HOMOTOPY BACKGROUND / SIMPLICIAL SETS

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Hi! In this document, I'm going to give a bunch of definitions, and hopefully a lot of intuition, as to what homotopy theory is and why things like simplicial sets, which look slightly esoteric, are actually very useful.

The point is that for the purposes of this study group, it's useful to know what a model category is, but you don't *really* need to understand them that well. The takeaway is that (co)chain complexes and topological spaces fit into a common (homotopical) framework, and that simplicial sets are "the same" as topological spaces in this framework.

1. Model Categories

1.1. **Warm up: cochain complexes.** Let's try to motivate homotopy theory by talking about cochain complexes. We could equally well talk about chain complexes, but oh well. First, fix a commutative ring R (with a unit) and consider the category Mod_R of R-modules. Often we'll add some additional structure, like a Galois action or something, but not right now. In homological algebra, we extract information about R and its modules by computing cohomology groups, which tells you why doing completely natural things (i.e. applying natural functors that you care about) can fail miserably and give you weird extra data. To compute cohomology groups, we need the category Ch_R^+ of bounded below cochain complexes of R-modules, i.e. sequences

$$0 \to M_0 \xrightarrow{d} M_1 \xrightarrow{d} M_2 \to \cdots$$

ranging over $n \in \mathbb{Z}$ such that $d^2 = 0$. We're usually lazy about labeling, because there's only ever one possible thing d can mean if you know what you're applying it to. The point is that what we really care about is the *derived category*, because cohomology theories show up as derived functors.

What does this mean? For instance, say you want to study Mod_R by fixing $M \in \mathsf{Mod}_R$ and then studying "maps to M", i.e. the contravariant functor $\mathrm{Hom}_R(-, M)$. Then strange things can happen: if

$$0 \to A \to B \to C \to 0$$

is exact, then you get

$$0 \to \mathrm{Hom}_R(C, M) \xrightarrow{\text{exact}} \mathrm{Hom}_R(B, M) \xrightarrow{\text{exact}} \mathrm{Hom}_R(A, M) \xrightarrow{\text{not necessarily exact!!!}} 0$$

Why isn't the sequence exact? Well, you can measure exactly how much it fails via the formalism of derived functors, which is set up so that you can complete the sequence to a long exact sequence:

$$0 \to \mathrm{Hom}_R(C,M) \to \mathrm{Hom}_R(B,M) \to \mathrm{Hom}_R(A,M) \to \mathrm{Ext}^1_R(C,M) \to \mathrm{Ext}^1_R(B,M) \to \mathrm{Ext}^1_R(A,M) \to \mathrm{Ext}^2_R(C,M) \to \cdots$$

So how do you actually define derived functors, i.e. what are these $\mathrm{Ext}^i_R(A,M)$? Note that applying $\mathrm{Hom}_R(-,M)$ to the chain complex

$$\cdots \to 0 \to A \to 0 \to \cdots$$

doesn't really give you much information on its own. On the other hand, it turns out that you can find an exact complex in Ch^+_R (called an **injective resolution**)

$$\cdots \to 0 \to A \to I_0 \to I_1 \to \cdots$$

where each I_k is injective, and then you remove A and apply $\mathrm{Hom}_R(-,M)$ and take the cohomology (i.e. "kernel mod the image") in each degree and it turns out that these are exactly the Ext groups $\mathrm{Ext}^i_R(A,M)$.

Importantly, it turns out that you can take *any* injective resolution, and you'll always get the same Ext groups! Why? Well, if you take two injective resolutions $I^\bullet$ and $J^\bullet$, you can prove by doing some homological algebra that they are *chain homotopic*. This means that there are maps $f : I^\bullet \to J^\bullet$ and $g : J^\bullet \to I^\bullet$ such that $f \circ g$ and $g \circ f$ are both "homotopic to the identity". This means that they are not quite the identity, but they are equal to the identity by a **chain homotopy** (think of a path between points in a space: here the points are chain complexes). In particular, this means that the induced maps $f : H^k(I^\bullet) \to H^k(J^\bullet)$ and $H^k(J^\bullet) \to H^k(I^\bullet)$ are isomorphisms[1]. We say that f and g are homotopy equivalences. But important(!) fact: if you call a map inducing isomorphisms on cohomology a **weak homotopy equivalence**, then there are weak homotopy equivalences which are not homotopy equivalences!

