ARITHMETIC LEVEL-RAISING FOR EVEN N

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Contents

1. Setup

We recall the basic setup.

(1) F/F^+ is a CM extension of a totally real field,

(2) $N = 2r > 1$ is an even integer,

(3) Π is a relevant ([LTX+19, Definition 1.1.3]) representation of $\mathrm{GL}_N(\mathbb{A}_F)$,

(4) E is a number field containing the normalized Satake parameters of Π at unramified places (the authors call this a "strong coefficient field" ([LTX+19, Definition 3.2.5]),

(5) $\Sigma_{\min}$ is a finite set of non-archimedean places of F^+ containing the places above primes which ramify in F, and such that Π is unramified outside $\Sigma_{\min}$,

(6) Σ_{lr} is a finite set of non-archimedean places in F^+ which are inert in F, away from $\Sigma_{\min}$,

(7) Σ is a finite set containing $\Sigma_{\min}$ and Σ_{lr},

(8) $\lambda \mid \ell$ is a finite place in E such that $\ell \nmid ||v||(||v||^2 - 1)$ for $v \in \Sigma_{\mathrm{lr}}$. Let $O = O_{E,\lambda}$ with residue field k

Then we get a system of Hecke eigenvalues

$$\phi_\Pi : \mathbb{T}^\Sigma \to O$$

where $\mathbb{T}^\Sigma$ is given by the restricted tensor product of the spherical Hecke algebras away from Σ. There is an associated Galois representation

$$\rho : \Gamma_F \to \mathrm{GL}_N(O)$$

satisfying $\rho^c \sim \rho^\vee(1-N)$ and the usual Frobenius-Satake compatibility. As Toby mentioned last week, $\overline{\rho}$ extends to a representation $\overline{r} : \Gamma_{F^+} \to \mathcal{G}_N(k)$. Our deformation problems will be deformations of $\overline{r}$.

We pick an odd place $\mathfrak{p} \mid p$ of F^+ such that

(1) $\mathfrak{p}$ is inert in F and p is unramified in F

(2) $F^+_{\mathfrak{p}} = \mathbb{Q}_p$

(3) $\Sigma \nmid p$

(4) $\ell \nmid p(p^2-1)$

(5) the Satake parameter at p contains $\{p, p^{-1}\}$ exactly once and does not contain $\{-1,-1\} \pmod{\lambda}$.

I'm omitting some other assumptions: see the beginning of [LTX+19, Section 6] for the details. Finally let

$$\mathfrak{m} := \mathbb{T}^{\Sigma\cup\Sigma_p} \cap \ker(\mathbb{T}^{\Sigma} \xrightarrow{\phi_\Pi} O_E \twoheadrightarrow O_E/\lambda)$$
$$\mathfrak{n} := \mathbb{T}^{\Sigma\cup\Sigma_p} \cap \ker(\mathbb{T}^{\Sigma} \xrightarrow{\phi_\Pi} O_E \twoheadrightarrow O_E/\lambda^m)$$

where $m > 1$ is some integer.

Recall that we have to choose a certain standard Hermitian space V° of rank N over F, which for us will be non-split at every place in Σ_{lr}, and we pick a nice enough level structure K°. For simplicity, write

$$\widetilde{\mathrm{H}}^i := \mathrm{H}^i_{\mathfrak{T}}(\overline{\mathrm{M}}_N, \mathrm{R}\Psi O) \text{ and } O[\mathrm{Sh}] := O[\mathrm{Sh}(\mathrm{V}^\circ, \mathrm{K}^\circ)].$$

We make two more assumptions ([LTX+19, Assumptions 6.1.(4,5)]).

(1) $\widetilde{\mathrm{H}}^i_{\mathfrak{m}} = 0$ for $i \neq N-1$, and $\widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}$ is finite free over O.

(2) $\overline{\rho}$ is absolutely irreducible.

