COLEMAN THEORY AND HIGHER COLEMAN THEORY

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Contents

Today I will discuss Coleman theory and higher Coleman theory. We will first show how to p-adically vary the weight of finite-slope cohomology classes, and then use this variation to construct the eigencurve in two different ways. There are natural sheaves arising in each of these constructions which admit a perfect pairing, and we will discuss how they interpolate Serre duality at classical points.

1. Interpolating the Sheaf

Recall from Juan's talk that in order to do Hida theory, one varies the weight p-adically via the Igusa tower. Recall that this is defined over the ordinary locus by first viewing the line bundle ω_E as a $\mathbb{G}_m$-torsor, and then taking a restriction of structure to a $\mathbb{Z}_p^\times$-torsor corresponding to the Hodge-Tate map $\mathrm{HT} : T_p(E)^{\mathrm{et}} \to \omega_E$. Then we defined a sheaf $\omega^{\kappa^{\mathrm{un}}} := (\mathscr{O}_{\mathrm{Ig}}\widehat{\otimes}\Lambda)^{\mathbb{Z}_p^\times}$. But let me be a bit more precise: in Louis's talk he first defined a moduli problem for characteristic p schemes at finite level n:

$$P_{\mathrm{Ig}_n} : S \mapsto \left\{E/T \text{ an elliptic curve with an isomorphism } H_n := \ker(F^n) \xrightarrow{\sim} \mu_{p^n}\right\}$$

which was represented by an étale cover of the ordinary locus. Here F is the Frobenius. The interpretation to keep is mind is that the varying H_n are the *level n canonical subgroups*. Let me describe a way of constructing H_n from H_{n-1} and Frobenius. Let $\pi_{n-1} : E \to E/H_{n-1}$ denote the natural projection. Then if $\widetilde{H_n} = \ker(F : E/H_{n-1} \to E/H_{n-1})$, then it is a straightforward check to show that

$$\pi_{n-1}^{-1}(\widetilde{H_n}) = H_n$$

This is tautological and seems overly complicated, but it will become clear why this is a useful perspective in a bit.

Now let's try to mimic this in the finite slope case. Recall that we now consider the modular curves now as adic spaces $X, X_0(p) \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ (I'm ignoring tame level everywhere). Recall also that we defined, for $v < 1$, a quasi-compact open $X_v \subseteq X$ whose rank points are exactly those $x \in X$ such that $v(\mathrm{Ha}(x)) < v$. We want to find an analog of the Igusa tower in this setting. In order to construct this, we will need to be able to talk about canonical subgroups of level n. Recall the following theorem from last week.

Theorem 1.0.1 (Lubin [Kat73, Theorem 3.1]). *If $v < \frac{p}{p+1}$ there exists a canonical subgroup $H_1 \subseteq E[p]$ over X_v.*

So at least if we work over the neighborhood X_v of the ordinary locus, we have access to a canonical subgroup. But what about level n subgroups for $n > 1$? Now the overly complicated construction I gave above comes in handy. First we note the following theorem.

Theorem 1.0.2 (Lubin [Kat73, Theorem 3.10.7])**.** *If $v < \frac{p}{p(p+1)}$ then E/H_1 admits a canonical subgroup $\widetilde{H_2} \subseteq E/H_1$.*

Proof. The idea is that if $v(\mathrm{Ha}(E)) < \frac{p}{p+1}$ then there is a canonical subgroup $H_1 \subseteq E[p]$ and then $v(\mathrm{Ha}(E/H_1)) = pv(\mathrm{Ha}(E))$, which is the theorem cited. □

Then letting $\pi_1 : E \to E/H_1$ be the projection map, we define $H_2 := \pi_1^{-1}(\widetilde{H_2})$.

Iterating this idea, we have thus proven the following proposition:

Proposition 1.0.3. *If $v < \frac{p}{p^{n-1}(p+1)}$, there exist canonical subgroups $H_n \subseteq E[p^n]$ of level n over X_v.*

Now we can use these to construct a version of the Igusa tower. Before giving Boxer-Pilloni's sketch of the construction, I should mention that the construction of modular sheaves at a specific p-adic weight in weight space is done explicitly in [Pil13], which is a good secondary reference. We now regard $\omega_E = e^\star \Omega^1_{E/X}$ as a $\mathbb{G}_m$-torsor. Let $\mathbb{G}_a^+ = \mathrm{Spa}(\mathbb{Q}_p \langle T \rangle, \mathbb{Z}_p \langle T \rangle)$ denote the adic closed unit disk.