In general, you can replace $\mathrm{Hom}_R(-,M)$ with any left-exact functor (there's a covariant version of this as well), and you get a notion of derived functors just as above.

So what's the derived category D^+_R? Well, there are actually two ways to define it:

- First, define objects in D^+_R to be objects in Ch^+_R all of whose terms are injective, and set

  $$\mathrm{Hom}_{D^+_R}(A,B) = \mathrm{Hom}_{\mathsf{Ch}^+_R}(A,B)/\text{chain homotopy}.$$

  By the construction I gave above, this is enough to compute derived functors/cohomlogy.
- The other way is to perform an construction which is simple to explain, but somewhat difficult to actually do because of annoying combinatorial issues. The idea is that a map $f : C^\bullet \to D^\bullet$ which induces isomorphisms $f : H^k(C^\bullet) \to H^k(D^\bullet)$ should actually be called "an isomorphism up to homotopy". So you construct the derived category by taking Ch^+_R and building a new category by adding inverses to the weak homotopy equivalences, and then adding extra maps so that the axioms of a category are again satisfied (e.g. composition). Note that this gives rise to set theoretic issues, so let's completely ignore them.

The striking thing is that these constructions give equivalent categories! I'm ignoring some serious set theoretic issues here.

1.2. **What is a model category?** Maybe this is all already familiar to you, in which case you're wondering why I'm saying all of this, and what this has to do with homotopy theory. The point is that we took a category (Ch^+_R), identified some well-behaved objects in it (injective ones), identified a notion of homotopy group (cohomology groups), singled out a class of maps which identify the homotopy groups (weak homotopy equivalences), and defined a category of "objects up to homotopy" given by inverting the aforementioned class of maps.

Definition 1.2.1. A *model category* is a category C admitting all limits/colimits along with

[1]This also means that $H^k(\mathrm{Hom}(I^\bullet,M)) \xrightarrow{\sim} H^k(\mathrm{Hom}(J^\bullet,M))$, hence the Ext groups are well-defined.

- a subclass of morphisms W called weak equivalences,
- a subclass of morphisms C called cofibrations,
- and a subclass of morphisms F called fibrations,

which satisfy some very natural axioms.

In particular C has an initial object $\circ$ and a terminal object $\bullet$. Then we call an object $c \in$ C *cofibrant* if the map $\circ \to c$ is a cofibration and *fibrant* if $c \to \bullet$ is a fibration.

Proposition 1.2.2. *There is a well-behaved notion of homotopy between morphisms of objects which are both fibrant and cofibrant.*

This may make it seem like you have to throw away a bunch of objects in order to have a nice homotopy theory of maps in C, but the axioms (which I have withheld from you) actually imply that every object has a "fibrant replacement" and a "cofibrant replacement", and roughly this means that you can always reduce to the case where you have a good homotopy theory of maps.

Definition 1.2.3. The *homotopy category* $\mathcal{H}$(C) of C is the category whose objects are the ones which are both fibrant and cofibrant, and the morphisms are homotopy classes of maps between them.

Theorem 1.2.4. *The homotopy category is equivalent to the category obtained by taking* C *and formally inverting all of the weak equivalences (again, ignore set theoretic issues here).*

This should remind you a lot of what happened with the definition of the construction of D_R^+: you can either restrict to a subclass of objects and take maps up to homotopy, or invert a bunch of weak equivalences, and you should get the same thing. In fact, D_R^+ is an example of a homotopy category of a model structure on Ch_R^+, as follows.

Theorem 1.2.5. *There exists a model structure on* Ch_R^+ *called the* ***injective model structure*** *such that*

- *every object is cofibrant*
- *the complexes of injective objects are exactly the fibrant objects*
- *taking a fibrant replacement is the same as taking an injective resolution*
- $\mathcal{H}(\mathsf{Ch}_R^+) \cong D_R^+$

1.3. **Topological Spaces.** The category Top of topological spaces has a few model structures, but here is the classical one, due to Quillen. For this we need one definition.

