

# THE EICHLER-SHIMURA ISOMORPHISM

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## 1. INTRODUCTION

We are studying the cohomology of arithmetic groups. Today, we will describe the case where when  $G = \mathrm{SL}_2$ , and  $\Gamma$  is a congruence subgroup, which is an important case showing up in the theory of modular forms.

I am mostly using [Bel] as a reference.

**Definition 1.1.** A subgroup  $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$  is a *congruence subgroup* if for some  $N$ ,

$$\Gamma(N) = \ker(\mathrm{SL}_2(\mathbf{Z}) \rightarrow \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})) \subseteq \Gamma.$$

So fix  $\Gamma$  as above and fix a weight  $k \geq 2$ . Let  $S_k(\Gamma)$  denote the space of cusp forms of level  $\Gamma$  and weight  $k$ , and let  $\mathcal{E}_k(\Gamma)$  denote the space of Eisenstein series of level  $\Gamma$  and weight  $k$ . The goal of today's talk is to prove the following result.

**Theorem 1.2** (Eichler-Shimura). *There is a Hecke-equivariant isomorphism*

$$S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \oplus \mathcal{E}_k(\Gamma) \xrightarrow{\sim} H^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbf{C}^2))$$

where  $\Gamma$  acts on  $\mathbf{C}^2$  via  $\Gamma \hookrightarrow \mathrm{GL}_2(\mathbf{C})$ .

Here  $\overline{S_k(\Gamma)}$  denotes the space of anti-holomorphic cusp forms, which in this case is actually isomorphic to  $S_k(\Gamma)$ . We will explain what “Hecke-equivariant” means later on in the talk.

## 2. MODULAR SYMBOLS

Modular symbols (which basically amount to homology classes) turn out to be a nice way of computationally accessing the link between spaces of modular forms and cohomology, so we will use them as our tool to construct the Eichler-Shimura map. They also turn out to be a nice way to construct  $p$ -adic  $L$ -functions, which is one of their primary uses.

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**Definition 2.1.** Let  $\Delta$  denote the group of divisors on  $\mathbf{P}^1(\mathbf{Q})$ . In other words, these are finite sums  $\sum_i n_i [r_i]$  with  $r_i \in \mathbf{Q} \cup \{\infty\}$ . Let

$$\Delta_0 = \left\{ D = \sum_i n_i [r_i] \in \Delta : \sum_i n_i = 0 \right\}$$

Note  $\Delta_0$  admits a left  $\Gamma$ -action via Möbius transformations. In other words if  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , we have an action on  $\mathbf{P}^1(\mathbf{Q})$  given by

$$\gamma \cdot [r] = \left[ \frac{ar + b}{cr + d} \right]$$

and this extends to all of  $\Delta_0$ .

**Remark 2.2.** Note that the set of  $\Gamma$ -equivalence classes of  $\mathbf{P}^1(\mathbf{Q})$  is exactly the set of cusps in the compactification of the modular curve. When we see the definition of a modular symbol associated to a modular form, we will need the interpretation of  $\Delta_0$  as the space of (finite sums of) equivalence classes of spaces of paths between cusps.

**Definition 2.3.** Let  $V$  be a group with a left  $\Gamma$ -action. We let

$$\text{Symb}_\Gamma(V) = \text{Hom}_\Gamma(\Delta^0, V)$$

and call  $\text{Symb}_\Gamma(V)$  the space of *modular symbols of level  $\Gamma$*  with values in  $V$ .

Note that we can put a left  $\Gamma$ -action on  $\text{Hom}(\Delta_0, V)$  which takes

$$\phi \mapsto [\gamma \cdot \phi : D \mapsto \gamma^{-1} \cdot \phi(\gamma \cdot D)]$$

Then we visibly have  $\text{Hom}(\Delta_0, V)^\Gamma = \text{Hom}_\Gamma(\Delta_0, V)$ .