Now recall from David's last talk that under these assumptions we have an injective map

$$(*) \qquad \mathrm{F}_{-1}\mathrm{H}^1(\mathrm{I}_{p^2}, \widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}(r)) \hookrightarrow \mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, \widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}(r)) := \mathrm{H}^1(\mathrm{I}_{p^2}, \widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}(r))^{\mathrm{Frob}_{p^2}}$$

and a surjective map

$$(*') \qquad \mathrm{F}_{-1}\mathrm{H}^1(\mathrm{I}_{p^2}, \widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}(r)) \twoheadrightarrow O[\mathrm{Sh}]_{\mathfrak{m}}/((p+1)\mathtt{R}^\circ_{\mathfrak{p}} - \mathtt{I}^\circ_{\mathfrak{p}})$$

where $\mathtt{R}^\circ_{\mathfrak{p}}$ and $\mathtt{I}^\circ_{\mathfrak{p}}$ are some explicit Hecke operators defined in [LTX+19, Proposition 5.7.(1,8)]. The goal is to prove:

Theorem 1.1 ([LTX+19, Theorem 6.3.4]). *Assume*

(1) $\ell \geq 2(N+1)$ *and* ℓ *is unramified in* F

(2) $\overline{r}$ *is rigid for* $(\Sigma_{\min}, \Sigma_{\mathrm{lr}})$*: this means that every lift of a place* $v \in \Sigma_{\min}$ *is minimally ramified, the set of generalized eigenvalues of* $\overline{r}(\mathrm{Frob}_v^2)$ *contains the pair* $\{||v||^{-N}, ||v||^{-N+2}\}$ *exactly once, and* $\overline{r}$ *is Fontaine-Laffaille at places dividing* ℓ*.*

(3) ϕ_Π *is cohomologically generic,*

(4) $O[\mathrm{Sh}]_{\mathfrak{m}} \neq 0$*.*

Then (among other things) ($*$) *and* ($*'$) *are both isomorphisms, and in particular induce isomorphisms*

$$\mathrm{F}_{-1}\mathrm{H}^1(\mathrm{I}_{p^2}, \widetilde{\mathrm{H}}^{N-1}(r)/\mathfrak{n}) \xrightarrow{\sim} O[\mathrm{Sh}]/\mathfrak{n}$$
$$\mathrm{F}_{-1}\mathrm{H}^1(\mathrm{I}_{p^2}, \widetilde{\mathrm{H}}^{N-1}(r)/\mathfrak{n}) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, \widetilde{\mathrm{H}}^{N-1}(r)/\mathfrak{n})$$

2. Proof

2.1. **Local Level-Raising Deformations.** Let $v \in \Sigma_{\mathrm{lr}} \cup \{\mathfrak{p}\}$ with $q = ||v||$ and let w denote the unique prime in F living over it. Fix a lift r_v of $\overline{r}_v$ to a coefficient ring R. Then by our assumption on the Satake parameters we can decompose

$$R^{\oplus N} = M_0 \oplus M_1$$

which is $r_v^\natural(\mathrm{Frob}_w)$-stable, and is such that $P_w(T) \equiv (T - q^{-N})(T - q^{-N+2}) \bmod \mathfrak{m}_R$ (here P_w is the characteristic polynomial of Frob_w on M_0). Now recall that

$$\mathscr{D}^{\mathrm{mix}} = \left\{\text{lifts such that } M_0 \oplus M_1 \text{ is stable under } r^\natural(\mathrm{I}_w), \mathrm{I}_w \text{ unipotent on } M_0, \mathrm{I}_w \text{ trivial on } M_1\right\}$$

We also have

$$\mathscr{D}^{\mathrm{unr}} = \left\{r \in \mathscr{D}^{\mathrm{mix}} : \mathrm{I}_w \text{ trivial on } M_0\right\}$$

and

$$\mathscr{D}^{\mathrm{ram}} = \left\{r \in \mathscr{D}^{\mathrm{mix}} : P_w(T) = (T - q^{-N})(T - q^{-N+2})\right\}$$

