COHOMOLOGY OF ARITHMETIC GROUPS: OVERVIEW

SAM MUNDY
NOTES BY ASHWIN IYENGAR

Contents

1. Arithmetic Groups

The setting is the same setting as for automorphic forms. Let G be a reductive group over $\mathbf{Q}$ (like GL_n or SL_n, or Sp_n, or $\mathrm{GSp}_n, \dots$). If R is a $\mathbf{Q}$-algebra (like $\mathbf{R}$, or $\mathbf{Q}_p$, or $\mathbf{A}_{\mathbf{Q}}, \dots$) then $G(R)$ form a group.

If R is a topological ring, we can topologize $G(R)$ by choosing a closed embedding (it's not obvious, but this is independent of the choice of embedding) $G \to \mathbf{A}^n$ and giving $G(R)$ the subspace topology of $\mathbf{A}^n(R) = R^n$.

Example 1.1.

(1) If $G = \mathrm{SL}_2$, then $\mathrm{SL}_2 \hookrightarrow M_{2\times 2} \cong \mathbf{A}^4$, and this is already a closed embedding (it's cut out by $\det = 1$). So $\mathrm{SL}_2(\mathbf{Q}_p)$ and $\mathrm{SL}_2(\mathbf{R})$ get their topology from $\mathbf{Q}_p^4$ and $\mathbf{R}^4$, etc.

(2) Let $G = \mathrm{GL}_1$: then the embedding $\mathrm{GL}_1 \hookrightarrow \mathbf{A}^1$ is not closed! To remedy this, one can take $\mathrm{GL}_1 \hookrightarrow \mathbf{A}^2$ via the embedding $x \mapsto (x, x^{-1})$: this is a closed embedding, and gives you the topology you want on $\mathrm{GL}_1(R)$. For instance, $\mathrm{GL}_1(\mathbf{A}_{\mathbf{Q}}) = \mathbf{I}_{\mathbf{Q}} \hookrightarrow \mathbf{A}_{\mathbf{Q}}^2$ gives the correct topology, whereas $\mathbf{I}_{\mathbf{Q}} = \mathbf{A}_{\mathbf{Q}}^\times \subseteq \mathbf{A}_{\mathbf{Q}}$ does *not* give you the right thing.

(3) If I choose a closed embedding $G \hookrightarrow \mathrm{GL}_n$, then the correct topology on $G(\mathbf{A}_{\mathbf{Q}})$ is given by its decomposition

$$G(\mathbf{A}_{\mathbf{Q}}) = \prod_{p \le \infty}{}' G(\mathbf{Q}_p)$$

with respect to the $G(\mathbf{Z}_p)$. Here, $G(\mathbf{Q}_p)$ gets its topology from $\mathrm{GL}_n(\mathbf{Q}_p)$: in this case, you can actually view this as an open subset of $M_{n\times n}(\mathbf{Q}_p)$.

Definition 1.2. Let $\Gamma, \Gamma' \subseteq G(\mathbf{Q})$ be subgroups, and say Γ, Γ' are *commensurable* if $\Gamma \cap \Gamma'$ has finite index in both Γ, Γ'.

Fact 1.3. *This is an equivalence relation.*

Definition 1.4. $\Gamma \subseteq G(\mathbf{Q})$ is **arithmetic** if for some (equivalently, every) closed embedding $G \hookrightarrow \mathrm{GL}_n$, Γ and $\mathrm{GL}_n(\mathbf{Z}) \cap G(\mathbf{Q})$ are commensurable.

Date: September 17, 2019.

Note that it's easy to see that such Γ are discrete in $G(\mathbf{R})$, and arithmetic subgroups are stable under $G(\mathbf{Q})$-conjugation.

Example 1.5.

(1) If $G = \mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$. Then $\mathrm{SL}_n(\mathbf{Q}) \cap \mathrm{GL}_n(\mathbf{Z}) = \mathrm{SL}_n(\mathbf{Z})$ is an arithmetic subgroup of $\mathrm{SL}_n(\mathbf{Q})$.

(2) Let $\Gamma_N = \ker(\mathrm{GL}_n(\mathbf{Z}) \twoheadrightarrow \mathrm{GL}_n(\mathbf{Z}/N\mathbf{Z}))$: this is arithmetic, and is called a "congruence subgroup" of level N.

(3) Let $G = \mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2$. We let
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \mod N \right\}$$
and
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \mod N, a \equiv d \equiv 1 \mod N \right\}$$
and
$$\Gamma(N) = \Gamma_N \cap \mathrm{SL}_2(\mathbf{Z}).$$
and these are arithmetic.