### 3. COHOMOLOGY

We now write down a cohomological interpretation of these modular symbols. First we make the technical assumption that  $\Gamma$  acts freely on  $\mathcal{H}$  (this is satisfied, for instance, when  $\Gamma(3) \subseteq \Gamma$ ). Note  $Y_\Gamma = \mathcal{H}/\Gamma$  is a classifying space for  $\Gamma$  (basically because  $\Gamma$  acts freely and discontinuously on  $\mathcal{H}$ , and  $\mathcal{H}$  is contractible), which (by general manipulations) implies that for  $V$  any left  $\Gamma$ -module,

$$H^i(\Gamma, V) \cong H^i(Y_\Gamma, \tilde{V})$$

where  $\tilde{V}$  is the local system associated to  $V$ , and where on the right we take singular cohomology. By abuse of notation, we let  $H_c^i(\Gamma, V)$  denote the compactly supported cohomology of  $Y_\Gamma$  with  $\tilde{V}$ -coefficients.

**Proposition 3.1.** *There is a canonical and functorial isomorphism*

$$\text{Symb}_\Gamma(V) \xrightarrow{\sim} H_c^i(\Gamma, V)$$

*Proof.* Let  $\bar{\mathcal{H}} = \mathcal{H} \cup \mathbf{P}^1(\mathbf{Q})$  denote the compactification of  $\mathcal{H}$  given by adding cusps. Note  $\mathcal{H}$  is not compact, so  $H_c^0(\bar{\mathcal{H}}, \tilde{V}) = 0$ . Note further that  $\bar{\mathcal{H}}$  is contractible, so  $H^1(\bar{\mathcal{H}}, \tilde{V}) = 0$ . Then if we consider the long exact sequence for a closed subset for compactly supported cohomology, and if we take Homs into  $V$  for the split short exact sequence  $0 \rightarrow \Delta_0 \rightarrow \Delta \rightarrow \mathbf{Z} \rightarrow 0$ , we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\bar{\mathcal{H}}, \tilde{V}) & \longrightarrow & H^0(\mathbf{P}^1(\mathbf{Q}), \tilde{V}) & \longrightarrow & H_c^1(\mathcal{H}, V) \longrightarrow 0 \\ & & \parallel & & \parallel & & \\ 0 & \longrightarrow & V & \longrightarrow & \text{Hom}(\Delta, V) & \longrightarrow & \text{Hom}(\Delta_0, V) \longrightarrow 0 \end{array}$$

The left vertical equality follows from the fact that  $\mathcal{H}$  is connected, and the middle equality follows from the fact that  $\Delta$  is the free group generated by  $\mathbf{P}^1(\mathbf{Q})$ . Therefore  $H_c^1(\mathcal{H}, V) = \text{Hom}(\Delta_0, V)$ , and so a spectral sequence computation, combined with the fact that  $H_c^0(\mathcal{H}\Gamma, \tilde{V}) = 0$ , implies that

$$\text{Symb}_\Gamma(V) = \text{Hom}(\Delta_0, V)^\Gamma = H_c^1(\mathcal{H}, \tilde{V})^\Gamma = H_c^1(\mathcal{H}/\Gamma, \tilde{V}).$$

□

But we care about the whole  $H^i$ , so we need to see to what extent the modular symbols contribute to it.

**Definition 3.2.** The *interior cohomology*  $H_!^i(\Gamma, V)$  is by definition the image of the natural map

$$H_c^i(\Gamma, V) \rightarrow H^i(\Gamma, V)$$

In fact, Poincaré duality induces a perfect pairing

$$H_!^1(\Gamma, V) \times H_!^1(\Gamma, V^\vee) \rightarrow \mathbf{C}$$

(here  $V$  is a  $\mathbf{C}$ -vector space and  $\Gamma$  acts  $\mathbf{C}$ -linearly).

#### 4. CUSP FORMS

In practice, we let

$$V = V_k(\mathbf{C}) = \text{Sym}^{k-2}(\mathbf{C}^2),$$

which is the exact analog of specifying the weight  $k$  in the definition of a modular form.

