[HT17]

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Contents

1. Venkatesh's Conjecture

Given a reductive group G with neat level $K = \prod_v K_v \subseteq G(\mathbf{A}_f)$ and maximal connected compact (mod center) $K_\infty^\circ \subseteq G(\mathbf{R})$, let

$$Y_K = G(\mathbf{Q})\backslash(G(\mathbf{R})/K_\infty^\circ \times G(\mathbf{A}_f)/K)$$

be the associated locally symmetric space. Away from a finite set of bad primes S (the ones where K_v is not hyperspecial: in the modular curve case think of "away from N") we define

$$\mathcal{H} = \bigotimes_{v \in S} \mathcal{H}_v = \bigotimes_{v \in S} \mathcal{C}_c(K_v\backslash G_v/K_v, \mathbf{Z})$$

and fix a character $\chi : \mathcal{H} \to \mathbf{Z}$ which comes from a cuspidal and tempered automorphic representation of $G(\mathbf{A})$, with some extra conditions as in Robin's talk.

Recall we defined $\ell_0 = \operatorname{rank} G(\mathbf{R}) - \operatorname{rank} K_\infty^\circ$, and $q_0 = \frac{\dim Y(K) - \delta}{2}$. Then $H^i(Y_K, \mathbf{Q})_\chi \neq 0$ exactly when $i \in [q_0, q_0 + \ell_0]$, we had a formula for the dimension, and

Conjecture 1.1. *$H^*(Y_K, \mathbf{Q})_\chi$ is generated over $H^{q_0}(Y_K, \mathbf{Q})_\chi$ by the action of the ℓ_0-exterior power of a motivic cohomology group (the subscript χ denotes the χ-eigenspace).*

The subject of this talk will be a variant of this conjecture for $G = \mathrm{GL}_n$ in terms of eigenvarieties.

2. Cohomology for GL_n

We now work with $G = \mathrm{GL}_n$. This is an appropriate context for Venkatesh's conjecture because $\ell_0 = \lfloor \frac{n-1}{2} \rfloor$, which is positive for $n > 2$. The level we work with throughout is

$$K = K_1(N; p) := K_1(N)^p I$$

with $(N, p) = 1$ where $K_1(N)$ is the mirabolic congruence subgroup of $\mathrm{GL}_n(\widehat{\mathbf{Z}})$ of matrices whose bottom row is congruent to $(0, \ldots, 0, 1)$ mod N and $I = (\mathrm{GL}_n(\mathbf{Z}_p) \to \mathrm{GL}_n(\mathbf{F}_p))^{-1}(B(\mathbf{F}_p))$ is the standard Iwahori. Maybe K isn't necessarily neat, but we can define

$$H^i(Y_K, -) := H^i(Y_{K'}, -)^K$$

for $K' \subseteq K$ neat normal open compact.

The Hecke algebra is

$$\mathbf{T}^{(N),p} = \mathbf{T}^{(N)} \otimes_{\mathbf{Z}} \mathbf{Z}[X_*(T)^-] = \left(\bigotimes_{\ell \nmid Np} \mathcal{C}_c(\mathrm{GL}_n(\mathbf{Z}_\ell) \backslash \mathrm{GL}_n(\mathbf{Q}_\ell) / \mathrm{GL}_n(\mathbf{Z}_\ell), \mathbf{Z}) \right) \otimes_{\mathbf{Z}} \mathbf{Z}[X_*(T)^-]$$

Here the second factor is for the p-part and acts via Hecke operators via

$$\mathbf{Z}[X_*(T)^-] \hookrightarrow \mathcal{C}_c(I \backslash G(\mathbf{Q}_p)/I, \mathbf{Z})$$
$$\mu \mapsto [I\mu(p)I]$$

If we denote $U_{p,i} = [I\mu_i(p)I]$ where $\mu_i : x \mapsto \mathrm{diag}(1, \ldots, 1, x, \ldots, x)$, then our U_p-operator (with respect to which we measure slopes and say "finite slope") is

$$U_p = U_{p,1} \cdots U_{p,n-1}$$

Let $L/\mathbf{Q}_p$ be finite. For each dominant character $\lambda \in X^*(T)^+$, we let $\mathcal{L}_{\lambda,L}$ denote the L-points of the algebraic representation of G of highest weight λ. In particular $\mathcal{L}_{\lambda,L}$ this has compatible L-linear actions of $G(\mathbf{Q})$ and $\Delta_p = \bigsqcup_{\mu \in X_*(T)^-} I\mu(p)I$, so it defines a local system on Y_K and we get U_p operators on $H^i(Y_K, \mathcal{L}_{\lambda,L})$.