Proposition 1.0.4 (Proposition 1.15 in [BP20])**.** *If $v < \frac{1}{p^{n-1}(p-1)}$, the $\mathbb{G}_m$-torsor ω_E has a natural reduction to a $\mathbb{Z}_p^\times(1 + p^{n-v\frac{p^n}{p-1}}\mathbb{G}_a^+)$-torsor $\mathcal{T}_v$ over X_v.*

Proof Sketch. First, note that the condition on v implies the existence of a canonical subgroup $H_n \subseteq E[p^n]$ over X_v. One then gets an exact sequence

$$0 \to \omega^+_{E/H_n} \to \omega^+_E \to \omega^+_{H_n} \to 0$$

which induces an isomorphism $\omega_E^+/p^{n-v\frac{p^n-1}{p-1}} \xrightarrow{\sim} \omega^+_{H_n}/p^{n-v\frac{p^n-1}{p-1}}$ (for an explanation of why this is true at a rank 1 point, see [Pil13, Proposition 3.1]). Then there is a Hodge-Tate map

$$(H_n)^D \xrightarrow{\mathrm{HT}} \omega^+_{H_n}$$

and corresponding linearization $(H_n)^D \otimes \mathscr{O}^+_{X_v} \to \omega^+_{H_n}$. This is not an isomorphism, but if we let

$$\omega_E^\sharp := \left\{\omega \in \omega_E^+ \mid r(w) \in \mathrm{im}(\mathrm{HT} \otimes 1)\right\} \subseteq \omega_E^+$$

then it is mod $p^{n-v\frac{p^n}{p-1}}$:

$$\mathrm{HT}_v : (H_n)^D \otimes \mathscr{O}^+_{X_v}/p^{n-\frac{p^n}{p-1}} \to \omega_E^\sharp/p^{n-v\frac{p^n}{p-1}}$$

and finally define the $\mathbb{Z}_p^\times(1 + p^{n-v\frac{p^n}{p-1}}\mathbb{G}_a^+)$-torsor

$$\mathcal{T}_v := \left\{\omega \in \omega_E^\sharp \mid \exists P \in (H_n)^D, p^{n-1}P \neq 0, \mathrm{HT}_v(P) \equiv \omega \mod p^{n-v\frac{p^n}{p-1}}\right\}$$

This admits a map $\mathcal{T}_v \hookrightarrow \omega_E$ which is equivariant for the analytic group morphism $\mathbb{Z}_p^\times(1+p^{n-v\frac{p^n}{p-1}}\mathbb{G}_a^+) \hookrightarrow \mathbb{G}_m$. □

Since $\mathcal{T}_v$ is meant to be the Coleman theory substitute for the Igusa tower, we want to use it to define a universal line bundle. Let $\mathcal{W} := \mathrm{Spa}(\Lambda, \Lambda) \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ denote the weight space, which is covered by an increasing union of affinoid closed unit balls:

$$\mathcal{W} = \bigcup_r \mathcal{W}_r$$

There is a universal character $\mathbb{Z}_p^\times \to \mathscr{O}_\mathcal{W}^\times$ which, for large enough $t(r)$, extends to a universal character

$$\mathbb{Z}_p^\times(1 + p^{t(r)}\mathscr{O}^+_{\mathcal{W}_r}) \to \mathscr{O}^\times_{\mathcal{W}_r}$$

Now fix a radius r and choose v small enough and n large enough so that $t(r) \leq n - v\frac{p^n}{p-1}$. The aforementioned actions combine to an action of $\mathbb{Z}_p^\times(1 + p^{n-\frac{p^n}{p-1}}\mathscr{O}^+_{X_v \times \mathcal{W}_r})$ on $\mathscr{O}_{\mathcal{T}_v \times \mathcal{W}_r}$. Then we let

$$\omega^{\kappa^{\mathrm{un}}} := (\mathscr{O}_{\mathcal{T}_v \times \mathcal{W}_r})^{\mathbb{Z}_p^\times(1+p^{n-v\frac{p^n}{p-1}}\mathscr{O}^+_{X_v \times \mathcal{W}_r})}$$

Fact: this is a locally free sheaf of $\mathscr{O}_{X_v \times \mathcal{W}_r}$-modules.