Definition 1.3.1. A *Serre fibration* is a continuous map $f : X \to Y$ in Top such that a lift h exists in any diagram of the following form:

$$\begin{array}{ccc} D^n & \longrightarrow & X \\ {\scriptstyle (\mathrm{id},0)}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle f} \\ D^n \times I & \longrightarrow & Y \end{array}$$

where D^n is an n-dimensional closed unit ball and $I = D^1$.

Examples are fiber bundles, a counterexample is a $+$ sign projecting to a $-$ sign: I'll leave this as an exercise.

Theorem 1.3.2. *There exists a model structure on* Top *such that*

- *the weak equivalences are maps $f : X \to Y$ inducing isomorphisms on all higher homotopy groups.*
- *the fibrations are Serre fibrations (indeed, this is where the word comes from),*
- *the cofibrations are "retracts of relative cell complexes".*

Exercise: every topological space is fibrant.

Note that homotopy equivalences are weak equivalences, but not vice versa: compare with the situation for chain complexes.

2. Simplicial Sets

Now I'll define another model category sSet of simplicial sets. The point will be that

$$\mathcal{H}(\mathsf{Top}) \cong \mathcal{H}(\mathsf{sSet})$$

but that sSet is slightly easier to study and contains less bizarre and weird and undesirable spaces than Top.

One way to approach this is to view the cofibrations in Top as somehow too complicated: in a simpler theory, every object should be a retract of a relative cell complex: if not, then the space is unnecessarily weird. This is further backed up by the fact that for any $X \in \mathsf{Top}$, there exists a weak equivalence $Z \to X$ from a CW-complex Z (this is called "CW-approximation"). Since we only care about objects up to weak equivalence, we could work with CW-complexes, but it turns out that simplicial sets have better formal properties.

Here's the definition.

Definition 2.0.1. The **simplex category** Δ is the category of finite totally ordered sets $[n] = \{0 \le 1 \le \cdots \le n\}$, whose morphisms are order preserving maps.

For instance the map $[2] \to [1]$ given by $(0,1,2) \mapsto (0,0,1)$ is a morphism in Δ, but the map $[2] \to [1]$ given by $(0,1,0)$ is not a morphism in Δ.

Definition 2.0.2. Given a category C, a *simplicial object in* C is a functor $\Delta^{\mathrm{op}} \to \mathsf{C}$. In particular, a simplicial set is a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$. Then the functor category

$$\mathsf{sC} := \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathsf{C})$$

is the category of simplicial objects in C, for example sSet.

Limits and colimits are computed in the functor category, and in particular, are computed "pointwise".

Let's unwind this a bit. First of all, it is an exercise to show that any map in Δ is the composition of maps of the form:

- **co-face maps** $\delta_i : [n] \to [n+1]$ for $i = 1, \dots, n+1$ which miss i (for instance, $\delta_2 : [2] \to [3]$ sends $(0,1,2) \mapsto (0,1,3)$) and
- **co-degeneracy maps** $\sigma_i : [n] \to [n-1]$ for $i = 1, \dots, n-1$ which double i (for instance $\sigma_1 : [2] \to [1]$ sends $(0,1,2) \mapsto (0,1,1)$).

So therefore, a simplicial set is equivalent to the data of a sequence $(S_n)_{n \in \mathbb{N}}$ of sets, along with **face maps** $d_i : S_{n+1} \to S_n$ for each δ_i and **degeneracy maps** $s_i : S_n \to S_{n+1}$, subject to a bunch of relations (which are induced by the relations satisfied by δ_i and σ_i). Elements of S_n are called n-**simplices**. Simplices in the image of some s_i are called degenerate, and nondegenerate otherwise.

2.1. **Simplices.** What are some examples? The most basic example are the representable functors.

Definition 2.1.1. The **standard n-simplex** Δ^n is the simplicial set defined by

$$\Delta^n_i := \mathrm{Hom}_\Delta([i],[n])$$

whose faces and degeneracies are given by composition. This is supposed to be the analog of the n-dimensional unit disk.

By Yoneda's lemma, if $S \in \mathsf{sSet}$ then $S_n = \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, S)$. Exercise: find all of the non-degenerate simplices in Δ^n (hint: there is only one nondegenerate n-simplex. can you find the rest of them?)