Proposition 2.1 (Corollary 4.4 in [HT17])**.** *Say π is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A})$ of conductor $(N,p) = 1$ satisfying a certain parity condition on the central character of π_∞ (this is so that the cohomology groups don't vanish, and can be made true by twisting π by a quadratic character). Let $\mathfrak{m}_\pi \subseteq \mathbf{T}^{(N)}$ be the associated maximal ideal (mention that this is generated by T_ℓ for $\ell \nmid Np$). Then there exists a unique $\lambda \in X^*(T)^+$ (the "weight" of π) such that*

$$H^*(Y_K, \mathcal{L}_{\lambda,L})_{\mathfrak{m}_\pi} \neq 0$$

Suppose the eigenvalues of the Satake parameter for π_p are distinct. Fix an ordering t for the eigenvalues of the Satake parameter. If $\mathfrak{m} \subseteq \mathbf{T}^{(N),p}$ is the maximal ideal associated to (π, t), then

$$\dim H^{q_0+i}(Y_K, \mathcal{L}_{\lambda,L})_{\mathfrak{m}} = \binom{\ell_0}{i}$$

Implicit in this statement is a correspondence between orderings of the eigenvalues of the Satake parameters for π and maximal ideals $\mathfrak{m} \subseteq \mathbf{T}^{(N),p}$ containing the maximal ideal $\mathfrak{m}_\pi \subseteq \mathbf{T}^{(N)}$ under the inclusion $T^{(N)} \hookrightarrow T^{(N),p}$.

3. Background on Eigenvarieties: Hida Families

One of the first constructions of eigenvarieties is due to Hida. We will give a very brief and sketchy explanation of what Hida did.

The idea is to put modular forms into p-adic families. More precisely, recall that a modular form of level $\Gamma_0(p)$ has a q-expansion $f = \sum_{n \geq 0} a_n q^n$, and it is a theorem that the a_n live in a common number field E. Suppose for simplicity that $E = \mathbf{Q}$. Then we can embed $\mathbf{Q} \hookrightarrow \mathbf{Q}_p$, and then using rigid analytic geometry we can study the p-adic variation of the a_n: rigid geometry is a version of analytic geometry where the coefficient rings are certain Banach $\mathbf{Q}_p$-algebras R, and one way to think about families of p-adic modular forms is to cook up q-expansions with $a_n \in R$ and then specialize to certain classical points of $\mathrm{Sp}\, R$, and hope to recover classical modular forms.

In particular, varying modular forms p-adically this way really involves varying the *weight* of the modular form (this line of reasoning was developed by Serre, Katz, Hida, Coleman, etc). In the p-adic world, this means that our family of modular forms lives over weight space.

Definition 3.1. Let $W : \mathsf{Aff}_{\mathbf{Q}_p} \to \mathsf{Grp}$ be the functor taking

$$W(R) = \left\{ \text{continuous characters } \mathbf{Z}_p^\times \to R^\times \right\}$$

Here $\mathsf{Aff}_{\mathbf{Q}_p}$ is the category of affinoid $\mathbf{Q}_p$-algebras, each of which has a canonical topology. Then this functor is represented by a $\mathbf{Q}_p$-rigid analytic group variety $\mathcal{W}$, called weight space.

The $\mathbf{C}_p$-points of weight space looks like this (draw a bunch of circles). Why? Because $\mathbf{Z}_p^\times \cong (\mathbf{Z}/p)^\times \times 1+p\mathbf{Z}_p$. Each $1+p\mathbf{Z}_p$ contributes a disk to $\mathcal{W}$ (e.g. via $\chi \mapsto \chi(1+p) - 1 \in \mathscr{O}_{\mathbf{C}_p}$) and there are $|(\mathbf{Z}/p)^\times| = p-1$ many of them. In particular, integers live in the weight space via the map