So how do you associate a modular symbol to a cusp form? We need to construct a map that takes a “path”  $[s] - [r]$  and spits out an element of  $V_k(\mathbf{C}) = \text{Sym}^{k-2}(\mathbf{C}^2)$ . First we give a more concrete description of  $V$ . Let  $P_k(\mathbf{C})$  denote the space of homogeneous polynomials of degree  $k - 2$  in two variables, with the action

$$(P \cdot \gamma)(X, Y) = P(aX + bY, cX + dY).$$

Then  $V_k(\mathbf{C}) \cong P_k(\mathbf{C})^\vee$ . Note also that there is an isomorphism

$$\Theta_k : V_k(\mathbf{C}) \xrightarrow{\sim} P_k(\mathbf{C})$$

which follows from the classification of highest weight representations of  $\text{SL}_2(\mathbf{C})$ . Combined with Poincaré duality, we get a perfect pairing

$$H_!^1(\Gamma, V) \times H_!^1(\Gamma, V) \xrightarrow{1 \times \Theta_k} H_!^1(\Gamma, V) \times H_!^1(\Gamma, V^\vee) \rightarrow \mathbf{C}$$

We also have a pairing on  $S_k(\Gamma)$ :

**Definition 4.1.** The Petersson inner product is given by

$$(f, g)_\Gamma = \int_{\mathcal{H}/\Gamma} f(z) \overline{g(z)} y^{k-2} dx dy$$

It is a perfect sesquilinear pairing.

In a moment we will relate the two pairings. First we associate a modular symbol to each cusp form.

**Definition 4.2.** Given a cusp form  $f \in S_k(\Gamma)$ , we may define

$$I_f([s] - [r])(P) = \int_r^s f(z) P(z, 1) dz$$

and extend  $\phi_k(f)$  to all of  $\Delta^0$  (exercise: check that this is independent of the path chosen between  $r$  and  $s$  on the modular curve, and that this “extension to  $\Delta_0$ ” is well-defined). Convergence of this integral relies on the fact that  $f$  decays rapidly as it approaches the cusps, eventually hitting 0.

**Lemma 4.3.** *The map  $I_f$  is  $\Gamma$ -equivariant.*

*Proof.* If  $D \in \Delta_0$  and  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , then we just compute:

$$\begin{aligned}
I_f(\gamma \cdot ([s] - [r]))(P) &= I_f(\gamma \cdot [s] - \gamma \cdot [r])(P) \\
&= \int_{\gamma \cdot r}^{\gamma \cdot s} f(z)P(z, 1)dz \\
&= \int_r^s f(\gamma \cdot z)P(\gamma \cdot z, 1)d(\gamma \cdot z) \\
&= \int_r^s (cz + d)^k f(z)P\left(\frac{az + b}{cz + d}, 1\right)(cz + d)^{-2}dz \\
&= \int_r^s (cz + d)^k f(z)(cz + d)^{-(k-2)}(P \cdot \gamma)(z, 1)(cz + d)^{-2}dz \\
&= \int_r^s f(z)(P \cdot \gamma)(z, 1)dz \\
&= I_f([s] - [r])(P \cdot \gamma) \\
&= [\gamma \cdot (I_f([s] - [r]))](P)
\end{aligned}$$

Here we used the automorphy condition on  $f$ , the homogeneity of  $P$ , and the chain rule for  $dz$ . □

So we get a map

$$\phi_k : S_k(\Gamma) \xrightarrow{f \mapsto I_f} \text{Symb}_\Gamma(V_k(\mathbf{C})) \xrightarrow{\sim} H_c^1(\Gamma, V_k(\mathbf{C})) \rightarrow H^1(\Gamma, V_k(\mathbf{C}))$$

Note the existence of the  $\mathbf{R}$ -vector subspace  $H_{(c)}^1(\Gamma, V_k(\mathbf{R}))$  which satisfies

$$H_{(c)}^1(\Gamma, V_k(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C} \cong H_{(c)}^1(\Gamma, V_k(\mathbf{C}))$$

so we define a map

$$\text{ES}_k : S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \cong S_k(\Gamma) \otimes_{\mathbf{R}} \mathbf{C} \xrightarrow{2\Re(\phi_k) \otimes 1} H_c^1(\Gamma, V_k(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C} = H_c^1(\Gamma, V_k(\mathbf{C})) \rightarrow H^1(\Gamma, V_k(\mathbf{C}))$$