2. Interpolating Cohomology

Continue with the same setup as above, so that v is sufficiently small to allow the existence of level n canonical subgroups. Recall from Chi-Yun's talk that for $0 < a \leq v/p$, there is a map $p_1 : X_0(p)_{[0,a]} \to X_v$, so we can pull back $\omega^{\kappa^{\mathrm{un}}}$ to an invertible sheaf over $X_0(p)_{[0,a]} \times \mathcal{W}_r$. Dually, for $1 - v \leq a < 1$, then we get a map $X_0(p)_{[a,1]} \to X_v$ and so we can pull back $\omega^{\kappa^{\mathrm{un}}}$ to an invertible sheaf over $X_0(p)_{[a,1]} \times \mathcal{W}_r$.

The cohomology groups that we want to consider are

- $R\Gamma_{X_0(p)_{[0,a)}}(X_0(p), \omega^{\kappa^{\mathrm{un}}}) := R\Gamma_{X_0(p)_{[0,a)}}(X_0(p)_{[0,a]}, \omega^{\kappa^{\mathrm{un}}})$
- $R\Gamma(X_0(p)_{[a,1]}, \omega^{\kappa^{\mathrm{un}}})$

Now we would like to define a U_p correspondence which induces a U_p operator on cohomology. Let's start with the $[a, 1]$ part. Recall from Chi-Yun's talk that we had a correspondence diagram

$$\begin{tikzcd} & C_{[a,1]} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ X_0(p)_{[a/p,1]} & & X_0(p)_{[a,1]} \end{tikzcd}$$

Lemma 2.0.1. *There is a natural isomorphism $p_2^\star\omega^{\kappa^{\mathrm{un}}} \to p_1^\star\omega^{\kappa^{\mathrm{un}}}$, and we can define a cohomological correspondence*

$$U_p : (p_1)_\star p_2^\star\omega^{\kappa^{\mathrm{un}}} \to \omega^{\kappa^{\mathrm{un}}}$$

which specializes to U_p in weight $k \geq 1$.

Proof. The idea is that over $C_{[a,1]}$ the universal isogeny $p_1^\star E \to p_2^\star E$ induces an isomorphism on canonical subgroups $p_1^\star H_n \to p_2^\star H_n$: this is because C parametrizes tuples (E, H, E', H') and on $C_{[a,1]}$ both H and H' are the canonical subgroup of E and E'. We end up with a commutative diagram of isomorphisms

$$\begin{tikzcd} p_2^\star(H_n)^D \otimes \mathscr{O}^+_{X_v}/p^{n-v\frac{p^n}{p-1}} \arrow[r] \arrow[d, "\mathrm{HT}"] & p_1^\star(H_n)^D \otimes \mathscr{O}^+_{X_v}/p^{n-v\frac{p^n}{p-1}} \arrow[d, "\mathrm{HT}"] \\ p_2^\star\omega_E^\sharp/p^{n-v\frac{p^n}{p-1}} \arrow[r] & p_1^\star\omega_E^\sharp/p^{n-v\frac{p^n}{p-1}} \end{tikzcd}$$

and one can check that this induces an isomorphism $p_2^\star\mathcal{T}_v \to p_1^\star\mathcal{T}_v$, which finally induces an isomorphism $p_2^\star\omega^{\kappa^{\mathrm{un}}} \to p_1^\star\omega^{\kappa^{\mathrm{un}}}$. We thus define

$$U_p : p_2^\star\omega^{\kappa^{\mathrm{un}}} \to p_1^\star\omega^{\kappa^{\mathrm{un}}} \xrightarrow{\frac{1}{p}\,\mathrm{tr}_{p_1}} p_1^!\omega^{\kappa^{\mathrm{un}}}$$

$\square$

This induces a compact operator

$$U_p : R\Gamma(X_0(p)_{[a,1]}, \omega^{\kappa^{\mathrm{un}}}) \xrightarrow{\mathrm{res}} R\Gamma(X_0(p)_{[a/p,1]}, \omega^{\kappa^{\mathrm{un}}}) \to R\Gamma(X_0(p)_{[a,1]}, \omega^{\kappa^{\mathrm{un}}})$$

