1-MOTIVES

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1. Introduction

Fix a perfect field k of any characteristic. Pol's talk gave an introduction to pure motives, and gave an indication that they form a universal Weil cohomology theory for smooth projective varieties over a field. Recall that objects in Mot_k (the category of pure numerical motives over a field k) are given by triples (X, p, n), where X is a smooth projective variety over k, $p \in \mathrm{Corr}(X, X)$ is a correspondence satisfying $p \circ p = p$, and n is an integer. We had a canonical functor

$$\mathsf{SPV}_k \to \mathsf{Mot}_k$$

from the category of smooth projective varieties over k to pure motives over k mapping $X \mapsto h(X) = (X, [\Delta_X], 0)$, where $\Delta_X \hookrightarrow X \times_k X$ is the diagonal of X.

With this description of a pure motive, we can begin to try to understand how to decompose a motive, and think about its ith degree components, for $0 \leq i \leq 2\dim(X)$. In the first part of this talk, we'll focus on the first component, which we'll call $h^1(X)$, and we will try to understand how exactly it parametrizes the first cohomology group in any Weil cohomology theory. We will start with the case of curves, and then move on to the case of a smooth projective variety in arbitrary dimension.

For singular and non-projective varieties, we need a different category MM_k of "mixed motives" in order to develop a similar theory for varieties that are neither smooth nor projective. For this second part of the talk, we will investigate a construction of Deligne of a category $\mathsf{MM}_k^{1,\mathrm{fr}}$, and a functor

$$\mathsf{Var}_k \to \mathsf{MM}_k^{1,\mathrm{fr}},$$

which parametrizes all of our usual cohomology theories, in degree 0 and degree 1. By usual (Weil) cohomology theories, we will specifically address

- Hodge (singular) cohomology
- de Rham cohomlogy
- étale cohomology
- crystalline cohomology

Deligne's construction is motivated by an idea of Grothendieck, which says that *pure 1-dimensional motives are abelian varieties, up to isogeny.* This observation is based on the idea that to extract the essential 0- and 1-cohomological data from a smooth projective variety X, all one needs is the Albanese variety, i.e. its "universal abelian variety" or its "abelian completion". We will spell this out for curves, and show that objects in $\mathsf{MM}_k^{1,\mathrm{fr}}$ really give us the cohomology theories we want.

At the end we will mention that although $\mathsf{MM}_k^{1,\mathrm{fr}}$ looks promising, it is not abelian, and we'll mention how to formally extend the category to MM_k^1, which is abelian.

2. Decomposing Pure Motives

Given $X \in \mathsf{SPV}_k$, we associate to it the motive $h(X) = (X, [\Delta_X], 0)$.

Assume that $X(k) \neq \varnothing$, to avoid some small technical problems.

We recall two constructions:

(1) If $f : X \to Y$ is a morphism of varieties, then h functorially gives a correspondence Γ_f
$$h(Y) \xrightarrow{f^*} h(X).$$

(2) There is a covariant version of this. If X and Y are purely d and e-dimensional, then we get a morphism
$$f_* : h(X) \to h(Y) \otimes \mathbf{L}^{\otimes(d-e)}$$
which is exactly Γ_f^T. Here $\mathbf{L} = (\operatorname{Spec} k, [\Delta_X], -1)$.

Supposing $X(k) \neq \varnothing$, we can pick a (non-canonical) section $\operatorname{Spec} k \xrightarrow{x} X$ of the structure map $X \xrightarrow{\alpha} \operatorname{Spec} k$. Thus,
$$h(\operatorname{Spec} k) \xrightarrow{\alpha^*} h(X) \xrightarrow{x^*} h(\operatorname{Spec} k),$$
is the identity map, so $h^0(X) = h(\operatorname{Spec} k)$ is a direct summand of $h(X)$. We have thus decomposed our motive into components
$$h(X) = h^0(X) \oplus h^{\geq 1}(X),$$
where $h^0(X)$ is the image of $h(\operatorname{Spec} k)$ in $h(X)$. Some work shows that
$$h^0(X) \cong (X, \{x\} \times X, 0),$$
Furthermore, $\alpha_* \circ x_* = \mathrm{id}$, so we get a split quotient map
$$h(X) \to \mathbf{L}^d.$$
Thus, we get another direct factor $h^{2d}(X)$ of $h(X)$, which, after some work is isomorphic to
$$h^{2d}(X) \cong (X, X \times \{x\}, 0).$$

