THE DÉCALAGE FUNCTOR

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1. Definition for Modules

We'll start by doing very explicit computations of the décalage functor for modules over a commutative ring A with unity.

Fix a nonzerodivisor (NZD) $f \in A$. Let $C(A)$ denote the category of cochain complexes of A-modules, let $C(A)_f$ denote the full subcategory of complexes whose terms are all f-torsion-free (i.e. for all i, if $f \cdot x = 0$ for $x \in C^i$ then $x = 0$), and let $D(A)$ denote the associated derived category of $C(A)$.

1.1. **Definition.** For any complex $C \in C(A)_f$, we may associate the complex $C[1/f]$, defined by $(C[1/f])^i = C^i[1/f]$: one checks that this is again a cochain complex.

Then we define $\eta_f C$ to be the sub-complex

$$(\eta_f C)^i = \{x \in f^i C^i : dx \in f^{i+1} C^{i+1}\}.$$

In other words, $\eta_f C$ is the largest sub-complex of $C[1/f]$ which in each degree i is contained in $f^i C^i$. One readily checks that $\eta_f : C(A)_f \to C(A)$ defines a functor.

Perhaps the most interesting feature of η_f is how it modifies cohomology groups.

1.2. **Proposition.** *The map $Z^i(C) \to Z^i(\eta_f C)$ defined by $x \mapsto f^i x$ induces an isomorphism*

$$H^i(C)/H^i(C)[f] \cong H^i(\eta_f C),$$

for all $i \in \mathbf{Z}$, which is natural in C.

Proof. If $x \in C^i$ is a coboundary, then $dx' = x$ for $x' \in C^{i-1}$, then $f^i x' \in (\eta_f C)^{i-1}$, and $d(f^i x') = f^i dx' = f^i x$, which is the image of x under $Z(C^i) \to Z((\eta_f C)^i)$, so we get a well defined map

$$H^i(C) \to H^i(\eta_f C).$$

It remains to compute the kernel. The following are equivalent:

- $f^i x$ is a coboundary for some $x \in Z(C^i)$
- $f^i x = d(f^{i-1} x') = f^{i-1} dx'$ for some $x' \in C^{i-1}$
- (C^i is f-torsion-free) $fx = dx'$ for some $x' \in C^{i-1}$
- $[fx] = 0 \in H^i(C)$.

Lastly, this is surjective because an element of $H^i(\eta_f C)$ is represented by $f^i x$ for some $x \in C^i$ with $d(f^i x) = f^i dx = 0$, but again C^i is f-torsion-free, so $dx = 0$. $\square$

So maybe this wasn't the most conceptual argument, but basically you want some functorial way of killing f-torsion on the level of complexes, so you invert f, but then maybe this is still too big, so you throw away everything that doesn't live in $f^i C^i$ in each degree, and voilà.

On the other hand, we wish to kill f-torsion on the level of cohomology for *any* complex. Since cohomology is only defined up to quasi-isomorphism of complexes, we use the following proposition to extend the definition to any complex.

1.3. **Definition.** A complex $C \in C(A)$ is called strongly K-flat if each C^i is a flat A-module, and for every acyclic complex $M \in C(A)$, the total complex $\mathrm{Tot}(M^\bullet \otimes_A C^\bullet)$ is acyclic.

1.4. **Proposition.** *Any complex $D \in C(A)$ admits a quasi-isomorphism $D \xrightarrow{\sim} C$ to a strongly K-flat complex $C \in C(A)$, (which in particular lives in $C(A)_f$).*