**Theorem 4.4.** *The map  $\text{ES}_k$  is an isomorphism.*

*Proof.* We first show injectivity of  $\Re(\phi_k) : S_k(\Gamma) \rightarrow H_c^1(V_k(\mathbf{R}))$ . To check injectivity, we use the following computation, due to Shimura:

$$(\Re(\phi_k)(f), \Re(\phi_k)(g)) = c_k((f, g)_\Gamma + (-1)^{k+1} \overline{(f, g)_\Gamma})$$

for some nonzero constant  $c_k \in \mathbf{C}$  (to do this computation, you need to trace through the cohomological formalism and figure out what the pairing is: basically it's given by a cup product, and by using a comparison with de Rham cohomology, you can get a handle on computing this cup product). Then if we assume  $\Re(\phi_k)(f) = 0$ , then applying the above formula to  $g$  and  $ig$  we see that  $\Re(f, g)_\Gamma = 0$  and  $\Im(f, g)_\Gamma = 0$ . Since  $(\cdot, \cdot)_\Gamma$  is perfect and  $(f, g)_\Gamma = 0$  for all  $g \in S_k(\Gamma)$ , we must have  $f = 0$ .

As for surjectivity, there is a dimension formula in Section 6 of [Hid93]. Following Bellaïche's book, we will sketch a proof when  $k = 0$ . In this case,  $S_2(\Gamma)$  can actually be identified with the space  $\Omega^1(X(\Gamma))$ , the space of holomorphic differential 1-forms on  $X_\Gamma$ . But by definition  $\dim_{\mathbf{C}} \Omega^1(X(\Gamma))$  is the genus of  $X_\Gamma$ , so  $\dim_{\mathbf{R}} S_2(\Gamma) = 2g$ . On the other hand we have a factorization (note  $V_0(\mathbf{R}) = \mathbf{R}$ )

$$\begin{array}{ccc}
H_c^1(Y_\Gamma, \mathbf{R}) & \longrightarrow & H^1(Y_\Gamma, \mathbf{R}) \\
\downarrow & \nearrow & \\
H^1(X_\Gamma, \mathbf{R}) & & 
\end{array}$$

but  $\dim_{\mathbf{R}} H^1(X_\Gamma, \mathbf{R}) = 2g$ . □

## 5. HECKE OPERATORS

As mentioned in the beginning, we should show that this map has “nice properties”. In this case, this means that the map respects Hecke operators.

First we define Hecke operators for modular forms. Take the monoid  $S = \mathrm{GL}_2^+(\mathbf{Q}) \cap M_2(\mathbf{Z})$ , and form the double coset  $\Gamma s \Gamma$ . If we pick a decomposition  $\Gamma s \Gamma = \bigsqcup_i \Gamma s_i$ , for  $s_i \in S$ , then we can define a right action on:

- $S_k(\Gamma)$ :

$$f \mapsto f|_{\Gamma s \Gamma} = \sum_i f[s_i]_k$$

where we define the slash operator

$$f[s_i]_k(z) = (\det s_i)^{k-1} j(s_i, z)^{-k} f(s_i \cdot z)$$

- $\mathrm{Symb}_\Gamma(V_k(\mathbf{R}))$ : note  $V_k(\mathbf{R})$  is actually an  $S$ -module because  $S$  acts on  $\mathbf{R}^2$  in the same way as  $\Gamma$ .

$$\phi \mapsto \phi|_{\Gamma s \Gamma} : D \mapsto \sum_i \phi(s_i \cdot D) \cdot s_i$$

**Remark 5.1.** When  $\Gamma = \Gamma_1(N)$ , the matrices  $1, 0, 0, p$  give you the  $T_p$  operators, which have the decomposition

**Proposition 5.2.** *The map  $\mathfrak{R}(\phi_k) : S_k(\Gamma) \rightarrow H^1(\Gamma, V_k(\mathbf{R}))$  respects the Hecke operators.*

*Proof.* First we show that  $S_k(\Gamma) \rightarrow \mathrm{Symb}(V_k(\mathbf{R}))$  respects the Hecke operators. Then the proposition follows from the fact that

$$\mathrm{Symb}_\Gamma(V(\mathbf{R})) \rightarrow H^1(\Gamma, V_k(\mathbf{R}))$$

is Hecke-equivariant, which follows from the fact that  $0 \rightarrow V \rightarrow \mathrm{Hom}(\Delta, V) \rightarrow \mathrm{Hom}(\Delta_0, V) \rightarrow 0$  is actually an exact sequence of  $S$ -modules, and the above map is the connecting homomorphism in the long exact sequence associated to taking  $\Gamma$ -invariants of this sequence.

