_F$.

Now fix ϖ again. To understand a point of $\mathcal{Y}_F$ we can look at its behavior at p and $[\varpi]$. More precisely, define

$$\kappa : \mathcal{Y}_F \to (0, \infty)$$
$$x \mapsto \frac{\log |[\varpi](\widetilde{x})|}{\log |p(\widetilde{x})|}$$

where $\widetilde{x}$ denotes the rank 1 generization of x, which is well-defined because we removed $V(p[\varpi])$. Note that the coordinate ring of $\mathcal{Y}_F$ is

$$B_b := A_{\text{inf}} \left[\frac{1}{p[\varpi]}\right] = \left\{ \sum_{n >> -\infty} [x_n] p^n \in W(F)[1/p] \mid |x_n| \text{ is bounded} \right\}.$$

For $\rho \in (0, \infty)$, we define a norm on B^b by

$$\left| \sum_{n>>-\infty} [x_n] p^n \right|_\rho = \max_n |x_n| |\varpi|^{n/\rho}.$$

Then if $I \subseteq (0, \infty)$ has endpoints in $\mathbf{Q}$, we let B_I denote the completion of B^b with respect to every $\rho \in I$ and we let B_I° denote the ring of power-bounded elements in B_I.

Proposition 3.4. *If $I = [a, b] \subseteq (0, \infty)$ is any closed interval with $a, b \in \mathbf{Q}$, then*

$$\mathcal{Y}_{F,I} := \kappa^{-1}(I)^\circ = \mathrm{Spa}(B_I, B_I^\circ)$$

and $\mathcal{Y}_{F,I}$ is an adic space. In particular, $\mathcal{Y}_F$ is an adic space.

Sketch of Proof. These rings are preperfectoid (i.e. there exists a perfectoid field K such that $B_I \widehat{\otimes}_{\mathbf{Q}_p} K$ is uniform and perfectoid) by the Ph.D thesis of Ryan Rodriguez, and rational localizations of preperfectoid rings are preperfectoid, so B_I is stably uniform, hence sheafy. □

The Frobenius $\varphi : \mathscr{O}_F \to \mathscr{O}_F$ lifts to a Witt vector Frobenius $\varphi : A_{\mathrm{inf}} \to A_{\mathrm{inf}}$. This induces a Frobenius φ on $\mathcal{Y}_F$, which on points is given by

$$|\cdot| \mapsto |\varphi(\cdot)|.$$

But note

$$\kappa(\varphi(x)) = \frac{\log |[\varpi]^p(\widetilde{x})|}{\log |p(\widetilde{x})|} = \frac{p \log |[\varpi](\widetilde{x})|}{\log |p(\widetilde{x})|} = p\kappa(x),$$

so actually we get induced maps $\varphi : \mathcal{Y}_{F,[a,b]} \to \mathcal{Y}_{F,[pa,pb]}$. Thus by choosing small enough intervals, we see that φ acts properly discontinuously, so we can take the quotient.

Definition 3.5. The **Fargues-Fontaine curve** is

$$\mathcal{X}_F = \mathcal{Y}_F / \varphi^{\mathbf{Z}}$$

So why did we call this a "curve"? We will see this by introducing a scheme theoretic version of the curve. We compute

$$B := \Gamma(\mathcal{Y}_F, \mathscr{O}_{\mathcal{Y}_F}) = \Gamma(\varinjlim_I \mathcal{Y}_{F,I}, \mathscr{O}_{\mathcal{Y}_F}) = \varprojlim_I \Gamma(\mathcal{Y}_{F,I}, \mathscr{O}_{\mathcal{Y}_F}) = \varprojlim B_I = \text{the Fréchet completion of } B^b \text{ for all } |\cdot|_\rho$$

Remark 3.6. We are motivated by the following analogous construction. Given a proper curve $X \to k$ over a field, if we want an explicit model for this, we can pick a very ample line bundle $\mathcal{L}$ and then

$$X = \mathrm{Proj} \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$$

Note that by descent, a line bundle on $\mathcal{X}_F$ should be the same as a φ-equivariant line bundle on $\mathcal{Y}_F$, which should be the same as an invertible B-module with a compatible φ-action. We will take this to be the free B-module of rank one, with the φ action given by $\varphi(e) = p^{-1}e$. Call this Be. Then we define

$$X_F := \mathrm{Proj} \bigoplus_{d \geq 0} (Be^{\otimes d})^{\varphi = 1} = \mathrm{Proj} \bigoplus_{d \geq 0} B^{\varphi = p^d}$$

Theorem 3.7 (Fargues-Fontaine)**.** *X_F is a one dimensional regular Noetherian scheme with associated adic space $\mathcal{X}_F$: in particular there is a morphism of locally ringed spaces $\mathcal{X}_F \to X_F$ which induces a bijection*

$$\{\text{classical points (arising from } (p - [\varpi]))\} \xrightarrow{\sim} \{\text{closed points}\}$$

and an isomorphism $\widehat{\mathscr{O}}_{\mathcal{X}_F, x} \cong \widehat{\mathscr{O}}_{X_F, x}$ for x classical.