$$\mathbf{Z} \mapsto \mathcal{W}$$
$$k \mapsto (x \mapsto x^k)$$

In fact the integers contained in a mod $p-1$-equivalence class of $\mathbf{Z}$ are dense (under the above map) in each piece of $\mathcal{W}$. So if we pick a point $k \in \mathcal{W}$ (draw this) and take a smaller disk $\mathcal{D}$ around it (draw this) then a Hida family $w : \mathcal{E} \to \mathcal{D}$ is another copy of $\mathcal{D}$, such that each point $x \in \mathcal{E}$ is a modular form of weight given by $w(x)$. Heuristically, if R is the ring of functions on $\mathcal{D}$, then a Hida family should be given by a q-expansion $\sum_{n\geq 0} a_n q^n \in R[[q]]$ such that if you specialize to an integer weight $k' \in \mathcal{D}$, you get a modular form of level $\Gamma_0(p)$ and weight k'.

Theorem 3.2 (Hida)**.** *If $f \in S_k(\Gamma_0(p))$ is an ordinary eigenform for the T_ℓ ($\ell \neq p$) and U_p operators then there exists a unique Hida family f_∞ specializing to f at the weight k.*

What does ordinary mean? Let $U_p f = \lambda_p f$. Then ordinary means that $v_p(\lambda_p) = 0$, i.e. $\lambda_p \in \mathbf{Z}_p^\times$. For non-ordinary forms, Coleman and Mazur (and numerous other people after that) have a various generalizations of this type of construction: one can define U_p-operators for general reductive groups and talk about p-adic variation of finite slope automorphic forms.

For instance, the eigencurve parametrizes all finite slope p-adic eigenforms of some tame level (draw the halo and some scribbles).

4. The GL_n-eigenvariety

Definition 4.1. The weight space $\mathcal{W}$ in this context is the rigid analytic space associated to

$$R \mapsto \left\{\text{continuous characters } T(\mathbf{Z}_p) \to R^\times\right\}$$

The eigenvariety $w : \mathcal{E} \to \mathcal{W}$ of tame level K^p is a rigid space whose fibers $w^{-1}(\lambda)$ parametrize Hecke eigensystems

$$\mathbf{T}^{(N),p} \twoheadrightarrow L$$

whose kernels appear in the support of $H^*(Y_K, \mathcal{D}_{\lambda,L})$; here $\mathcal{D}_{\lambda,L}$ is some huge p-adic coefficient system with an action of Δ_p, which I won't explicitly describe. However, note that there is a canonical Δ_p-equivariant map $\mathcal{D}_{\lambda,L} \to \mathcal{L}_{\lambda,L}$, which basically follows from the fact that $\mathcal{D}_{\lambda,L}$ is dual to a space of locally analytic functions, and $\mathcal{L}_{\lambda,L}$ is dual to a space of polynomials (in fact this isn't true on the nose: you have to twist the action of U_p by a character determined by λ, but let's agree to ignore this for exposition purposes).

Recall π is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A})$ of conductor N, satisfying a parity condition. Suppose π_p is unramified and that the Satake parameter of π_p is regular. Fix an ordering t of the eigenvalues of the Satake parameter and let $\mathfrak{m} \subseteq \mathbf{T}^{(N),p}$ be the maximal ideal corresponding to (π, t).

Definition 4.2. Let $\lambda = (k_1 \geq \cdots \geq k_n)$ be the weight. We say that $\mathfrak{m}$ is "numerically non-critical" if

$$v_p(U_{p,i}) < 1 + k_{n-i} - k_{n+1-i} \text{ for } 1 \leq i \leq n-1$$

Proposition 4.3. *If $\mathfrak{m}$ is numerically non-critical, then*

$$H^*(Y_K, \mathcal{D}_{\lambda,L})_{\mathfrak{m}} \to H^*(Y_K, \mathcal{L}_{\lambda,L})_{\mathfrak{m}}$$

is a Hecke-equivariant isomorphism.

Since points on $\mathcal{E}$ are given by Hecke eigensystems appearing in $H^*(Y_K, \mathcal{D}_{\lambda,L})$, we get a point $x \in \mathcal{E}$ associated to π, such that $w(x) = \lambda$.

