BREUIL-MÉZARD AND AUTOMORPHY

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Contents

These notes are sketchy, use at your own risk

The goal is to explain the relationship between automorphy lifting theorems in the global context and the Breuil-Mézard conjecture, which is purely local statement. We will look at the work of Gee-Kisin showing that automorphy lifting theorems imply the Breuil-Mézard conjecture when $n = 2$ and for potentially Barsotti-Tate representations. When $n = 2$ and $K = \mathbf{Q}_p$, Kisin shows the other direction.

The magic ingredient to go between these two things is the Taylor-Wiles-Kisin patching method: we won't give details, but we'll give the context.

1. Recollection of the (numerical) Breuil-Mézard conjectures

Let $K/\mathbf{Q}_p$ be a p-adic field with ring of integers $\mathscr{O}_K$ and residue field k. We'll fix $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbf{F}_p})$, and we work with potentially crystalline lifting rings in this lecture. We fix λ a Hodge type, i.e. a tuple $(\lambda_{1,\iota} \geq \lambda_{2,\iota} \geq \cdots \geq \lambda_{n,\iota})$ for each $\iota : K \hookrightarrow \overline{\mathbf{Q}_p}$, and we fix $\tau : I_K \to \mathrm{GL}_n(\overline{\mathbf{Q}_p})$ a representation with open kernel. This allows us to define the complete local Noetherian $\mathscr{O}$-algebra ($\mathscr{O}$ is the integers in some p-adic coefficient ring E)

$$R_{\overline{\rho}}^{\lambda,\tau}$$

which classifies potentially crystalline lifts of $\overline{\rho}$ with Hodge-Tate weights $\lambda + \eta = (\lambda_{1,\iota} + n - 1 > \lambda_{2,\iota} + n - 2 > \cdots > \lambda_{n,\iota})$ and with inertial type τ. The ring $R_{\overline{\rho}}^{\lambda,\tau}$ is reduced and equidimensional, and $R_{\overline{\rho}}^{\lambda,\tau}[1/p]$ is regular.

To (λ, τ) we associate the locally algebraic representation $\sigma^{\mathrm{cris}}(\lambda, \tau)$ of $\mathrm{GL}_n(\mathscr{O}_K)$ on a finite dimensional E-vector space. Recall that

$$\sigma^{\mathrm{cris}}(\lambda, \tau) = \sigma_{\mathrm{alg}}(\lambda) \otimes \sigma_{\mathrm{sm}}^{\mathrm{cris}}(\tau)$$

The point is that $\sigma_{\mathrm{sm}}^{\mathrm{cris}}(\tau)$ detects irreducible representations π of $\mathrm{GL}_n(K)$ where $\mathrm{rec}(\pi)|_{I_K} \cong \tau$ and $N = 0$. We choose $L_{\lambda,\tau} \subseteq \sigma^{\mathrm{cris}}(\lambda, \tau)$ a $\mathrm{GL}_n(\mathscr{O}_K)$-stable $\mathscr{O}$-lattice. Then

$$(L_{\lambda,\tau} \otimes_{\mathscr{O}} \overline{\mathbf{F}_p})^{ss} \cong \bigoplus_{V \text{ irreducible } \overline{\mathbf{F}_p}\text{-reps of } \mathrm{GL}_n(k)} V^{\oplus m_V(\lambda,\tau)}$$

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[1]Notes taken by Ashwin Iyengar, and have not been looked at or edited by the speaker

Conjecture 1.0.1. *There exist non-negative integers $\mu_V(\overline{\rho})$ such that*

$$e(R_{\overline{\rho}}^{\lambda,\tau} \otimes \overline{\mathbf{F}_p}) = \sum_V m_V(\lambda,\tau)\mu_V(\overline{\rho})$$

2. Patching Functors

As before we have $\mathscr{O} \subseteq E$. The patching method spits out $R_\infty = R_{\overline{\rho}}^{\square}[[x_1,\dots,x_h]]$, with some auxiliary patching variables. Let $\mathcal{C}$ be the category of finitely generated $\mathscr{O}$-modules with a continuous action of $\mathrm{GL}_n(\mathscr{O}_K)$. Some examples of objects are $L_{\lambda,\tau}$ and V a Serre weight. Let $X_\infty = \operatorname{Spec} R_\infty$ and $R_\infty^{\lambda,\tau} = R_{\overline{\rho}}^{\lambda,\tau} \otimes_{R_{\overline{\rho}}} R_\infty$, and let $X_\infty^{\lambda,\tau} = \operatorname{Spec} R_\infty^{\lambda,\tau}$.