Proof. This is [1, Tag 06Y4], but we sketch the argument. If D is bounded above, you can take a quasi-isomorphism to a strongly K-flat complex. If D is not bounded above, then you can fix any n and form a diagram

$$\begin{array}{ccccc} \tau_{\le n} D & \longrightarrow & \tau_{\le n+1} D & \longrightarrow & \cdots \\ \uparrow\sim & & \uparrow\sim & & \\ C_n & \longrightarrow & C_{n+1} & \longrightarrow & \cdots \end{array}$$

such that each C_i is strongly K-flat. This induces a map $\varinjlim_n C_n \to D$, which by exactness of filtered colimits is still a quasi-isomorphism, and furthermore one can show that $C = \varinjlim_n C_n$ is still strongly K-flat. $\square$

Furthermore, note that

1.5. **Lemma.** *If $C \xrightarrow{\sim} D$ is a quasi-isomorphism of f-torsion-free complexes, then $\eta_f C \xrightarrow{\sim} \eta_f D$ is a quasi-isomorphism as well.*

Proof. This follows from naturality of

$$\begin{array}{ccc} H^i(C)/H^i(C)[f] & \xrightarrow{\sim} & H^i(\eta_f C) \\ \downarrow\sim & & \downarrow\sim \\ H^i(D)/H^i(D)[f] & \xrightarrow{\sim} & H^i(\eta_f D) \end{array}$$

$\square$

1.6. **Corollary.** *The functor $\eta_f : C(A)_f \to C(A)$ extends to a functor $L\eta_f : D(A) \to D(A)$ on the derived category.*

In [2], they give a more general definition: in particular, they are situated in some ringed topos, i.e. they consider a site (e.g. the pro-étale site for an adic space, big/small Zariski site topology for a formal scheme) with a sheaf of rings $\mathscr{O}$ defined on this site, and instead of defining $L\eta_f$ for $f \in A$, they define $L\eta_{\mathcal{I}}$ for $\mathcal{I}$ an ideal sheaf in $\mathscr{O}$. But in the paper, one can really restrict our attention to a constant sheaf of rings (they use A_{inf}) and look at principal ideals, and even in this setting we don't lose much if we study categories of modules: the proofs generalize easily.

2. Almost Quasi-Isomorphisms

Fix a ring A and an ideal $I \subseteq A$.

2.1. **Definition.** If $\varphi : M \to N$ is a map in Mod_A, then φ is an **I-almost isomorphism** if the kernel and cokernel of φ are killed by I.

2.2. **Definition.** If $\varphi : C \to D$ is a map in $C(A)$, then φ is an **I-almost quasi-isomorphism** if the maps $H^i(C) \to H^i(D)$ have kernel and cokernel killed by I.

One of the magical facts about $L\eta_f$ is that in good enough circumstances, it can turn almost quasi-isomorphisms into actual quasi-isomorphisms.

2.3. **Lemma.** *Fix an ideal $I \subseteq A$ and a NZD $f \in I$. Suppose $M \in \mathsf{Mod}_A$ and M/fM have no non-zero elements killed by every element of I. Then if $\varphi : M \to N$ is an I-almost isomorphism, $M/M[f] \to N/N[f]$ is an isomorphism.*

Proof. Since M has no I-torsion, $\ker \varphi = 0$, so we can regard $M \subseteq N$ as a submodule. Clearly $M \cap N[f] = M[f]$, proving injectivity.

For surjectivity, given an $n \in N$, we need to find $m \in M$ such that $n - m \in N[f]$, i.e. $fn = fm$. Note N/M is killed by I, so $fn \in M$. The image $\overline{fn} \in M/fM$ is killed by I (because $\overline{gfn} = \overline{f(gn)}$ and $gn \in M$), so $fn \in fM$. □

2.4. **Corollary.** *If $\varphi : C \to D$ is an I-almost-quasi-isomorphism, and $H^i(C)$ and $H^i(C)/fH^i(C)$ have no I-torsion, then $\eta_f C \to \eta_f D$ (for any $f \in I$) is a quasi-isomorphism.*

The remarkable part about this is that we only need to know about C for this to work. We don't need to check anything about D to get this kind of result.

3. Further Properties

We list some properties that will be important for later talks.