2.1. **Example.** For example, given $X = \mathbf{P}^1$, one can show that the direct sum of $\{x\} \times \mathbf{P}^1$ and $\mathbf{P}^1 \times \{x\}$ is rationally equivalent to the diagonal on $X \times X$, so
$$h(\mathbf{P}^1) \cong h^0(\mathbf{P}^1) \oplus h^2(\mathbf{P}^1) \cong \mathbb{1} \oplus \mathbf{L}.$$

2.2. **Remark.** In general, the standard conjectures predict that we have a decomposition

$$h(X) = \bigoplus_{i=1}^{2d} h^i(X)$$

in a way that lets us recover the ith cohomology groups from the motive h^i. This is known for curves, surfaces, and for abelian varieties (and probably other cases that I'm not aware of).

3. Pure 1-Motives

Since we can easily describe the h^0 and h^{2d} part of any motive, the interesting part lies in the rest of the splitting. For curves, it turns out this is easy to describe.

3.1. **Proposition.** *If $X \in \mathsf{SPV}_k$, then there is a decomposition*

$$h(X) \cong h^0(X) \oplus h^1(X) \oplus h^2(X),$$

where

$$h^1(X) = \ker(h^{\geq 1}(X) \to h(X) \to h^2(X)),$$

or equivalently

$$h^1(X) = (X, \mathrm{id} - p_0, p_2, 0),$$

where $h^0(X) = (X, p_0, 0)$ and $h^{2d}(X) = (X, p_2, 0)$.

Note we called this $h^1(X)$, so we need to justify the "one"-ness of this component. Although we should expect this given our conjectural decomposition, we don't know this a priori, but the following proposition justifies this terminology:

3.2. **Theorem.** *If $X, X' \in \mathsf{SPV}_k$ with Jacobian varieties $J(X), J(X')$, then*

$$\mathrm{Hom}_{\mathsf{Mot}_k}(h^1(X), h^1(X')) = \mathrm{Hom}_{\mathsf{AbVar}_k}(J(X), J(X')) \otimes \mathbf{Q}.$$

This more or less follows from the remarkable fact that

$$A^1(X \times X') = A^1(X) \oplus A^1(X') \oplus \mathrm{Hom}_{\mathsf{AbVar}_k}(J(X), J(X')) \otimes \mathbf{Q}.$$

3.3. **Corollary.** *If $\mathsf{Mot}_k^{1,c}$ is the full subcategory of Mot_k whose objects are direct summands of objects of the form $h^1(X)$ for $X \in \mathsf{SPV}_k$ a curve, then $\mathsf{Mot}_k^{1,c}$ is equivalent to the category of abelian varieties over k up to isogeny.*

This almost justifies Grothendieck's statement: to fully justify it, we need to show that we can recover all of the first cohomology groups from the Jacobians in a natural way.

But before that, let's briefly mention what happens for more general smooth projective varieties. In fact, 3.4. **Proposition.** *For any $X \in \mathsf{SPV}_k$, there is a decomposition*

$$h(X) = h^0(X) \oplus h^1(X) \oplus M \oplus h^{2d-1}(X) \oplus h^{2d}(X),$$

where $M \in \mathsf{Mot}_k$.

3.5. **Remark.** Conjecturally M should always be $M = \bigoplus_{i=2}^{2d-2} h^i(X)$, but this is not fully known.

As before, we need a justification for the naming $h^1(X)$ and $h^{2d-1}(X)$. For this, we need to generalize the notion of the Jacobian variety. It turns out that there are two generalizations, called the Picard variety, and Albanese variety.