Definition 2.0.1. A **patching functor** is a nonzero covariant exact $\mathscr{O}$-linear functor

$$M_\infty : \mathcal{C} \to \mathrm{Coh}(X_\infty).$$

satisfying:

- For any (λ,τ), the action of R_∞ on $M_\infty(L_{\lambda,\tau})$ factors through $R_\infty^{\lambda,\tau}$, and $M_\infty(L_{\lambda,\tau})$ is maximal Cohen-Macaulay over $R_\infty^{\lambda,\tau}$. This implies that the support of $M_\infty(L_{\lambda,\tau})$ is equal to a union of irreducible components in $X_\infty^{\lambda,\tau}$. It also implies that $M_\infty(L_{\lambda,\tau})[1/p]$ is locally free over $X_\infty^{\lambda,\tau}[1/p]$.
- $M_\infty(L_{\lambda,\tau})[1/p]$ is locally free of rank 1 over its support.
- If V is an irreducible $\overline{\mathbf{F}_p}$-representation of $\mathrm{GL}_n(k)$, then $M_\infty(V)$ is Cohen-Macaulay of dimension $\dim(R_\infty^{\lambda,\tau} \otimes \overline{\mathbf{F}_p}) =: d$, i.e. the support is equidimensional of dimension d.

Let $X_\infty(V)$ be the closed subscheme of X_∞ cut out by $M_\infty(V)$. A very special example is when $K = \mathbf{Q}_p$ and $n = 2$. Then P was the universal deformation of $\Pi^\vee$. We define an exact covariant functor

$$\sigma \mapsto P \otimes_{\mathscr{O}[[\mathrm{GL}_2(\mathbf{Z}_p)]]} \sigma.$$

More generally, one has to use the Taylor-Wiles-Kisin patching method, but the functor still looks like

$$\sigma \mapsto \sigma \otimes_{\mathscr{O}[[\mathrm{GL}_n(\mathscr{O}_K)]]} M_\infty$$

where M_∞ is constructed globally using spaces of automorphic forms for a unitary group.

So how do you construct these M_∞? Step 0 is to first globalize $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbf{F}_p})$ to

$$\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}_p})$$

where F is a CM number field. In other words, there should be one conjugate pair $v, \overline{v}$ lying above p such that $F_v \cong K$ and

$$\overline{r}|_{G_{F_v}} \cong \overline{\rho}.$$

Furthermore, $\overline{r}$ should come from an automorphic representation of a unitary group.

Consequences: when M_∞ is built by patching spaces of automorphic forms, then

(1) $M_\infty(V) \neq 0$ if and only if $S(V^\vee)_{\overline{r}} \neq 0$ (here S denotes some appropriate space of automorphic forms of weight $V^\vee$, I think). In other words, if and only if $\overline{r}$ is automorphic of weight V.

(2) If $\operatorname{supp} M_\infty(L_{\lambda,\tau}) = X_\infty^{\lambda,\tau}$, then $S(L_{\lambda,\tau}^\vee)_{\overline{r}}$ is supported on all of $\operatorname{Spec} R_{\overline{r}}^{\lambda,\tau}$, which is equivalent to proving an automorphy lifting theorem.

Lemma 2.0.2 (BM vs aut. lifting)**.** *Suppose M_∞ is as above. Then*

$$e(M_\infty(L_{\lambda,\tau}) \otimes \overline{\mathbf{F}_p}, R_\infty^{\lambda,\tau} \otimes \overline{\mathbf{F}_p}) \leq e(R_\infty^{\lambda,\tau} \otimes \overline{\mathbf{F}_p})$$

(where $e(M,R) = \sum_{\mathfrak{p} \in \mathrm{Spec}(R) max\ dim} e(R/\mathfrak{p})\ell(M_{\mathfrak{p}})$) with equality if and only if $\operatorname{supp}(M_\infty(L_{\lambda,\tau})) = X_\infty^{\lambda,\tau}$.

Theorem 2.0.3 (Gee-Kisin)**.** *Suppose M_∞ is a patching functor and suppose $M_\infty(L_{\lambda,\tau})$ is supported on $X_\infty^{\lambda,\tau}$. Then the conjecture holds for type λ, τ:*

$$e(R_{\overline{\rho}}^{\lambda,\tau} \otimes \overline{\mathbf{F}_p}) = \sum_V m_V(\lambda,\tau) e(M_\infty(V), R_\infty(V))$$

Theorem 2.0.4. *Suppose the Breuil-Mézard conjecture is true for some fixed λ, τ. Suppose M_∞ is a patching functor and $e(M_\infty(V), R_\infty(V)) \geq \mu_V(\overline{\rho})$ for all V such that $m_V(\lambda,\tau) > 0$. Then* $\operatorname{supp} M_\infty(L_{\lambda,\tau}) = X_\infty^{\lambda,\tau}$*. In other words, we have proven an automorphy lifting theorem in type (λ, τ).*

Proof. Compute $e(M_\infty(L_{\lambda,\tau}) \otimes \overline{\mathbf{F}_p})$ in terms of $e(M_\infty(V))$ and apply the lemma. □