Example 2.1.2. If X is a topological space, then the singular simplicial set is

$$\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathsf{Top}}(|\Delta^n|, X)$$

where $|\Delta^n| = \left\{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ and } \sum x_i = 1\right\}$ is the topological n-simplex. One can use this simplicial set to define singular homology of X.

2.2. **Homotopy.** Remember that we want to attach a model structure to simplicial sets. To do this, we need a notion of higher homotopy groups. Since this will be discussed in more detail in the next talk, I'll just mention the fundamental group.

Note that the co-face maps $\delta_0, \delta_1 \in \mathrm{Hom}_\Delta([0],[1]) = \Delta^1_0$ can also be thought of as maps $\delta_0, \delta_1 : \Delta^0 \to \Delta^1$. The corresponding maps $d_0, d_1 : S_1 \to S_0$ should be thought of as "taking a 1-simplex and picking out its endpoints". This naturally generalizes to higher n-simplices, where instead of endpoints, you take the various faces of an n-simplex, which are all $(n-1)$-simplices (draw a picture of a 2-simplex and you'll see how this works).

Definition 2.2.1. If $f, g : S \to T$ are two morphisms of simplicial sets, then we say they are homotopic if there exists a morphism $h : S \times \Delta^1 \to T$ making this diagram commute:

$$\begin{array}{ccc}
S \times \Delta^0 \cong S & & \\
\downarrow{\scriptstyle \mathrm{id} \times \delta^1} & \searrow^{f} & \\
S \times \Delta^1 & \overset{h}{\dashrightarrow} & T \\
\uparrow{\scriptstyle \mathrm{id} \times \delta^0} & \nearrow_{g} & \\
S \times \Delta^0 \cong S & &
\end{array}$$

Note that this isn't necessarily symmetric, so it doesn't necessarily define an equivalence relation. How do we fix this?

Definition 2.2.2. A morphism $f : S \to T$ is a **Kan fibration** if there is a lift in any diagram of the following form:

$$\begin{array}{ccc}
\Lambda^n_i & \longrightarrow & S \\
\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle f} \\
\Delta^n & \longrightarrow & T
\end{array}$$

where Λ^n_i is the smallest simplicial set containing all of the nondegenerate $(n-1)$-simplices in Δ^n except for the one missing i. The simplicial set Λ^n_i is called the ith horn of Δ^n: if you try to draw a picture of it, you'll see why.

Lemma 2.2.3. *If $T \to \Delta^0$ is a Kan fibration, then homotopy of maps $S \to T$ is an equivalence relation.*

This allows us to define $\pi_1(S, v)$ for any $v \in S_0$ (vertex) as the set of homotopy classes of maps $\Delta^1 \to S$ such that both $\Delta^0 \xrightarrow{\delta^0} \Delta^1 \to S$ and $\Delta^0 \xrightarrow{\delta^1} \Delta^1 \to S$ correspond to v under Yoneda. The π_0 and higher π_n are defined similarly.

Lemma 2.2.3 is also great because notice that Δ^0 is the terminal object: this suggests that if $T \to \Delta^0$ is a Kan fibration, then T should be called "Kan fibrant", and reinforces the notion that fibrant objects (I'll get to cofibrant ones in a second, don't worry) are the ones which admit good homotopy theory.

Theorem 2.2.4. *There exists a model structure on* sSet *such that*

- *the weak equivalences are maps $f : S \to T$ inducing isomorphisms on all higher homotopy groups.*
- *the fibrations are Kan fibrations,*
- *the cofibrations are just injective maps.*

In particular, every simplicial set is cofibrant (hint, what is the initial object in sSet?)! The fibrant ones are exactly the ones which admit good homotopy theory.

2.3. **Self-Enrichment.** One nice feature of simplicial sets is that the set $\mathrm{Hom}_{\mathsf{sSet}}(S, T)$ can be upgraded to a simplicial set. So sSet is enriched over itself! To do this, you take

$$\underline{\mathrm{Hom}}_{\mathsf{sSet}}(S, T)_n := \mathrm{Hom}_{\mathsf{sSet}}(S \times \Delta^n, T)$$

with faces and degeneracies induced by the δ_i and σ_i interpreted as maps $\Delta^i \to \Delta^n$ by Yoneda: I'll leave this as an exercise for you to work out that this defines a simplicial set. Also exercise: $\underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Delta^0, S) = S$.

