DEFORMATION THEORY BACKGROUND

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1. Introduction

Deformation theory is about studying infinitesimal local properties of geometric objects, which are usually moduli spaces. In practice we will take a point on a space and try to extract as much information as we can from the point without looking at any other points, and instead only relying on "non-reduced structure" in a sense we will make precise.

To say more precisely what we mean by this, let's immediately dive into the technical details and then give some examples.

Fix Λ be a complete local Noetherian ring with residue field k. For our purposes, this will eventually be $\Lambda = \mathbf{Z}_p$ and $k = \mathbf{F}_p$ (which allows us to deform in the p-adic direction), but one could also take $\Lambda = k = \mathbf{Q}_p$, for instance.

Definition 1.1. We define a category of pairs (R, π_R) where R is a complete local Noetherian Λ-algebra and $\pi_R : R \twoheadrightarrow k$ is a surjective map, so that it induces an isomorphism $R/\mathfrak{m}_R \cong k$. We typically omit π_R from the notation. Morphisms are maps $f : R \to S$ respecting the map to k: these are automatically local homomorphisms. Let Art_Λ denote the full subcategory of Artinian objects.

Note that $\operatorname{Spec} k \to \operatorname{Spec} A$ induces an isomorphism on the underlying topological space (which is a point), so the only interesting information is in the non-reduced structure of A. For instance, as we will see, $k[x]/x^2$ will tell us information about the tangent space at points of the moduli problem.

Definition 1.2. A *deformation functor* is a functor

$$D : \mathsf{Art}_\Lambda \to \mathsf{Set}$$

such that $D(k) = \{\bullet\}$.

The idea is that $D(k)$ contains the object $\bullet$ we want to deform, and each $D(A)$ contains the deformations of $\bullet$ over A.

Remark 1.3. A nice way to see why this definition might be useful is the following observation. Let $\Lambda = k$ some field. Given a Noetherian scheme X and a point $\iota_x : \operatorname{Spec} k \to X$, there is a deformation functor

$$D_{X,x}(A) = \left\{ \begin{array}{ccc} \operatorname{Spec} A & \longrightarrow & X \\ \uparrow & \nearrow_{\iota_x} & \\ \operatorname{Spec} k & & \end{array} \right\} = \left\{ \begin{array}{ccc} A & \longleftarrow & \mathscr{O}_{X,x} \\ \downarrow & \swarrow & \\ k & & \end{array} \right\} = \left\{ \begin{array}{ccc} A & \longleftarrow & \widehat{\mathscr{O}}_{X,x} \\ \downarrow & \swarrow & \\ k & & \end{array} \right\} = \operatorname{Hom}_{\mathsf{Noe}_\Lambda}(\widehat{\mathscr{O}}_{X,x}, A)$$

The second equality follows from adjunction, and the third from the fact that A is complete.

This motivates the following general definition.

Definition 1.4. If D is a deformation functor, then we say that D is *pro-representable* if $D \cong \operatorname{Hom}_{\mathsf{Noe}_\Lambda}(R, -)$ for some $R \in \mathsf{Noe}_\Lambda$.

Thus in the above definition $D_{X,x}$ is pro-represented by $\widehat{\mathscr{O}}_{X,x}$. The point to emphasize now is that if you develop enough of the theory of deformation functors, this gives you information about $\widehat{\mathscr{O}}_{X,x}$, which means that you now have a nice formalism for understanding local properties of your moduli space, like dimension, singularities, etc.

2. Pro-Representability

We will now recall Schlessinger's criterion, which is a way, given a deformation functor, to tell when it's pro-representable.

Theorem 2.1 (Grothendieck)**.** *A deformation functor D is pro-representable if and only if*

(1) *For any pair of maps $A \to C$ and $B \to C$ in Art_Λ, the natural map*
$$D(A \times_C B) \to D(A) \times_{D(C)} D(B)$$
is a bijection. In other words, D preserves finite products.