Now we have to tell the same story for $R\Gamma_{X_0(p)_{[0,a)}}(X_0(p), \omega^{\kappa^{\mathrm{un}}})$. I will simply remark that the same proof goes through, but now we note that the U_p-correspondence is actually supported on

$$\begin{tikzcd} & C^{\mathrm{can}}_{[0,a]} \arrow[dl, "p_2^{\mathrm{can}}"'] \arrow[dr, "p_1^{\mathrm{can}}"] & \\ X_0(p)_{[0,pa]} & & X_0(p)_{[0,a]} \end{tikzcd}$$

where p_1^{can} is an isomorphism. The point is that the universal isogeny is the canonical isogeny $E \to E/H_1$, but its dual $E/H_1 \to E$ induces an isomorphism on the canonical subgroups, which exist by the assumption on a.

Lemma 2.0.2. *There is a natural isomorphism $(p_2^{\mathrm{can}})^\star \omega^{\kappa^{\mathrm{un}}} \to (p_1^{\mathrm{can}})^\star \omega^{\kappa^{\mathrm{un}}}$, and we can define a cohomological correspondence*

$$U_p : (p_1^{\mathrm{can}})_\star (p_2^{\mathrm{can}})^\star \omega^{\kappa^{\mathrm{un}}} \to \omega^{\kappa^{\mathrm{un}}}$$

which specializes to U_p in weight $k \leq 1$.

3. Eigencurves

What are eigencurves and how are they constructed? The point is that we now have constructed complexes of Banach spaces over $\mathcal{O}_{\mathcal{W}_r}$ equipped with an action of a compact U_p-operator. In Coleman theory, one is interested in finite-slope vectors appearing in the complex, i.e. eigenvectors for the U_p-operator which have nonzero eigenvalue. There are two steps in constructing an eigenvariety:

- First, construct the spectral variety $\mathcal{Z}_r \to \mathcal{W}_r$ which, in some sense, parametrizes the reciprocals of the nonzero eigenvalues of U_p.
- Next, construct the eigenvariety itself, which parametrizes systems of Hecke eigenvalues for the whole algebra of Hecke operators away from p and the level N, which is implicit in this whole discussion.

The general construction goes as follows. Take $M^\bullet$ a bounded complex of $\mathscr{O}_{\mathcal{W}_r}$-modules. We first define the characteristic power series

$$P(T) = \det(1 - XU_p | M^\bullet) = \prod_i \det(1 - XU_p | M^i) \in \mathcal{O}_{\mathcal{W}_r}[[T]]$$

There is some work to be done to make sense of this, but one can do so if U_p is compact and if the terms in the complex $M^\bullet$ satisfy a technical condition called "property (Pr)" by Buzzard (which resembles, but is not equivalent to, projectivity in the categorical sense) which we now impose. Then we define

$$\widetilde{\mathcal{Z}}_r := V(P) \subseteq \mathbb{A}^1 \times \mathcal{W}_r \to \mathcal{W}_r$$

Over $\mathcal{Z}_r$ one has a bounded complex of coherent sheaves $\mathcal{M}^\bullet$, which is the universal nonzero generalized eigenspace of $M^\bullet$ for the U_p-operator, and for $x = (\kappa, \alpha) \in \mathcal{Z}_r$, it satisfies

$$\mathcal{M}^i_x = (M^i_x)^{U_p = \alpha^{-1}}$$

In the cases we care about, $\mathcal{M}^\bullet$ will have cohomology concentrated in one degree, so we work with $H^*(\mathcal{M})$. If $\mathcal{I} \subseteq \mathscr{O}_{\widetilde{\mathcal{Z}}}$ denotes the annihilator of $H^*(\mathcal{M})$, we define the *spectral variety*

$$\mathcal{Z} := V(\mathcal{I}) \subseteq \widetilde{\mathcal{Z}}.$$

Now suppose $\mathcal{M}$ has an action of an algebra of Hecke operators $\mathbb{T}$. Then $\mathbb{T}$ generates a sub-$\mathscr{O}_{\mathcal{Z}_r}$-algebra of $\mathrm{End}_{\mathcal{Z}_r}(\mathcal{M})$ which we call $\mathscr{O}_{\mathcal{C}_r}$. Its relative adic spectrum

$$\mathcal{C}_r \to \mathcal{Z}_r \to \mathcal{W}_r$$

is the eigencurve. Clearly the sheaf $\mathcal{M}$ extends to a sheaf $\mathcal{M}_r$ over $\mathcal{C}_r$.