3.6. **Definition.** Given $X \in \mathsf{SPV}_k$ with a k-rational point, we can define a functor

$$\mathrm{Pic} : \mathsf{Var}_k \to \mathsf{Grp}, V \mapsto \mathrm{Pic}(X \times_k V)/\,\mathrm{Pic}(V),$$

which, for smooth projective X with $X(k) \neq 0$, is representable by a variety $\mathrm{Pic}\, X$. The connected component of the identity $\mathrm{Pic}^0 X$ is an abelian variety in characteristic 0, while in characteristic p, this variety may be non-reduced, so we need to take the reduction $\mathrm{Pic}^{0,\mathrm{red}} X$, which is then an abelian variety over k perfect. We will let $\mathbf{P}(X) = \mathrm{Pic}^{0,\mathrm{red}} X$.

3.7. **Definition.** Given $X \in \mathsf{SPV}_k$ with a choice of a k-rational point x (there exists one by assumption), there is abelian variety $\mathbf{A}(X)$ and a map $X \to \mathrm{Alb}(X)$ taking $x \mapsto 0$ called the Albanese variety, which is the solution to the universal problem

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{A}(X) \\ & \searrow & \downarrow \\ & & A \end{array}$$

for any morphism $X \to A$ to an abelian variety A taking $x \mapsto 0$.

In fact, $\mathbf{P}(X)$ and $\mathbf{A}(X)$ are dual abelian varieties, which we'll show later.

With these definitions we have, for all $X, X' \in \mathsf{SPV}$,

$$\mathrm{Hom}_{\mathsf{Mot}_k}(h^1(X), h^1(X')) = \mathrm{Hom}_{\mathsf{Var}_k}(\mathbf{P}(X), \mathbf{P}(X')) \otimes \mathbf{Q}$$

and

$$\mathrm{Hom}_{\mathsf{Mot}_k}(h^{2d-1}(X), h^{2d-1}(X')) = \mathrm{Hom}_{\mathsf{Var}_k}(\mathbf{A}(X), \mathbf{A}(X')) \otimes \mathbf{Q}$$

4. Cohomology in Degree 1

Grothendieck's statement is justified by some miraculous isomorphisms between Weil cohomologies of a curve and of its Jacobian variety. In some sense one should expect these, by the Torelli theorem, which says that a curve is uniquely determined by its Jacobian variety.

In fact, the notion of a Jacobian variety generalizes to the notion of a Picard variety, which can be defined for a smooth projective variety of arbitrary dimension, so we'll treat this more general case, and then mention how it restricts to the Jacobian variety of a curve. The surprising fact is that the Weil cohomology theories we care about, for arbitrary smooth projective varieties over k, factor through the Picard variety!

We now show that we can recover H^1 for all of the Weil cohomologies we're interested in, just by considering $\mathbf{P}(X)$. The way this works is to associate

$$X \rightsquigarrow \mathbf{P}(X) \rightsquigarrow T_{(-)}\mathbf{P}(X) = H^1_{(-)}(X)(1),$$

where $T_{(-)}$ is some sort of "Tate module" of an abelian variety, which is an object that depends on the chosen cohomology theory, for which there should exist a pairing

$$T_{(-)}A \times T_{(-)}A^\vee \to \mathbf{L},$$

where $\mathbf{L}$ is the Tate object for a given category. For example, for ℓ-adic cohomology, the "Tate module" is the usual ℓ-adic Tate module.