In particular, closed points of X_F really correspond to characteristic 0 untilts of F.

4. Vector bundles

We want to talk about vector bundles on the Fargues-Fontaine curve. Note that for schemes, you can't build a theory of coherent sheaves unless you work over a locally Noetherian scheme. For rigid analytic spaces, the situation is worse, because you don't always have an analogue of the Hilbert basis theorem for Tate algebras. Therefore, we restrict ourselves to this case:

Definition 4.1. We say that a Huber ring A is **strongly Noetherian** if $A\langle T_1, \dots, T_n\rangle$ is Noetherian for $n \geq 0$.

Theorem 4.2 (Kedlaya)**.** *B_I is strongly Noetherian.*

From this, Kedlaya shows that one can develop a good theory of coherent sheaves on $\mathcal{Y}_F$ and $\mathcal{X}_F$, and in particular of vector bundles.

Let's describe all of the vector bundles on $\mathcal{X}_F$. We can pick an embedding $\overline{\mathbf{F}_p} \hookrightarrow \mathscr{O}_F$, which induces a map $\breve{\mathbf{Q}}_p = W(\overline{\mathbf{F}}_p)[1/p] \to A_{\mathrm{inf}}[1/p[\varpi]] \to B$, which by duality gives a structure map

$$\mathcal{Y}_F \to \mathrm{Spa}(\breve{\mathbf{Q}}_p, \mathscr{O}_{\breve{\mathbf{Q}}_p}).$$

Note a φ-equivariant vector bundle on $\mathrm{Spa}(\breve{\mathbf{Q}}_p, \mathscr{O}_{\breve{\mathbf{Q}}_p})$ is the same as an isocrystal. Then pullback of vector bundles induces a functor

$$\{\text{isocrystals}\} \to \{\varphi\text{-equivariant vector bundles on } \mathcal{Y}_F\} \xrightarrow{\sim} \{\text{vector bundles on } \mathcal{X}_F\}$$

Theorem 4.3 (Fargues-Fontaine)**.** *The above functor is an equivalence of categories.*

Remark 4.4. In particular, $\mathrm{Pic}(\mathcal{X}_F) \cong \mathbf{Z}$ by the classification of rank 1 isocrystals, so we have a well-defined notion of degree of any vector bundle via the determinant. If we define the **slope** to be $\deg \mathcal{E} / \operatorname{rank} \mathcal{E}$ and say that $\mathcal{E}$ is semistable if there are no subbundles of higher slope. Then the vector bundle attached to the isocrystal $N_{r/s}$ is semistable of slope r/s.

First note there is a GAGA theorem.

Theorem 4.5. *The pullback along $\mathcal{X}_F \to X_F$ induces an equivalence of categories between vector bundles.*

So it suffices to study vector bundles on X_F. Fix a closed point $\infty \in X_F$. Then let $B_e = \Gamma(X_F \setminus \infty, \mathscr{O}_{X_F})$, and we let $B_{\mathrm{dR}}^+ = \widehat{\mathscr{O}}_{X_F,\infty}$ and $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t]$ (recall X_F is 1-d regular). It turns out that B_e is a PID. It turns out that one can do a version of the Beauville-Laszlo gluing procedure to get an equivalence of categories

$$\{\text{vector bundles on } X_F\} \xrightarrow{\sim} \left\{\text{pairs } (M, W) \text{ where } M \text{ is a finite free } B_e\text{-module and } \Lambda \text{ is a } B_{\mathrm{dR}}^+\text{-lattice in } M_e \otimes_{B_e} B_{\mathrm{dR}}\right\}$$
$$\mathcal{E} \mapsto (\Gamma(X_F \setminus \infty, \mathcal{E}), \widehat{\mathcal{E}}_\infty)$$

5. G-bundles

Definition 5.1. Let G be a connected reductive group over $\mathbf{Q}_p$. Then the **Kottwitz set** is defined as

$$B(G) = G(\breve{\mathbf{Q}}_p)/(gh\sigma(g)^{-1} \sim h).$$

Definition 5.2. A G-isocrystal is an exact tensor functor

$$\mathsf{Rep}_{\mathbf{Q}_p} G \to \mathrm{Iso}$$

and similarly, a G-bundle on $\mathcal{X}_F$ is an exact tensor functor

$$\mathsf{Rep}_{\mathbf{Q}_p} G \to \mathsf{Bun}_{\mathcal{X}_F}$$

Theorem 5.3 (Fargues)**.** *We have bijections*

$$\{G\text{-}bundles\ on\ X_F\}_{/\cong} \xrightarrow{\sim} \{G\text{-}isocrystals\}_{/\cong} \xrightarrow{\sim} B(G)$$

For $G = \mathrm{GL}_n$, the idea is that a G-isocrystal is really just a rank n isocrystal, and the linearization of the Frobenius map gives a matrix in $\mathrm{GL}_n(\breve{\mathbf{Q}}_p)$, which is well-defined up to φ-conjugacy, since F was φ-linear.