In fact, if we let $r_{v,\mathrm{mix}} : \Gamma_{F_v^+} \to \mathcal{G}_N(\mathsf{R}_v^{\mathrm{mix}})$ denote the universal lift, then (up to conjugation)

$$r^\natural_{v,\mathrm{mix}}(\mathrm{Frob}_w) = \begin{pmatrix} q^{-N}\frac{1+x}{1+y} & \\ & q^{-N+2}\frac{1+y}{1+x} \end{pmatrix}$$

$$r^\natural_{v,\mathrm{mix}}(t) = \begin{pmatrix} 1 & 0 \\ x_0 & 1 \end{pmatrix}$$

for some $x, y, x_0 \in \mathfrak{m}_{\mathsf{R}_v^{\mathrm{mix}}}$ such that $x_0(x - y) = 0$, where t is a generator of the inertia part. So

$$\mathsf{R}_v^{\mathrm{unr}} = \mathsf{R}_v^{\mathrm{mix}}/x_0, \quad \mathsf{R}_v^{\mathrm{ram}} = \mathsf{R}_v^{\mathrm{mix}}/(x - y)$$

2.2. **Global Deformations.** For $? \in \{\mathrm{mix}, \mathrm{ram}, \mathrm{unr}\}$, let $\mathsf{R}^?$ denote the global deformation ring parametrizing deformations

- with fixed similitude character ϵ_ℓ^{1-N}
- which are minimally ramified at $\Sigma_{\min}$
- which land in $\mathscr{D}^{\mathrm{ram}}$ at Σ_{lr}
- which land in $\mathscr{D}^?$ at $\mathfrak{p}$
- which are Fontaine-Laffaille at Σ_ℓ
- and are unramified everywhere else.

As above we can find $\mathsf{x}, \mathsf{y}, \mathsf{x}_0 \in \mathfrak{m}_{\mathsf{R}^{\mathrm{mix}}}$ and $\mathsf{v}, \mathsf{v}' \in (\mathsf{R}^{\mathrm{mix}})^{\oplus N}$ with eigenvalues $\mathsf{s} = p^{-N}(1+\mathsf{x})/(1+\mathsf{y})$ and $\mathsf{s}' = p^{-N+2}(1+\mathsf{y})/(1+\mathsf{x})$. We again have

$$\mathsf{R}^{\mathrm{unr}} = \mathsf{R}^{\mathrm{mix}}/(\mathsf{x}_0), \quad \mathsf{R}^{\mathrm{ram}} = \mathsf{R}^{\mathrm{mix}}/(\mathsf{x} - \mathsf{y}) = \mathsf{R}^{\mathrm{mix}}/(\mathsf{s} - p^{-N}), \quad \mathsf{R}^{\mathrm{cong}} := \mathsf{R}^{\mathrm{unr}} \otimes_{\mathsf{R}^{\mathrm{mix}}} \mathsf{R}^{\mathrm{ram}}$$

2.3. **Comparison with Cohomology.** Now if we let

$$\mathsf{T}^{\mathrm{unr}} = \mathrm{im}(\mathbb{T}^\Sigma \to \mathrm{End}_O(O[\mathrm{Sh}]))$$

then there exists a canonical isomorphism $\mathsf{R}^{\mathrm{unr}} \xrightarrow{\sim} \mathsf{T}^{\mathrm{unr}}_{\mathfrak{m}}$ which makes $O[\mathrm{Sh}]_{\mathfrak{m}}$ a free $\mathsf{T}^{\mathrm{unr}}_{\mathfrak{m}}$-module of rank d_{unr}.