(2) *$D(k[x]/x^2)$ is a finite set.*

To explain this theorem, first note that a representable functor preserves fiber products. Then recall that the Zariski cotangent space of a Noetherian k-scheme X is given by $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$, where $\mathfrak{m}_{X,x}$ is the maximal ideal in the local ring $\mathscr{O}_{X,x}$. In fact, it is an exercise to show that
$$\operatorname{Hom}_{\mathsf{Noe}_\Lambda}(R, k[x]/x^2) \cong \operatorname{Hom}_k(\mathfrak{m}_R/(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda), k)$$
for any $R \in \mathsf{Noe}_\Lambda$, so in particular if $\Lambda = k$, we have $D_{X,x}(k[\epsilon])$ is the Zariski tangent space of X at the point x. Here we let $k[\epsilon] = k[x]/x^2$. So the second condition says that the tangent space is finite dimensional.

But this condition is hard to check in practice, so there is a slightly weaker condition that one can check to achieve the same result.

Definition 2.2. A morphism $A \to B$ in Art_Λ is small if it is surjective and $\ker(A \to B)$ is principal and killed by $\mathfrak{m}_A$.

Theorem 2.3 (Schlessinger's Criterion)**.** *Let D be a deformation functor and consider a pair of maps $A \to C$ and $B \to C$ in the category Art_Λ and let $(*) : D(A \times_C B) \to D(A) \times_{D(C)} D(B)$. Then D is pro-representable if and only if the following four conditions are satisfied.*

(1) *If $A \to C$ is small then $(*)$ is surjective.*

(2) *If $C = k$ and $B = k[\epsilon]$, then $(*)$ is bijective. In this case, one can actually show that $F(k[\epsilon])$ acquires the structure of a k-vector space.*

(3) *Assume* $F(k[\epsilon])$ *is finite dimensional.*

(4) *If* $A = B$ *and* $A \to C$ *and* $B \to C$ *are the same and are small, then* $(*)$ *is bijective.*

Since we just need to check that D interacts well with products in these simplified situations, this is simpler to check than Grothendieck's criterion.

If we have a pro-representable deformation functor D with representing ring $R \in \mathsf{Noe}_\Lambda$, then we wish to understand the ring theoretic properties of R.

Lemma 2.4. *If* $d = \dim D(k[\epsilon]) = \dim \mathfrak{m}_R/(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda)$*, then there exists a surjection*

$$\Lambda[[x_1, \ldots, x_d]] \to R$$

This gives an upper bound on the dimension of R, but if we want finer information about R, we need to understand the kernel of the above surjection. In nice enough situations, the above map will be an isomorphism, but if not, we can obtain at least a lower bound on the dimension of R by giving an upper bound on a minimal generating set of the kernel.

Rather than talk about this in general, which is the formalism of "tangent-obstruction theory", we will now talk about Galois deformation theory.

3. Galois Deformation Theory

Let G be a profinite group and fix a prime p. Let $L/\mathbf{Q}_p$ be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathscr{O}$ and residue field k. Fix a continuous representation $\overline{\rho} : G \to \mathrm{GL}_n(k)$, equivalently an n-dimensional k-vector space V_k with a continuous action of G and a basis B_k. We define functors

$$\begin{aligned} D_{\overline{\rho}}^{\square} &: \mathsf{Art}_{\mathscr{O}} \to \mathsf{Set} \\ A &\mapsto \left\{\rho : G \xrightarrow{\text{cts}} \mathrm{GL}_n(A) \mid \pi_A \circ \rho = \overline{\rho}\right\} \\ &= \{V \text{ a finite free rank } n \text{ } A\text{-module with a continuous action of } G \text{ with } V_A \otimes_A k \cong V,\ B \text{ a basis for } V \text{ lifting } B_k\} \end{aligned}$$

$$\begin{aligned} D_{\overline{\rho}} &: \mathsf{Art}_{\mathscr{O}} \to \mathsf{Set} \\ A &\mapsto \left\{\rho : G \xrightarrow{\text{cts}} \mathrm{GL}_n(A) \mid \pi_A \circ \rho = \overline{\rho}\right\} / \sim \\ &= \{V \text{ a finite free rank } n \text{ } A\text{-module with a continuous action of } G \text{ with } V_A \otimes_A k \cong V\} / \cong \end{aligned}$$

The equivalence relation $\sim$ is the one given by *strict equivalence*, which means that $\rho \sim \rho'$ if and only if they are conjugate by a matrix in $1 + M_n(\mathfrak{m}_A)$.