2. Their Cohomology Groups

Let $\Gamma \subseteq G(\mathbf{Q})$ be arithmetic, and let (ρ, V) be a finite dimensional algebraic representation of G. The cohomology groups we want to study are the group cohomology groups
$$H^*(\Gamma, V(\mathbf{C}))$$
One of the key points is that these have a Hecke action. Let $\Gamma', \Gamma'' \subseteq \Gamma$ be two subgroups of finite index such that there is an isomorphism $\varphi : \Gamma' \xrightarrow{\sim} \Gamma''$. Then we define $T_\varphi \in \mathrm{End}(H^i(\Gamma, V(\mathbf{C})))$ by
$$H^i(\Gamma, V(\mathbf{C})) \xrightarrow{\text{res}} H^i(\Gamma'', V(\mathbf{C})) \xrightarrow{\varphi^*} H^i(\Gamma', V(\mathbf{C})) \xrightarrow{\text{cores}} H^i(\Gamma, V(\mathbf{C}))$$
For instance if $G = \mathrm{SL}_2$ and $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ and $\Gamma' = \Gamma_0(p)$, and $\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b \equiv 0 \mod N \right\}$. The isomorphism is conjugation by $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. We will soon see that T_φ corresponds to the T_p-operator.

3. Preview of upcoming topics

The reason why these are so important is that these spaces $H^*(\Gamma, V(\mathbf{C}))$ can be interpreted as a space of automorphic forms on G of level Γ.

(1) (Geometric interpretation of the $H^*(\Gamma, V(\mathbf{C}))$) Let $K \subseteq G(\mathbf{R})$ be a maximal compact subgroup. Let $X = G(\mathbf{R})/K$, and view this as a real manifold. Then we consider
$$X_\Gamma := \Gamma \backslash G(\mathbf{R})/K,$$
which is a real manifold if Γ is torsion-free. One can make a local system $\widetilde{V}$ on X_Γ out of $V(\mathbf{C})$, and then:
$$H^*(\Gamma, V(\mathbf{C})) \cong H^*(X_\Gamma, \widetilde{V})$$

Example 3.1. Let $G = \mathrm{SL}_2$, and let $K = \mathrm{SO}_2(\mathbf{R}) \hookrightarrow \mathrm{SL}_2(\mathbf{R})$. Note $\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R}) \cong \mathcal{H}$ (the Poincaré upper half plane). Then if $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ is a finite index subgroup, then Γ acts on $\mathcal{H}$ by Möbius transformations.

In general, we can view automorphic forms as functions on certain X_Γs.

(2) (Matsushima's Formula) Assume X_Γ is compact (equivalently $\Gamma\backslash G(\mathbf{R})$ is compact). Then $G(\mathbf{R})$ acts on $L^2(\Gamma\backslash G(\mathbf{R}))$ by right translation.

Theorem 3.2 (Gelfand-(Piatetski-Shapiro))**.**

$$L^2(\Gamma\backslash G(\mathbf{R})) = \widehat{\oplus}_\pi \pi^{m(\pi)}$$

with $m(\pi) \in \mathbf{Z}_{\geq 0}$, and π running over irreducible unitary representations of $G(\mathbf{R})$.

Then Matsushima's formula says that

$$H^*(X_\Gamma, \widetilde{V}) = \bigoplus_\pi H^*(\mathfrak{g}, K, \pi \otimes \xi_V)$$

where ξ_V is the character of $Z(\mathcal{U}(\mathfrak{g}))$ on V. Note these $H^*(\mathfrak{g}, K, \pi \otimes \xi_V)$ can be computed explicitly.

(3) (Eisenstein classes) If $\Gamma\backslash G(\mathbf{R})$ is not compact mod center, then $H^*(X_\Gamma, \widetilde{V})$ has subspaces corresponding to cuspidal automorphic forms, but they don't exhaust the entire $H^*(X_\Gamma, \widetilde{V})$. Other cohomology classes come from cusp forms on Levi's of proper parabolic subgroups, which are known as "Eisenstein classes".

Theorem 3.3 (Franke)**.** *These cuspidal and Eisenstein classes exhaust $H^*(X_\Gamma, \widetilde{V})$.*

4. Why study automorphic forms in this way?

(1) (The "Chao Li"-style answer) $H^*(\Gamma, V(\mathbf{C}))$ has an obvious rational structure, namely $H^*(\Gamma, V(\mathbf{Q}))$. This leads to rationality results for automorphic forms. For instance, we can use this (co)homological structure to prove rationality results for L-values for modular forms.

(2) (The "Michael Harris"-style answer) X_Γ will often have the structure of not only a real manifold, but also a complex manifold, but not only a complex manifold, but also an algebraic structure: in particular, it will often admit the structure of a variety over a number field F. If we fix an isomorphism $\overline{\mathbf{Q}_\ell} \cong \mathbf{C}$, then these $H^*(X_\Gamma, \widetilde{V})$ can be compared with $H^*_{\text{et}}((X_\Gamma)_{\overline{F}}, \widetilde{V})$. Then there's a Hecke action on the singular cohomology group, and a G_F-action on the étale cohomology group, and these actions together lead to the construction of Galois representations (with a lot of work!)

(3) (The "Eric Urban"-style answer) It's hard to directly p-adically interpolate automorphic forms. However, it is much easier to interpolate the local systems V. Then suitably taking cohomology of the interpolated Vs is the first step on the way to constructing eigenvarieties.