But the proof that  $\mathrm{Symb}_\Gamma(V_k(\mathbf{R})) \rightarrow H^1(\Gamma, V_k(\mathbf{R}))$  is Hecke-equivariant is an only slightly more general version of Lemma 4.3. □

## 6. CORRESPONDENCES

There is a more geometric perspective on Hecke operators. To explain this, we use the formalism of *cohomological correspondences*. We will do this in a very elementary way, and in particular we will not discuss derived categories or the six functor formalism.

Again assume for simplicity that  $\Gamma$  acts freely on  $\mathcal{H}$ . Let  $F = \mathbf{R}, \mathbf{C}$ . As before, we fix an element  $s \in \mathrm{GL}_2^+(\mathbf{Q}) \cap M_2(\mathbf{Z})$ . Then we let

$$\Phi = \Gamma \cap s \Gamma s^{-1}$$

This allows us to define the map

$$H^i(\Gamma, V_k(F)) \xrightarrow{\text{res}} H^i(\Phi, V_k(F)) \xrightarrow{u \mapsto u(s^{-1} \cdot s)} H^i(s^{-1}\Phi s, V_k(F)) \xrightarrow{\text{cores}} H^i(\Gamma, V_k(F))$$

Note restriction is well-defined for any morphism of groups, and corestriction is an operation on Galois cohomology which is well-defined because  $[\Gamma : s^{-1}\Phi s] < \infty$ . The corestriction map is the map on cohomology induced by the norm map: if  $H \leq G$  is a finite index subgroup and  $A$  is a left  $G$ -module, then the norm map is (exercise: check this is well-defined)

$$\begin{aligned} A^H &\rightarrow A^G \\ a &\mapsto \sum_{gH \in G/H} g \cdot a \end{aligned}$$

Fix cocycle representatives  $\bar{g}$  for each  $g \in H \backslash G$ . Explicitly on cocycles then, the map is given by

$$\begin{aligned} H^i(H, A) &\rightarrow H^i(G, A) \\ u(\sigma_0, \dots, \sigma_i) &\mapsto \sum_{Hg \in H \backslash G} \bar{g}^{-1} x(\bar{g}\sigma_0[\bar{g}\sigma_0]^{-1}, \dots, \bar{g}\sigma_i[\bar{g}\sigma_i]^{-1}) \end{aligned}$$

**Exercise 6.1.** Show that the map

$$S_k(\Gamma) \xrightarrow{I_f} \text{Symb}_\Gamma(V_k(\mathbf{C})) \xrightarrow{\sim} H_c^1(\Gamma, V_k(\mathbf{C})) \rightarrow H^1(\Gamma, V_k(\mathbf{C}))$$

is given by

$$f \mapsto \left( \gamma \mapsto \int_{z_0}^{\gamma(z_0)} f(z)(zX + Y)^{k-2} dz \right)$$

and then show that this map is Hecke equivariant (you will use the fact that

$$\begin{aligned} \Phi \backslash \Gamma &\rightarrow \Gamma \backslash \Gamma s \Gamma \\ \gamma &\mapsto s\gamma \end{aligned}$$

is a bijection)

**Remark 6.2.** There is a more geometric picture of what's going on: we'll illustrate it for smooth complex curves. If we have the diagram of complex curves

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow g \\ X & & Y \end{array}$$

with nonconstant holomorphic maps, then you may take

$$H^1(X, \Omega^1) \xrightarrow{f^*} H^1(C, \Omega^1) \xrightarrow{g^*} H^1(Y, \Omega^1)$$

So for instance if we take  $X = Y = Y_\Gamma$  and  $C = Y_\Phi$  then we recover the situation above, with  $f$  the natural inclusion, and  $g = f \circ c_s$ , where  $c_s$  denotes conjugation. If  $s = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ , this should recover the  $T_p$  operator.