Proposition 3.1. *If* G *satisfies the* p*-finiteness condition (which says that for all open subgroups* $H \leq G$ *the maximal pro-*p*-quotient of* H *is topologically finitely generated) then* $D_{\overline{\rho}}^{\square}$ *is pro-representable.*

If furthermore $\mathrm{End}_{k[G]}(\overline{\rho}) = k$*, then* $D_{\overline{\rho}}$ *is pro-representable as well.*

Proof Sketch. To prove this, one can either do the work of directly checking Schlessinger's criterion, or there is a more direct proof for $D_{\overline{\rho}}^{\square}$ by first expressing G as an inverse limit of finite groups, then choosing a presentation of each finite group, then explicitly constructing R from this presentation for each finite group, and then taking a limit. Then for $D_{\overline{\rho}}$ there is a way to take the quotient by the conjugation action of $\widehat{PGL_n}$, which is the formal completion of PGL_n at the identity section. ☐

Typically moduli problems are not representable when there are "too many automorphisms", and this is an instance of this kind of obstruction.

Examples of groups that satisfy the p-finiteness condition are G_F where $F/\mathbf{Q}_\ell$ is a p-adic local field, or $G_{K,S}$ where K is a number field, S is a finite set of primes, and $G_{K,S}$ is the Galois group of the maximal extension of K unramified outside of S.

In fact, if the p-finiteness property is not satisfied, then $D^{\square}_{\overline{\rho}}$ will still be represented by some ring but it need not be Noetherian.

Now we can describe the tangent space and the defining ideal for $R_{\overline{\rho}}$. From now on assume $\overline{\rho}$ is absolutely irreducible for simplicity, so that in particular $D_{\overline{\rho}}$ is representable.

Proposition 3.2. *There is an isomorphism*

$$D_{\overline{\rho}}(k[\epsilon]) \cong H^1(G, \operatorname{ad}\overline{\rho})$$

and if $I = \ker(\mathscr{O}[[x_1, \ldots, x_n]] \to R_{\overline{\rho}})$*, an injection*

$$(I/(\pi_l, x_1, \ldots, x_n)I)^{\vee} \hookrightarrow H^2(G, \operatorname{ad}\overline{\rho})$$

Here $\operatorname{ad}\overline{\rho} \cong M_n(k)$ *with the conjugation action of* G *via* $\overline{\rho}$*.*

Proof. The H^1 part is a direct calculation using the fact that one can write a deformation $\rho \in D_{\overline{\rho}}(k[\epsilon])$ as $\rho = \overline{\rho}(1 + c\epsilon)$ where $c : G \to M_n(k)$ is some function, which one can explicitly check is a continuous 1-cocycle. The second part is more involved, and we omit the proof. □

In particular, the minimal number of generators of I is bounded above by $\dim_k H^2(G, \operatorname{ad}\overline{\rho})$. For example, if $H^2(G, \operatorname{ad}\overline{\rho}) = 0$, then the deformation problem is represented by $\mathscr{O}[[x_1, \ldots, x_n]]$.

4. Local Conditions

There is a sensible way to impose conditions on deformations. For instance, if we're deforming local representations, we might want to say "unramified", "having fixed determinant". You also want to say things like "crystalline" at the prime p with certain fixed Hodge-Tate weights.

Definition 4.1. A *deformation condition* is a property P of continuous representations $G \to \mathrm{GL}_n(A)$ for $A \in \mathsf{Art}_{\mathscr{O}}$ such that

(1) $\overline{\rho}$ satisfies P
(2) If $f : A \to B$ is a map in $\mathsf{Art}_{\mathscr{O}}$ and $\rho \in D_{\overline{\rho}}(A)$ satisfies P, then $D(f)(\rho)$ satisfies P
(3) If $f : A \to C$ and $g : B \to C$ are as before and $\rho \in D_{\overline{\rho}}(A \times_C B)$ satisfies P then so does $D(f)(\rho)$ and $D(g)(\rho)$.

If these conditions are satisified then the subfunctor

$$D_{\overline{\rho},P}(A) = \{\rho \in D_{\overline{\rho}}(A) \mid \rho \text{ satisfies } P\}$$

is also pro-representable.

Example 4.2. If we fix a character $\delta : G \to \mathscr{O}^{\times}$ then the condition "has determinant δ" is a deformation condition.