(1) Consider the exact sequence

$$0 \to A/f \to A/f^2 \to A/f \to 0.$$

If $C \in C(A)$ is f-torsion-free, then tensoring with C gives an exact sequence

$$0 \to C/fC \to C/f^2C \to C/fC \to 0.$$

Taking the long exact sequence associated to this complex gives "Bockstein" (boundary) homomorphisms

$$H^i(C/fC) \xrightarrow{\beta} H^{i+1}(C/fC).$$

One can show that $\beta^2 = 0$, so this turns $(H^\bullet(C/fC), \beta)$ into a complex. There is a natural map of complexes

$$\eta_f C \to (H^\bullet(C/fC), \beta)$$

given as follows: if $f^i x \in (\eta_f C)^i$ for $x \in C^i$ and $d(f^i x) = f^{i+1}x'$ for some $x' \in C^{i+1}$, so since C is f-torsion-free, $dx = fx'$, and so $d\overline{x} = 0$ where $\overline{x} \in C^i/fC^i$, so $\overline{x}$ gives a class in $H^i(C/fC)$. One can check the map forms a map of complexes, and furthermore, one can check that

$$\eta_f C \otimes_A A/fA \to (H^\bullet(C/fC), \beta)$$

is a quasi-isomorphism. More generally if D is any complex,

$$L\eta_f D \otimes^{\mathbf{L}}_A A/fA \to (H^\bullet(D \otimes^{\mathbf{L}}_A A/fA), \beta)$$

is a quasi-isomorphism.

(2) The functor η is multiplicative, i.e. $\eta_f \eta_g = \eta_{fg}$, and induces a natural equivalence $L\eta_f \circ L\eta_g = L\eta_{fg}$.

(3) (Base Change) Suppose $\alpha : A \to B$ is a map of rings such that $f \in A$ and $\alpha(f) \in B$ are NZD. For $D \in C(A)$, we want

$$(L\eta_f D) \otimes^{\mathbf{L}}_A B \to L\eta_{\alpha(f)}(D \otimes^{\mathbf{L}}_A B)$$

To do this, we take a K-flat resolution $C \xrightarrow{\sim} D$. From here we can take $\eta_f C$, but then to represent a derived tensor product with B by an ordinary one, we need to take another K-flat resolution $C_0 \to \eta_f C$. On the other hand, we can first form $C \otimes_A B$ (this represents $D \otimes^{\mathbf{L}} B$ since C is K-flat), and then apply $\eta_{\alpha(f)}$ (note $C \otimes_A B$ is still a complex of flat modules). We summarize this as follows:

$$\begin{array}{ccccccccc}
D & \xleftarrow{\sim} & C & \rightsquigarrow & \eta_f C & \xleftarrow{\quad\sim\quad} & C_0 & \rightsquigarrow & C_0 \otimes_A B \\
\| & & \| & & & & & & \\
D & \xleftarrow{\sim} & C & \rightsquigarrow & C \otimes_A B & \rightsquigarrow & \eta_{\alpha(f)}(C \otimes_A B) & &
\end{array}$$

At each level i, there is a map $C_0^i \otimes_A B \to (\eta_f C)^i \otimes_A B$. The A-bilinear map $(\eta_f C)^i \times B \to (\eta_{\alpha(f)}(C \otimes_A B))^i$ given by $(f^i x, b) \mapsto \alpha(f)^i(x \otimes b)$ induces a map $(\eta_f C)^i \otimes_A B \to (\eta_{\alpha(f)}(C \otimes_A B))^i$, which finally induces

$$(L\eta_f C) \otimes_A^{\mathbf{L}} B \to L\eta_{\alpha(f)}(C \otimes_A^{\mathbf{L}} B)$$

In general, this map is *not* an isomorphism. But this simplifies considerably if α is flat: in particular, we need only take one K-flat resolution, so the situation looks like

$$\begin{array}{ccccccc} D & \xrightarrow{\sim} & C & \rightsquigarrow & \eta_f C & \rightsquigarrow & \eta_f C \otimes_A B \\ \| & & \| & & & & \\ D & \xrightarrow{\sim} & C & \rightsquigarrow & C \otimes_A B & \rightsquigarrow & \eta_{\alpha(f)}(C \otimes_A B), \end{array}$$

and one can directly show that $(\eta_f C) \otimes_A B \to \eta_{\alpha(f)} C$ is an isomorphism. In general, this extends to a map of ringed topoi.