4.1. **Hodge (Singular) Cohomology.** Suppose we're working over $k = \mathbf{C}$ (or more generally, a subfield of $\mathbf{C}$, before base changing). Then taking the exponential exact sequence of sheaves

$$0 \to \mathbf{Z}(1) \to \mathscr{O}_X \to \mathscr{O}_X^\times \to 0$$

and taking cohomology, we get

$$0 \to H^1_{\mathrm{sing}}(X, \mathbf{Z}(1)) \to \ker(\mathrm{Lie}(\mathbf{P}(X)(\mathbf{C})) \to \mathbf{P}(X)(\mathbf{C})) \to 0,$$

This lets us compute the singular cohomology in terms of the Picard variety, and we can put a Hodge filtration on this H^1 by setting

$$F^0 H^1_{\mathrm{sing}}(X, \mathbf{Z}(1)) = \ker(H^1_{\mathrm{sing}}(X, \mathbf{Z}(1)) \otimes \mathbf{C} \to \mathrm{Lie}(\mathbf{P}(X)(\mathbf{C})))$$

This description is bit complicated, but one can show that this is the same as the Hodge filtration one gets from the Hodge decomposition.

4.2. **de Rham Cohomology.** Now let k be a field of characteristic 0. One can show that there is an isomorphism

$$H^1_{\mathrm{dR}}(X) \cong \mathrm{Lie}(\mathbf{P}(X)^\natural),$$

where $\mathbf{P}(X)^\natural \to \mathbf{P}(X)$ represent the functors

$$\left\{ \begin{array}{c} \text{isomorphism classes of line bundles} \\ \text{with an integrable connection} \end{array} \right\} \longrightarrow \{\text{isomorphism classes of line bundles}\}$$

4.3. **Étale Cohomology.** Let $\operatorname{char} k = 0$. Starting with the Kummer exact sequence of étale sheaves on the étale site of schemes over X

$$0 \to \mu_{\ell^n} \to \mathbf{G}_{m,X} \xrightarrow{\cdot \ell^n} \mathbf{G}_{m,X} \to 0,$$

one can take étale cohomology to get an exact sequence

$$0 \to H^1_{\text{ét}}(\overline{X}) \to H^1_{\text{ét}}(\overline{X}, \mathbf{G}_{m,X}) \to H^1_{\text{ét}}(\overline{X}, \mathbf{G}_{m,X}) \to 0.$$

Continuing this argument, one ends up with an isomorphism

$$H^1_{\text{ét}}(X, \mathbf{Z}_\ell)(1) \cong T_\ell(\mathbf{P}(X)).$$

4.4. **Crystalline Cohomology.** If k is a perfect field of characteristic p (e.g. a finite field), then let X/k be a smooth projective variety. Then $H^1_{\text{crys}}(X)(1)$ is isomorphic to the Dieudonné module attached to the Barsotti-Tate group of $\mathbf{P}(X)$.

Now the Tate objects and Tate modules are:

(1) **Singular:** The Tate object is $\mathbf{Z}(1) := 2\pi i\mathbf{Z}$ which has Hodge structure given by placing $2\pi i\mathbf{Z} \otimes \mathbf{C}$ in bidegree $(-1,-1)$. The Tate module is
$$T_{\text{sing}}A = \ker(\text{Lie}(A(\mathbf{C})) \to A(\mathbf{C})).$$

(2) **de Rham:** The Tate object is k itself, with filtration $F^{<-2} = 0$ and $F^{\geq -2} = k$. The Tate module is
$$T_{\text{dR}}A = \text{Lie}(A^\natural).$$

(3) **Étale:** The Tate object is $\mathbf{Z}_\ell(1)$ acting through the cyclotomic character. The Tate module is the ℓ-adic Tate module
$$T_\ell A = \varprojlim_n A(\overline{k})[\ell^n]$$

(4) **Crystalline:** The Tate object is $W(k)$ with some extra data, and the Tate module $T_{\text{crys}}A$ is the Dieudonné module attached to the p-divisible group associated with A.

By Poincaré duality in a Weil cohomology theory, there is an isomorphism
$$H^1_{(-)}(X)^\vee \xrightarrow{\sim} H^{2d-1}_{(-)}(X)(d)/\text{torsion}.$$
Then since we have a pairing
$$T_{(-)}\mathbf{P}(X) \times T_{(-)}\mathbf{A}(X) \to \mathbf{L},$$
we get isomorphisms
$$T_{(-)}\mathbf{A}(X) \xrightarrow{\sim} (T_{(-)}\mathbf{P}(X)(-1))^\vee \xrightarrow{\sim} (H^1_{(-)}(X))^\vee = H^{2d-1}_{(-)}(X)(d)/\text{torsion}.$$

This shows that if we can formulate this theory of "Tate objects" correctly, then we basically get that $H^{2d-1}_{(-)}(X)/\text{torsion}$ factorizes through the Albanese variety for free.