One can show that the inclusion $D_{\overline{\rho},P} \hookrightarrow D_{\overline{\rho}}$ induces a map $R_{\overline{\rho}} \twoheadrightarrow R_{\overline{\rho},P}$ and thus a closed embedding $\operatorname{Spec} R_{\overline{\rho},P} \hookrightarrow \operatorname{Spec} R_{\overline{\rho}}$: this reflects the intuition that P is a closed condition.

Unfortunately in general there is no good notion of "crystalline" for representations valued in objects of the category $\mathsf{Art}_{\mathscr{O}}$. If you restrict the Hodge-Tate weights, you can use Fontaine-Lafaille theory, which is a form of integral p-adic Hodge theory, to define a deformation functor for crystalline deformation rings with

Hodge-Tate weights in a certain range. In general Kisin proved the following. Let $G = G_F$ for F a p-adic local field.

Theorem 4.3 (Kisin). *There exists a unique torsion-free quotient $R_{\overline{\rho},\delta,cris,H}$ such that a map $R_{\overline{\rho}} \to \overline{\mathbf{Q}}_p$ factors through $R_{\overline{\rho},\delta,cris,H}$ if and only if the associated p-adic Galois representation is crystalline with Hodge-Tate weights in some finite range $H = [a,b]$ and determinant δ.*

Now, for instance, if we have a representation $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{F}_p)$, then we can consider something like

$$R^{\square}_{\overline{\rho},\delta} \otimes_{R^{\square}_{\overline{\rho}_p,\delta}} R^{\square}_{\overline{\rho}_p,\delta,cris,H}$$

which pro-represents

$$D^{\square}_{\overline{\rho},\delta} \times_{D^{\square}_{\overline{\rho}_p,\delta}} D^{\square}_{\overline{\rho}_p,\delta,cris,H}$$

which should parametrize framed deformations of $\overline{\rho}$ which are crystalline with Hodge-Tate weights H and determinant δ after restriction to $G_{\mathbf{Q}_p}$.

5. Taylor-Wiles and Derived Information

So what is the point of these constructions? The idea is really to prove modularity lifting theorems. These are theorems of the form

$$R \xrightarrow{\sim} \mathbf{T}_{\mathfrak{m}}$$

Here R is a global deformation ring with local conditions, as above, and $\mathbf{T}_{\mathfrak{m}}$ is a certain Hecke algebra (acting on a properly defined space of automorphic forms) localized at a maximal ideal $\mathfrak{m}$ which corresponds to $\overline{\rho}$. Then in particular, any lift ρ of $\overline{\rho}$ to characteristic 0 is actually induced by a map $\mathbf{T}_{\mathfrak{m}} \to \overline{\mathbf{Q}}_p$, and thus is modular.

But to prove this kind of theorem, you have to have set up R correctly in the first place. Galois representations coming from automorphic forms will have a certain shape: they will be unramified outside of a finite set of primes determined by the level of the automorphic form, at the primes dividing the level the representation will be of a certain form, and at the prime p the representation will satisfy certain p-adic Hodge theoretic properties, so in particular you need to choose the right ones.

Why does a derived story enter the picture? Briefly, the expected dimension of the intersection of Spec $R^{\square}_{\overline{\rho}_p,\delta,cris,H}$ and Spec $R^{\square}_{\overline{\rho},\delta}$ inside of Spec $R^{\square}_{\overline{\rho}_p,\delta}$ is $-\ell_0$, where ℓ_0 is this defect determining the range of degrees over which nice automorphic forms contribute to cohomology.

Suppose $\overline{\rho}$ and $\overline{\rho}_p$ are absolutely irreducible. Then one expects the dimension of the global thing (by a computation of Mazur) to be $N(N+1)/2 - 1 - \ell_0$, where ℓ_0 is the defect for SL_n. One expects the local thing to have dimension $N^2 - 1$, and one expects the crystalline local thing to have dimension $N(N-1)/2$. Adding these up gives you $-\ell_0$.

So when $\ell_0 > 0$, then you expect the global deformations and the crystalline deformations to rarely intersect, but you still expect them to intersect, but now non-transversally. So a more sensible thing to do, to really keep track of all of the derived information, is to take

$$R^{\square}_{\overline{\rho},\delta} \otimes^{\mathbf{L}}_{R^{\square}_{\overline{\rho}_p,\delta}} R^{\square}_{\overline{\rho}_p,\delta,cris,H}$$

and to interpret this, we need to introduce the formalism of simplicial commutative rings and work with derived deformation functors.