Theorem 4.4 (Hansen-Thorne). *Let $\mathbf{T}_x = \widehat{\mathscr{O}}_{\mathcal{E},x}$ and let $\Lambda = \widehat{\mathscr{O}}_{\mathcal{W},\lambda}$. Then*

$$\dim \mathbf{T}_x \geq \dim \Lambda - \ell_0$$

If equality holds, then

(1) *The natural map $\Lambda \to \mathbf{T}_x$ is surjective and $\mathbf{T}_x$ is a complete intersection ring.*

(2) *If we let $V_x := \ker(\Lambda \to \mathbf{T}_x) \otimes_\Lambda L$ (which has dimension ℓ_0 by our equality assumption), then $H^*(K, \mathcal{L}_{\lambda,L})_{\mathfrak{m}_{(\pi,t)}}$ is a free rank 1 $\bigwedge^* V_x$-module.*

Sketch. There exists a faithful graded $\mathbf{T}_x$-module H_x^* which is finite over Λ and concentrated in degrees $[0, \dim Y_K]$, and there exists a spectral sequence

$$E_2^{i,j} = \mathrm{Tor}_{-i}^\Lambda(H_x^j, L) \implies H^{i+j}(Y_K, \mathcal{D}_{\lambda,L})_{\mathfrak{m}_{(\pi,t)}}$$

Furthermore, there exists a complex $C^\bullet$ of finite free Λ-modules such that such that $H_x^* = H^*(C^\bullet)$ and

$$H^*(C^\bullet \otimes_\Lambda L) = H^*(Y_K, \mathcal{L}_{\lambda,L})_{\mathfrak{m}_{(\pi,t)}}$$

Some general commutative algebra then allows you to deduce that $\dim H^*(C^\bullet) \geq \dim \Lambda - \ell_0$ and that equality happens when H_x^* is concentrated in degree $q_0 + \ell_0$.

When equality happens, the spectral sequence already degenerates on the second page to

$$\mathrm{Tor}_i^\Lambda(H_x, L) \cong H^{q_0+\ell_0-i}(Y_K, \mathcal{L}_{\lambda,L})_{\mathfrak{m}}$$

If $i = 0$ we get

$$\dim H_x \otimes_\Lambda L = \dim H^{q_0+\ell_0}(Y_K, \mathcal{L}_{\lambda,L})_{\mathfrak{m}} = 1$$

By Nakayama's lemma $H_x \cong \Lambda/I_x$ and in fact H_x is free of rank 1 over $\mathbf{T}_x$, so $\Lambda \to \mathbf{T}_x$ is surjective. Then

$$H^{q_0+\ell_0-1}(Y_K, \mathcal{L}_{\lambda,L})_{\mathfrak{m}_{(\pi,t)}} = \mathrm{Tor}_1^\Lambda(H_x, L) = I_x \otimes_\Lambda L$$

But the left side has dimension $\binom{\ell_0}{\ell_0-1} = \ell_0$ so using Nakayama again, we see I_x can be generated by ℓ_0 elements. Since $\dim \mathbf{T}_x = \dim \Lambda - \ell_0$, we conclude that $\mathbf{T}_x$ is a complete intersection ring. The conclusion then follows from some other commutative algebra lemmas that they prove in Section 2 of [HT17], which I didn't have time to work out. □

Attached to π (by Harris-Lan-Taylor-Thorne) is a Galois representation

$$\rho_\pi : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(L)$$

Conjecturally, ρ_π is crystalline with Hodge-Tate weights determined by λ and ϕ-eigenvalues of $D_{\mathrm{cris}}(\rho_\pi)$ given by

$$(\alpha_1, \ldots, \alpha_n) = p^{(n-1)/2} \iota^{-1}(t_1, \ldots, t_n)$$

where the t_i run through the eigenvalues of the Satake parameter of π (here $\iota : \overline{\mathbf{Q}}_p \cong \mathbf{C}$ is some isomorphism).

Then an ordering t of the α_i determines something called a "triangulation" of ρ_π. We can deform both ρ_π along with the triangulation. Assume

$$\mathbf{T}_x \cong R_{\rho_\pi, \alpha}$$

Under these assumptions, [HT17] proves the existence of a canonical isomorphism

$$V_x \cong H_f^1(\mathbf{Q}, \mathrm{ad}\, \rho_\pi(1))$$

The H_f^1 is a Bloch-Kato Selmer group which should be p-adic realization of a motivic cohomology group.

References

[HT17] David Hansen and Jack A. Thorne. On the GL_n-eigenvariety and a conjecture of Venkatesh. *Selecta Math. (N.S.)*, 23(2):1205–1234, 2017.