Similarly if we let

$$\mathsf{T}^{\mathrm{ram}} = \mathrm{im}(\mathbb{T}^{\Sigma \cup \Sigma_p} \to \mathrm{End}_O(\widetilde{\mathrm{H}}^{N-1}))$$

By surjectivity of ($*'$) we get that $\mathsf{T}^{\mathrm{ram}}_{\mathfrak{m}} \neq 0$ and an isomorphism $\mathsf{R}^{\mathrm{ram}} \xrightarrow{\sim} \mathsf{T}^{\mathrm{ram}}_{\mathfrak{m}}$ making $\widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}$ a finite free $\mathsf{R}^{\mathrm{ram}}$-module. Now we let

$$\mathsf{H} = \mathrm{Hom}_{\mathsf{R}^{\mathrm{ram}}[\Gamma_F]}(r^{\natural,\mathsf{c}}_{\mathrm{ram}}, \widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}})$$

This is still a free R^{ram}-module of some rank d_{ram}, and in fact we have

$$\widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}} \simeq \mathsf{H} \otimes_{\mathsf{R}^{\mathrm{ram}}} r^{\natural,\mathsf{c}}_{\mathrm{ram}}$$

which follows from [LTX+19, Hypothesis 3.2.9], which is a multiplicity one result for discrete automorphic representations of an indefinite unitary group.

Lemma 2.4. *There exist isomorphisms*

$$O[\mathrm{Sh}]_{\mathfrak{m}}/((p+1)\mathtt{R}_{\mathfrak{p}}^{\circ}-\mathtt{I}_{\mathfrak{p}}^{\circ}) = O[\mathrm{Sh}]_{\mathfrak{m}}/(\mathsf{s}-p^{-N}) = O[\mathrm{Sh}]_{\mathfrak{m}} \otimes_{\mathsf{R}^{\mathrm{unr}}} \mathsf{R}^{\mathrm{cong}}$$
$$\mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, \widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}(r)) \simeq \mathsf{H} \otimes_{\mathsf{R}^{\mathrm{ram}}} \mathsf{R}^{\mathrm{cong}}$$

Proof. First of all, an explicit computation in the Hecke algebra, done in appendix B, shows that

$$(p+1)\mathtt{R}_{\mathfrak{p}}^{\circ}-\mathtt{I}_{p}^{\circ} = p^{-r^2}\prod_{i=1}^{r}(\alpha_i+\alpha_i^{-1}-p-p^{-1})$$

where $\{p^{1-N}\alpha_i, p^{1-N}\alpha_i^{-1}\}$ runs over the eigenvalues of $r^{\natural}_{\mathrm{unr}}(\mathrm{Frob}_p^2)$ and we take $\alpha_r = \mathsf{s}p^{N-1}$. But by our assumption on the Satake parameters, almost all of these will act invertibly, except for the α_r term. In this case the factor simplifies to the form $u(\mathsf{s}-p^{-N})$ for some unit u. But this shows that

$$((p+1)\mathtt{R}_{\mathfrak{p}}^{\circ}-\mathtt{I}_{\mathfrak{p}}^{\circ})O[\mathrm{Sh}]_{\mathfrak{m}} = (\mathsf{s}-p^{-N})O[\mathrm{Sh}]_{\mathfrak{m}}$$

and so we're done with the definite case.

For the indefinite case, we have

$$\mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, \widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}(r)) = \mathsf{H} \otimes_{\mathsf{R}^{\mathrm{ram}}} \mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, r^{\natural,c}_{\mathrm{ram}}(r))$$

But taking $\mathrm{H}^1(\mathrm{I}_{p^2}, -)$ is the same as taking coinvariants for the inertia action (i.e. kill the action of x_0) and twisting by -1. Then we take Frob_w-invariants. If you do this correctly, you find that

$$\mathsf{H} \otimes_{\mathsf{R}^{\mathrm{ram}}} \mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, r^{\natural,c}_{\mathrm{ram}}(r)) \cong \mathsf{H} \otimes_{\mathsf{R}^{\mathrm{ram}}} \mathsf{R}^{\mathrm{ram}}v'/\mathsf{x}_0\mathsf{R}^{\mathrm{ram}}v' = \mathsf{H} \otimes_{\mathsf{R}^{\mathrm{ram}}} \mathsf{R}^{\mathrm{cong}}$$

□

2.5. **Equality of Ranks.** It remains to show that $d^{\mathrm{ram}} = d^{\mathrm{unr}}$. This is done using automorphic methods, and we sketch the proof here.