5. Arbitrary Schemes

Now we let X be non-smooth and non-projective, and see what happens.

First of all, what happens to our Weil cohomology theories? Here are some facts

(1) Singular (Hodge) cohomology is still possible, except now each $H^1_{\text{sing}}(X)$ comes with a mixed Hodge structure instead of a pure Hodge structure. In other words, we have an increasing filtration $W^\bullet$ such that W^i/W^{i-1} is a pure Hodge structure of weight i.

(2) de Rham cohomology generalizes through singular cohomology using hypercohomology.

(3) Taking ℓ-adic cohomology still works in exactly the same way.

(4) Crystalline cohomology doesn't generalize in the way we want (no longer finitely generated), need something called rigid cohomology, which we won't discuss.

To motivate 1-motives, we can ask the naive question: given a variety X over a perfect field k, can we find some (more complicated) abelian variety $A(X)$ such that for $\ell \neq \operatorname{char} k$, we have a natural isomorphism

$$T_\ell(\operatorname{Pic} X) \cong H^1_{\text{ét}}(X, \mathbf{Z}_\ell)(1),$$

or similarly for other cohomology theories?

The answer is no. For example, $H^1_\ell(\mathbf{G}_m)(1)$ is 1-dimensional, while $T_\ell(A)$ is always even dimensional. But let's consider some examples to see how to fix this.

5.1. **Example.** First let's look at something that is smooth, but not projective. Let $X = \mathbf{G}_m$. Embed $X \hookrightarrow \mathbf{P}^1$ so that $D = \mathbf{P}^1 \setminus \mathbf{G}_m = \{0, \infty\}$. Then the long exact sequence of relative cohomology groups for $(X, \mathbf{P}^1, D)$ gives us

$$0 \to H^1_{\text{ét}}(\mathbf{P}^1, \mathbf{Z}_\ell(1)) \to H^1_{\text{ét}}(\mathbf{G}_m, \mathbf{Z}_\ell(1)) \to \ker(H^2_D(\mathbf{P}^1, \mathbf{Z}_\ell(1)) \to H^2_{\text{ét}}(\mathbf{P}^1, \mathbf{Z}_\ell(1))) \to 0$$

The left hand term vanishes, and this becomes

$$0 \to H^1_{\text{ét}}(\mathbf{G}_m, \mathbf{Z}_\ell(1)) \to \ker(\deg : \operatorname{Div}_D(\mathbf{P}^1) \to \mathbf{Z}) \otimes \mathbf{Z}_\ell \to 0.$$

But $\operatorname{Div}_D(\mathbf{P}^1) = \mathbf{Z}^2$ (divisors supported on $\{0, \infty\}$). Therefore, our étale H^1 can be constructed from this kernel, which depends only on D.

5.2. **Example.** Now let's look at something that is projective, but not smooth. In particular, choose a projective nodal curve. Using the Kummer exact sequence again, we can get an isomorphism

$$H^1_{\text{ét}}(X, \mathbf{Z}_\ell)(1) \cong T_\ell \operatorname{Pic}^{0,\text{red}}(X).$$

However, one can also show that

$$\operatorname{Pic}^{0,\text{red}}(X) \cong \mathbf{G}_m.$$

So these pathologies suggest that we need to add tori to account for singularities, and free abelian groups for "compatification" (embedding in projective space).

6. 1-Motives

Deligne defined a category that encompassed abelian varieties, tori, and free abelian groups.

