Let G be a connected reductive group $/\mathbb{F}_q$, let $k = \overline{\mathbb{F}}_q$

Last time: fix F-stable max. torus $T \leq G$ and temporarily

fix $B \supseteq T$, then we get

$$Y_{T\subset B} \longrightarrow X_{T\subset B}$$

T^F-torsor

G^F-equivariant

Fix $\theta : T^F \to \overline{\mathbb{Q}}_\ell$. Then

$$\rightsquigarrow R_T^\theta = R_{T\subset B}^\theta \in R(G^F)$$

Today: define an $\sim$ on pairs (T, θ)

Geometric conjugacy

$(T,\theta) \sim (T', \theta')$ "geometrically conjugate pairs"

Theorem: (next talk)

if (T,θ) and (T', θ') are no geometrically conj.

then $\langle R_T^\theta, R_{T'}^{\theta'} \rangle = 0$

and if $(T,\theta) \sim (T', \theta')$ then there is a 'simple formula'

for $\langle R_T^\theta, R_{T'}^{\theta'} \rangle \neq 0$

§ Geometric conjugacy.

Consider an F-stable maxl torus T, where $F : T \to T$ is the q-Frobenius. This induces a map

$$F : X_*(T) \longrightarrow X_*(T)$$

(where $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$) We have an F-eq. isomorphism

$$X_*(T) \otimes_{\mathbb{Z}} k^\times \xrightarrow{\sim} T$$
$$n \otimes \alpha \longmapsto n(\alpha)$$

$$\rightsquigarrow \quad 0 \to T^F \to X_*(T) \otimes_{\mathbb{Z}} k^\times \xrightarrow{F-\mathrm{id}} X_*(T) \otimes_{\mathbb{Z}} k^\times \to 0$$

But can use $k^\times \cong (\mathbb{Q}/\mathbb{Z})_{p'}$ + clever application of snake lemma:

'dual' exact sequence

$$0 \to X_*(T) \xrightarrow{F-\mathrm{id}} X_*(T) \to T^F \to 0$$

Def: For $n > 0$, the norm map

$$N = \frac{F^n - \mathrm{id}}{F - \mathrm{id}} = \sum_{i=0}^{n-1} F^i : T^{F^n} \to T^F$$

Lemma:
$$\begin{tikzcd} X_*(T) \arrow[r] \arrow[rr, bend right] & T^{F^n} \arrow[r, "N"] & T^F \end{tikzcd}$$

in exact sequence we can take map to T^F or T^{F^n} compatible under N.

Prop: Let (T,θ), (T',θ') be 2 pairs where T, T' F-stable max. tori. $\theta, \theta' : T^F, T'^F \longrightarrow \bar{\mathbb{Q}}_\ell$

TFAE

(1) $\exists\, g \in G$ s.t. $gTg^{-1} = T'$ and

$$\begin{tikzcd} X_*(T) \arrow[r] \arrow[d, "\mathrm{ad}(g)"'] & T^F \arrow[dr, "\theta"] \arrow[d, phantom, "\circlearrowleft"] & \\ X_*(T') \arrow[r] & T'^F \arrow[r, "\theta'"'] & \bar{\mathbb{Q}}_\ell \end{tikzcd}$$

(2) for some $n \gg 0$ $\exists\, g \in G^n$ s.t. $gTg^{-1} = T'$

$$\begin{tikzcd} T^{F^n} \arrow[r, "N"] \arrow[d, "\mathrm{ad}(g)"'] & T^F \arrow[dr, "\theta"] \arrow[d, phantom, "\circlearrowleft"] & \\ T'^{F^n} \arrow[r, "N"] & T'^F \arrow[r, "\theta'"'] & \bar{\mathbb{Q}}_\ell \end{tikzcd}$$

$\rightarrow$ go to high enough extension so that T, T' split.

Def: Two pairs (T,θ) and (T',θ') are <u>geometrically conjugate</u> if the equivalent conditions of the prop above hold.

↳ How many geom. conj classes are there?

Now fix absolute torus $\mathbb{T}$. Now given a pair (T,θ), we get a character of $X_*(\mathbb{T})$ by

$$X_*(\mathbb{T}) \xrightarrow{\sim} X_*(T) \to T^F \xrightarrow{\theta} \bar{\mathbb{Q}}_\ell$$

Then θ defines an elt of

$$\mathrm{Hom}(X_*(\mathbb{T}), \bar{\mathbb{Q}}_\ell) = \mathrm{Hom}(X_*(\mathbb{T}), \mu_\infty(\bar{\mathbb{Q}}_\ell)) = \mathrm{Hom}(X_*(\mathbb{T}), k^\times)$$
$$= X^*(\mathbb{T}) \otimes k^\times \circlearrowleft F$$

by construction this is fixed by F.