If we take geometric $\overline{\mathbb{Q}}_\ell$-points η_1 and η_2 of $\operatorname{Spec}\mathsf{R}^{\mathrm{unr}}$ and $\operatorname{Spec}\mathsf{R}^{\mathrm{ram}}$ which are respectively contained in the support of $O[\mathrm{Sh}]_{\mathfrak{m}}$ and $\widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}$, then these are the systems of Hecke eigenvalues for certain relevant representations Π_1 and Π_2 of $\mathrm{GL}_N(\mathbb{A}_F)$ and we get

$$d_{\mathrm{unr}} = \dim \overline{\mathbb{Q}}_\ell[\mathrm{Sh}][\phi_{\Pi_1}]$$
$$Nd_{\mathrm{ram}} = \dim(\widetilde{\mathrm{H}}^{N-1})_{\overline{\mathbb{Q}}_\ell}[\phi_{\Pi_2}] = \dim \mathrm{H}^{N-1}_{\text{ét}}(\mathrm{Sh}(\mathrm{V}'), \overline{\mathbb{Q}}_\ell)$$

(here V' is the indefinite Hermitian space chosen as part of the indefinite uniformization data [LTX+19, Definition 5.1.6]) Furthermore, since both of these automorphic representations have residual system of Hecke eigenvalues given by $\mathfrak{m}$, we have an isomorphism of representations valued in $\overline{\mathbb{F}}_\ell$:

$$\overline{\rho}_{\Pi_1} \cong \overline{\rho}_{\Pi_2} \cong \overline{\rho}$$

Lemma 2.6. *For each $v \in \Sigma_{\min}$, write V_1, V_2 for the underlying vector spaces of $\Pi_{1,v}$ and $\Pi_{2,v}$. Then there exists a $\mathrm{GL}_N(O_{F_v})$-equivariant isomorphism $i: V_1 \xrightarrow{\sim} V_2$ which commute with certain intertwining operators A_1, A_2 (i.e. linear maps which realize the conjugate self-duality).*

Then [LTX+19, Proposition D.2.3] gives the result by computing the desired dimensions in terms of traces of endoscopic transfers of certain functions composed with A_1 and A_2.

We have thus deduced that $(*)$ and $(*')$ are both isomorphisms.

2.7. **Killing $\mathfrak{n}$.** A bit more work (done in [LTX+19, Section 6.4]) involving a more detailed analysis of the weight spectral sequence in the even case (specifically concerning the cohomology of the ground stratum $\overline{\mathrm{M}}_N$ and the eigenvalues of Frobenius acting on it) shows that

$$\mathrm{F}_{-1}\mathrm{H}^1(\mathrm{I}_{p^2}, \widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}(r))/\mathfrak{n} \to \mathrm{F}_{-1}\mathrm{H}^1(\mathrm{I}_{p^2}, \widetilde{\mathrm{H}}^{N-1}(r)/\mathfrak{n})$$

and

$$\mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, \widetilde{\mathrm{H}}^{N-1}_{\mathfrak{m}}(r))/\mathfrak{n} \to \mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, \widetilde{\mathrm{H}}^{N-1}(r)/\mathfrak{n})$$

are isomorphisms, so the main result follows.

References

[LTX+19] Yifeng Liu, Yichao Tian, Liang Xiao, Wei Zhang, and Xinwen Zhu. On the Beilinson-Bloch-Kato conjecture for Rankin-Selberg motives. *arXiv e-prints*, page arXiv:1912.11942, December 2019.