## 7. EISENSTEIN SERIES

To finish the proof of the Eichler-Shimura isomorphism, we need to address the Eisenstein part.

First we construct a map. Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , and define

$$\begin{aligned} \mathcal{E}_k(\Gamma) &\rightarrow H^1(\Gamma, V_k(\mathbf{C})) \\ f &\mapsto u_f : (\gamma \mapsto g|_{\gamma^{-1}, -(k-2)} - g) \end{aligned}$$

where  $g$  is some analytic function with  $g^{(k-1)} = f$ . One can show that in fact

$$(g|_{\gamma, -(k-2)})^{(k-1)}(z) = f|_{\gamma, k}(z),$$

Modularity of  $f$  implies that  $u_f(\gamma)^{(k-1)} = 0$ , so  $u_f(\gamma)$  is actually a polynomial landing in  $V_k(\mathbf{C})$ . One can check that  $u_\gamma$  in fact defines a cocycle, and that the map  $f \mapsto u_f$  is Hecke-equivariant.

**Lemma 7.1.** *The map  $f \mapsto u_f$  is injective.*

*Proof.* If  $u_f = 0$ , then  $g$  satisfies the weight  $-(k-2)$  automorphy condition, and since  $g$  is holomorphic at the cusps, it really is a modular form of weight  $-(k-2)$ . If  $k > 2$  then  $g = 0$ , and if  $k = 2$  then  $g$  is constant. Either way,  $f = 0$ .  $\square$

Thus we get a Hecke-equivariant embedding

$$H_1(\Gamma, V_k(\mathbf{C})) \oplus \mathcal{E}_k \hookrightarrow H^1(\Gamma, V_k(\mathbf{C}))$$

Another dimension computation in [Hid93] shows that this is actually an isomorphism.

**Remark 7.2.** The exact sequence

$$0 \rightarrow V \rightarrow \text{Hom}(\Delta, V) \rightarrow \text{Hom}(\Delta_0, V) \rightarrow 0$$

and the isomorphism above give

$$0 \rightarrow B\text{Symb}_\Gamma(V_k(\mathbf{C})) := \text{Hom}(\Delta, V_k(\mathbf{C}))^\Gamma / V_k(\mathbf{C})^\Gamma \rightarrow H_c^1(\Gamma, V_k(\mathbf{C})) \rightarrow H^1(\Gamma, V_k(\mathbf{C})) \rightarrow \mathcal{E}_k(\Gamma) \rightarrow 0$$

Note Poincaré duality (and an identification  $V_k(\mathbf{C}) \cong V_k(\mathbf{C})^\vee$ ) gives a pairing

$$H_c^1(\Gamma, V_k(\mathbf{C})) \times H^1(\Gamma, V_k(\mathbf{C})) \rightarrow \mathbf{C}$$

Fact:  $(c|_{\Gamma_s\Gamma}, d) = (c, d|_{\Gamma_{s'}\Gamma})$  where  $s' = \det(s)s^{-1}$ , so this pairing “respects Hecke operators”. Then one can define a pairing

$$\begin{aligned} B\text{Symb}_\Gamma(V_k(\mathbf{C})) \times \mathcal{E}_k(\Gamma) &\rightarrow \mathbf{C} \\ (x, y) &\mapsto (x, \tilde{y}) \end{aligned}$$

where  $\tilde{y}$  is any lift of  $y$  to  $H^1(\Gamma, V_k(\mathbf{C}))$ , and one can show this is well-defined. This pairing also respects Hecke operators. This induces a Hecke-equivariant isomorphism

$$B\text{Symb}_\Gamma(V_k(\mathbf{C})) \cong \mathcal{E}_k(\Gamma)^\vee.$$

## REFERENCES

- [Bel] Joel Bellaïche. *Eigenvarieties, families of Galois representations,  $p$ -adic  $L$ -functions*.
- [Hid93] Haruzo Hida. *Elementary theory of  $L$ -functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.