4. Application to A_{inf} Cohomology

Briefly, we mention the important of $L\eta$. The basic construction goes as follows: I'll go into some detail, but not too much, as this will be the focus of later talks in the study group. Let $\mathfrak{X}$ be a smooth and proper formal scheme over $\mathcal{O}$, the ring of integers in $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}_p}$. To $\mathfrak{X}$ we can associate the rigid generic fiber X, which is an adic space on which we can define the pro-étale site X_{proet}. As mentioned in Pol's talk we can define a sheaf

$$\mathbf{A}_{\mathrm{inf},X} = W(\widehat{\mathscr{O}}^+_{X^\flat}) = W(\varprojlim_{\phi} \mathscr{O}^+_X/p)$$

on the pro-étale site: this is a sheaf of algebras over the constant sheaf of rings A_{inf}. There is a map $\nu : X_{proet} \to \mathfrak{X}_{Zar}$, and we take

$$A\Omega_{\mathfrak{X}} := L\eta_\mu(R\nu_* \mathbf{A}_{\mathrm{inf},X}),$$

where $\mu = [\epsilon] - 1$ is some specific element of A_{inf}, where $\epsilon = (1, \mu_p, \mu_{p^2}, \dots)$.

Locally on $\mathfrak{X}_{Zar}$, one can assume $\mathfrak{X} = \operatorname{Spf} R$ for R a p-adically complete, formally smooth $\mathcal{O}$-algebra, where $\mathfrak{X}$ is connected, and R is formally étale over $\mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle$. We can define

$$R_\infty = R \widehat{\otimes}_{\mathcal{O}\langle T^{\pm 1}\rangle} \mathcal{O}\langle T^{1/p^\infty}\rangle,$$

which has an action of $\mathbf{Z}_p^d = \operatorname{Gal}(\operatorname{Frac}(R_\infty)/\operatorname{Frac}(R))$. By the almost purity theorem of Faltings, we get an $[\mathfrak{m}^\flat]$-almost quasi-isomorphism

$$R\Gamma_{cont}(\mathbf{Z}_p^d, A_{\mathrm{inf}}(R_\infty)) \to R\Gamma_{proet}(X, \mathbf{A}_{\mathrm{inf},X})$$

of complexes of $\mathbf{A}_{\mathrm{inf}}$-modules, where $\mathfrak{m}^\flat \subseteq \mathcal{O}^\flat$ is the maximal ideal in the tilt. The point is that we can show that the left side is "good", in the sense that it has no non-zero elements killed by $[\mathfrak{m}^\flat]$, and neither does the quotient by μ. Then applying $L\eta_\mu$ tells us that

$$L\eta_\mu R\Gamma_{cont}(\mathbf{Z}_p^d, A_{\mathrm{inf}}(R_\infty)) \to L\eta_\mu R\Gamma_{proet}(X, \mathbf{A}_{\mathrm{inf},X})$$

is a quasi-isomorphism on the nose. But the right hand side, it turns out, is $A\Omega_{\mathrm{Spf}\,R}$, and the left hand side can be computed more easily.