Note that $W = N(\mathbb{T})/\mathbb{T}$ acts on $X^*(T)$ (character lattice) by precomposition. We let

$$\mathcal{S} = \left[(X^*(T) \otimes k^*)/W \right]^F$$

Denote the image of θ in $\mathcal{S}$ by $[\theta] \in \mathcal{S}$

Rk: $X^*(\mathbb{T}) = X_*(\mathbb{T}^\vee)$ where $\mathbb{T}^\vee$ is the dual lattice inside the dual group $G^\vee$. (Langlands?)

Prop: The association $(T, \theta) \longmapsto [\theta]$ induces a bijection from $\{(T,\theta)\}/\sim \; \longrightarrow \mathcal{S}$ and $|\mathcal{S}| = |(Z^\circ)^F|\, q^r$

Z° is the connected center of G, r is the semisimple rank of G i.e. this is the rank of a max'l torus in $G/R(G)$

pf well-defined by (1). Injectivity if $(T,\theta), (T',\theta')$ satisfy

$$[\theta] = [\theta'] \subset X^*(T) \otimes k^*/W$$

$$\begin{tikzcd}
X_*(\mathbb{T}) \arrow[r, "\mathrm{ad}\, g"] \arrow[d, "\mathrm{ad}\, w"'] & X_*(T) \arrow[r] & T^F \arrow[r] & \overline{\mathbb{Q}}_\ell \\
X_*(\mathbb{T}) \arrow[r, "\mathrm{ad}\, g'"'] & X_*(T') \arrow[r] & T'^F \arrow[ur] &
\end{tikzcd}$$

Surjectivity: If W-orbit of $\theta : X_*(\mathbb{T}) \to \overline{\mathbb{Q}}_\ell$ s.t. $\exists\, w \in W$ s.t. $F.w.\theta = \theta$, so θ descends to a map

$\mathbb{T}(w)^F \to \overline{\mathbb{Q}}_\ell$, then $\mathbb{T}(w)^F = T^F$ for some F-stable max'l. torus in G.

WTS: $|\mathcal{S}| = |(Z^\circ)^F| q^r$

$\mathcal{S} = (\mathbb{T}^\vee/W)^F$

$|(\mathbb{T}^\vee/W)^F| = \sum (-1)^i \, tr(F, H^i_c(\mathbb{T}^\vee/W))$

$= \sum (-1)^i \, tr(F, H^i_c(\mathbb{T}^\vee)^W)$

But $\exists$ isogeny $\mathbb{T}^\vee \simeq (\mathbb{T}')^\vee \times (Z^\circ)^\vee$ $\quad$ $\mathbb{T}'$ - image of $\mathbb{T}$ under $G \to G/Z^\circ$

so $|(\mathbb{T}^\vee/W)^F| = |(Z^\circ)^F| \sum (-1)^i \, tr(F, H^i_c((\mathbb{T}')^\vee)^W))$

[Prop 5.7.] use Poincare duality explicit comp of coh. of $\mathbb{T}'$ $\quad X_*(\mathbb{T}')^W = 0$

Ex lets do $GL_2/\mathbb{F}_q$. T = standard $\begin{pmatrix} k^* & \\ & k^* \end{pmatrix}$

$X_*(T) = \mathbb{Z}^2 \qquad (a,b) \longleftrightarrow \left(x \longmapsto \begin{pmatrix} x^a & \\ & x^b \end{pmatrix}\right)$

Note $F = (x \mapsto x^p)$ $\quad F: \mathbb{Z}^2 \to \mathbb{Z}^2$
$\qquad (a,b) \mapsto (pa, pb)$

$\operatorname{coker}(\mathbb{Z}^2 \xrightarrow{F-id} \mathbb{Z}^2) = \mathbb{Z}/p-1 \oplus \mathbb{Z}/p-1 \cong \mathbb{F}_p^\times \times \mathbb{F}_p^\times = T^F$

$(a,b) \mapsto ((p-1)a, (p-1)b)$

$$\mathcal{S} = [(k^\times)^2/W]^F = \begin{cases} (x,x) & x \in \mathbb{F}_p^\times \\ (x,y) & x,y \in \mathbb{F}_p^\times \text{ distinct} \\ (x,x^p) & x \in \mathbb{F}_{p^2}^\times \setminus \mathbb{F}_p^\times \end{cases}$$

$W = S_2 \qquad$ st $S_2 \qquad s \cdot \begin{pmatrix} a & \\ & b \end{pmatrix} = \begin{pmatrix} b & \\ & a \end{pmatrix}$, $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$|\mathcal{S}| = p-1 + \binom{p-1}{2} + \frac{p^2-1-p+1}{2} = p(p-1)$

$r=1$ here the max'l torus in PGL_2

$$|Z^\circ(GL_2)^F| = |\mathbb{G}_m^F| = |\mathbb{F}_p^\times| = p-1$$

The rest of § 5 is more about relationship between geometric conjugacy in G and $G^\vee$