Now let's put ourselves back in the scenario we care about. There are two cases. In the first case, we take $M^\bullet = M = H^0(X_0(p)_{[a,1]}, \omega^{\kappa^{\mathrm{un}}})$ to get the first eigencurve $\mathcal{C}_r \to \mathcal{Z}_r \to \mathcal{W}_r$. In the second case, we take

$N^\bullet = R\Gamma_{X_0(p)_{[0,a)}}(X_0(p), \omega^{2-\kappa^{\mathrm{un}}}(-D))$ where D is the divisor of cusps (this has to do with Serre duality, as we'll see) and we get $\mathcal{D}_r \to \mathcal{Z}_r \to \mathcal{W}_r$. We have generalized eigensheaves $\mathcal{M}_r \to \mathcal{C}_r$ and $\mathcal{N}_r \to \mathcal{D}_r$. The authors remark that the constructions are compatible as r varies, and do not depend on auxiliary choices like a, n, v, etc, so we can glue together these eigencurves as r approaches 1, and we obtain two eigencurves

$$\mathcal{C} \to \mathcal{Z} \to \mathcal{W} \text{ and } \mathcal{D} \to \mathcal{X} \to \mathcal{W}$$

which carry generalized eigensheaves $\mathcal{M}$ and $\mathcal{N}$.

4. Serre Duality

Lastly I'll sketch Serre duality over the eigencurve. There is a classical Serre duality pairing via the Kodaira-Spencer isomorphism:

$$H^0(X_0(p), \omega^k) \times H^1(X_0(p), \omega^{2-k}(-D)) \xrightarrow{\mathrm{KS}} H^0(X_0(p), \omega^k) \times H^1(X_0(p), \omega^{-k} \otimes \Omega^1_{X_0(p)/\mathbb{Q}_p}) \to \mathbb{Q}_p.$$

After slightly modifying this by the Atkin-Lehner involution, one obtains the identity

$$\langle f, U_p g\rangle = \langle U_p f, g\rangle$$

Now let's interpolate this p-adically to a pairing between $\mathcal{M}$ and $\mathcal{N}$. In the paper, they work with the dagger spaces

$$X_0(p)^{m,\dagger} := \mathrm{colim}_{a\to 1} X_0(p)_{[a,1]} \text{ and } X_0(p)^{\mathrm{et},\dagger} := \mathrm{colim}_{a\to 0} X_0(p)_{[0,a]}$$

and construct a pairing

$$H^0(X_0(p)^{m,\dagger}, \omega^{\kappa^{\mathrm{un}}}) \times H^1_c(X_0(p)^{\mathrm{et},\dagger}, \omega^{2-\kappa^{\mathrm{un}}}(-D)) \to \mathscr{O}_{\mathcal{W}_r}$$

satisfying $\langle U_p f, g\rangle = \langle f, U_p g\rangle$. Combined with the fact that by definition

$$H^0(X_0(p)^{m,\dagger}, \omega^{\kappa^{\mathrm{un}}}) = \mathrm{colim}_{a\to 1} H^0(X_0(p)_{[a,1]}, \omega^{\kappa^{\mathrm{un}}})$$

and by [BP20, Lemma 5.24]

$$H^1_c(X_0(p)^{\mathrm{et},\dagger}, \omega^{2-\kappa^{\mathrm{un}}}(-D)) = \lim_{a\to 0} H^1_{X_0(p)_{[0,a)}}(X_0(p), \omega^{2-\kappa^{\mathrm{un}}}(-D))$$

one gets compactness of U_p on the dagger cohomology groups, and the duality pairing ensures that the characteristic power series of U_p are same in both cases and thus $\mathcal{Z} = \mathcal{X}$. These identifications also imply the existence of a pairing

$$\mathcal{M} \times \mathcal{N} \to w^{-1}\mathscr{O}_{\mathcal{W}_r}$$

such that $\langle zf, g\rangle = \langle f, zg\rangle$ and $\langle hf, g\rangle = \langle f, hg\rangle$ for $z \in \mathscr{O}_Z$ and h any Hecke operator in $\mathbb{T}$. This pairing specializes to the classical Serre duality statement given above. One can deduce from this that $\mathcal{C} = \mathcal{D}$.

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