One application of the Bockstein homomorphism and multiplicativity of $L\eta_\bullet$ is a comparison theorem.

$$\begin{aligned}
A\Omega_{\mathfrak{X}} \otimes^{\mathbf{L}}_{\mathbf{A}_{\mathrm{inf},\theta}} \mathcal{O} &= (L\eta_\mu R\nu_* \mathbf{A}_{\mathrm{inf},X}) \otimes^{\mathbf{L}}_{A_{\mathrm{inf},\theta}} \mathcal{O} \\
&\xrightarrow{\sim}^{\varphi} (L\eta_{\varphi(\mu)} R\nu_* \mathbf{A}_{\mathrm{inf},X}) \otimes^{\mathbf{L}}_{A_{\mathrm{inf},\widetilde{\theta}}} \mathcal{O} \\
&= L\eta_{\widetilde{\psi}}(L\eta_\mu R\nu_* \mathbf{A}_{\mathrm{inf},X}) \otimes^{\mathbf{L}}_{A_{\mathrm{inf},\widetilde{\theta}}} \mathcal{O} \\
&= L\eta_{\widetilde{\xi}} A\Omega_{\mathfrak{X}} \otimes^{\mathbf{L}}_{A_{\mathrm{inf},\widetilde{\theta}}} \mathcal{O} \\
&= H^\bullet(A\Omega_{\mathfrak{X}}/\widetilde{\xi}) = \Omega^{\bullet,cont}_{\mathfrak{X}/\mathcal{O}} = \varprojlim_n \Omega^i_{(\mathfrak{X}/p^n)/(\mathcal{O}/p^n)}
\end{aligned}$$

5. de Rham-Witt Complex

I'll say something briefly about the historical origin of this functor.

5.1. **Definition.** Let $A \to B$ be a morphism of $\mathbf{Z}[1/p]$-algebras. An F-V-procomplex is the data $(\mathcal{W}^\bullet_r, R, F, V, \lambda_r)$, where

(1) Each $\mathcal{W}^\bullet_r$ is a commutative d.g. $W_r(A)$-algebra for each $r \geq 1$.
(2) Morphisms $R : \mathcal{W}^\bullet_{r+1} \to R_*\mathcal{W}^\bullet_r$ of d.g. $W_{r+1}(A)$-algebras.
(3) Morphisms $F : \mathcal{W}^\bullet_{r+1} \to F_*\mathcal{W}^\bullet_r$ of d.g. $W_{r+1}(A)$-algebras.
(4) Morphisms $V : F_*\mathcal{W}^\bullet_r \to \mathcal{W}^\bullet_{r+1}$ of d.g. $W_{r+1}(A)$-algebras.
(5) Morphisms $\lambda_r : W_r(B) \to \mathcal{W}^0_r$ commuting with F, V, R.

such that R commutes with F, V, $FV = p$, $FdV = d$, $V(F(x)y) = xV(y)$, and

$$Fd\lambda_{r+1}([b]) = \lambda_r([b])^{p-1} d\lambda_r([b])$$

for $b \in B$, $r \geq 1$.

One can show there is an initial object in the category of F-V-procomplexes, and this is called the de Rham-Witt complex.

5.2. **Proposition.** *Let k be a perfect field of characteristic p and R a smooth k-algebra. Then (Illusie 1979) The Frobenius $\varphi : W\Omega^\bullet_{R/k} \to W\Omega^\bullet_{R/k}$ is injective and has image $\eta_p W\Omega^\bullet_{R/k}$. In particular (Berthelot-Ogus), there is a Frobenius semi-linear isomorphism*

$$R\Gamma_{crys}(R/W(k)) \xrightarrow{\sim} L\eta_p R\Gamma_{crys}(R/W(k))$$

References

[1] The Stacks Project. Available at `http://stacks.math.columbia.edu`.
[2] Bhatt, B., Morrow, M., Scholze, P. 2016, *Integral p-adic Hodge theory*, arXiv:1602.03148.
[3] Morrow, M. 2016, *Notes on the A_{inf}-cohomology of Integral p-adic Hodge theory* arXiv:1608.00922.