6.1. **Definition.** The category of free 1-motives over k, denoted $\mathsf{MM}_k^{1,\text{fr}}$ is the category of 2-term complexes

$$[L \to G],$$

of commutative group schemes over k, where L is an étale locally constant sheaf, such that $L(\overline{k})$ is a free finitely generated abelian group, and G is a semi-abelian variety over k, which is an extension

$$0 \to T \to G \to A \to 0,$$

where T is a torus (i.e. isomorphic to $\mathbf{G}_m^n$) and A is an abelian variety. Morphisms in this category are given by morphisms of complexes.

As expected from a theory of motives, the category $\mathsf{MM}_k^{1,\mathrm{fr}}$ comes with various realization functors. They are quite technical, and instead of doing them all, we will just discuss the Hodge realization, and the étale realization, referring the reader to the work of Deligne or Barbieri-Viale for further elucidation.

6.2. **Hodge Realization.** Associated to a 1-motive $[L \to G]$ is the following mixed Hodge structure. Note we have a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(G(\mathbf{C})) & \longrightarrow & \mathrm{Lie}(G(\mathbf{C})) & \longrightarrow & G(\mathbf{C}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^1(G(\mathbf{C})) & \longrightarrow & T(M) & \longrightarrow & L(\mathbf{C}) & \longrightarrow & 0 \end{array}$$

$T(M)$ is the vector space. The Hodge filtration is given by

$$F^0 = \ker(T(M) \otimes \mathbf{C} \to \mathrm{Lie}(G(\mathbf{C}))).$$

The weight filtration is given by

$$W_{-1} = H^1(G(\mathbf{C})) \text{ and } W_{-2} = \mathrm{im}(H^1(T(\mathbf{C})) \hookrightarrow H^1(G(\mathbf{C})))$$

6.3. **ℓ-adic Realization.** A 1-motive is a 2-term complex of group schemes $M = [L \xrightarrow{u} G]$ which we will say are placed in degrees $-1, 0$. We can consider the multiplication by ℓ^n map on this complex, which looks like

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \cdot\ell^n} & & \downarrow{\scriptstyle \cdot\ell^n} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Taking the mapping cone of this complex gives us

$$\cdots \longrightarrow 0 \longrightarrow L \xrightarrow{\begin{pmatrix} -u \\ \cdot\ell^n \end{pmatrix}} G \times L \xrightarrow{\begin{pmatrix} \cdot\ell^n & u \end{pmatrix}} G \longrightarrow 0 \longrightarrow \cdots$$

We then take the 0th cohomology of this complex, which gives

$$T_{\mathbf{Z}/\ell^n\mathbf{Z}}(M) = \left\{(g,x) \in G(\overline{k}) \times L(\overline{k}) : \ell^n g = u(x)\right\} / \left\{(nx, u(x)) : x \in L(\overline{k})\right\}$$

Then we let

$$T_\ell(M) = \varprojlim_n T_{\mathbf{Z}/\ell^n\mathbf{Z}}(M)$$

Note in particular that

$$T_\ell([0 \to \mathbf{G}_m]) = \mathbf{Z}_\ell(1)$$

and $T_\ell([0 \to A])$ is the usual ℓ-adic Tate module of an abelian variety.

6.4. **Defining 1-motives.** Now that we have realizations, it remains to ask: is there a canonical functor $\mathsf{Var}_k \to \mathsf{MM}_k^{1,\mathrm{fr}}$? In fact, Deligne does this in the case of curves by taking a normalization of a curve and embedding it into a projective space.

Barbieri-Viale and Srinivas have constructed 1-motives in characteristic 0 that correctly give Hodge, de Rham, and ℓ-adic cohomology. Andreatta and Barbieri-Viale additionally have defined $M^1(X)$ in positive characteristic for perfect fields.

7. Abelianizing $\mathsf{MM}_k^{1,\mathrm{fr}}$

If one allows $L(\overline{k})$ to be a finitely generated abelian group (not necessarily free), one gets the notion of 1-*motives with torsion*, or *effective* 1-*motives*. Then after localizing at quasi-isomorphisms of 1-motives (as complexes), we get MM_k^1, which is an abelian category.