3. Geometric Realization

Now we can talk about geometric realization, which compares simplicial sets to topological spaces.

First, a fact from category theory:

Proposition 3.0.1. *Let* C, D *be two categories. If* C *is small and* D *admits small limits and colimits, then so does* $\mathrm{Funct}(\mathsf{C}, \mathsf{D})$.

Proof. Exercise. □

Now let $\mathsf{D} = \mathsf{Set}$.

Proposition 3.0.2. *Let $F : \mathsf{C} \to \mathsf{Set}$ be a functor from a small category C. Consider the category*

$$\mathsf{D}_F := \{(c, \phi_c) \mid c \in \mathsf{C} \text{ and } \varphi_c : \mathrm{Hom}_{\mathsf{C}}(c, -) \to F\}$$

with morphisms given by maps $\varphi : c \to c'$ such that $\phi_{c'} \circ \varphi = \phi_c$. Then D_F is small, and the natural map $\mathrm{colim}_{\mathsf{D}_F} \mathrm{Hom}(c, -) \to F$ is an isomorphism.

Proof. Exercise, using the Yoneda lemma. □

Corollary 3.0.3. *Now let $\mathsf{C} = \Delta^{\mathrm{op}}$. The natural map*

$$\mathrm{colim}_{\Delta^n \to S} \Delta^n \to S$$

is an isomorphism of simplicial sets.

In other words, simplicial sets are simplices glued together, and the gluing rules are given by the face and degeneracy maps in the simplicial set. The idea behind topological realization is to replace the standard n-simplices Δ^n with their topological incarnations $|\Delta^n|$ so that now you glue together topological simplices using the same faces and degeneracy maps in the simplicial sets: the result is an honest topological space now, instead of a simplicial set.

Definition 3.0.4. If S is a simplicial set, then its **topological realization** is

$$|S| = \operatorname{colim}_{\Delta^n \to S} |\Delta^n|$$

Proposition 3.0.5. *Topological realization is left adjoint to* $\operatorname{Sing}(\cdot)$.

Proof.

$$\begin{aligned} \operatorname{Hom}_{\mathsf{Top}}(|S|, X) &= \operatorname{Hom}_{\mathsf{Top}}(\operatorname{colim}_{\Delta^n \to S} |\Delta^n|, X) \\ &= \lim_{\Delta^n \to S} \operatorname{Hom}_{\mathsf{Top}}(|\Delta^n|, X) \\ &= \lim_{\Delta^n \to S} \operatorname{Hom}_{\mathsf{sSet}}(\Delta^n, \operatorname{Sing}(X)) \\ &= \operatorname{Hom}_{\mathsf{sSet}}(\operatorname{colim}_{\Delta^n \to S} \Delta^n, \operatorname{Sing}(X)) \\ &= \operatorname{Hom}_{\mathsf{sSet}}(S, \operatorname{Sing}(X)) \end{aligned}$$

The equalities are (1) by definition (2) relationship between homs and (co)limits (3) definition of $\operatorname{Sing}(X)$ (see above) (4) same as 2 and. (5) Corollary 3.0.3. ☐

A corollary is that Kan fibrations are sent to Serre fibrations under $|\cdot|$.

In fact, more is true! Our goal was to identify the homotopy categories of sSet and Top. Note equivalence of homotopy categories is weaker than equivalence of the model categories themselves, and it turns out that to find an equivalence of homotopy categories, all you need is an adjunction between the model category which respects the model structure.

Definition 3.0.6. A pair of adjoint functors (F, G) of model categories is a **Quillen adjunction** if F sends cofibrations to cofibrations and sends cofibrations that are also weak equivalences to cofibrations that are also weak equivalences, or equivalently, if G sends fibrations to fibrations and sends fibrations that are also weak equivalences to fibrations that are also weak equivalences.

Proposition 3.0.7. *If (F, G) are a Quillen adjunction of model categories* C *and* D*, then F induces an equivalence* $\mathcal{H}(\mathsf{C}) \xrightarrow{\sim} \mathcal{H}(\mathsf{D})$.

Theorem 3.0.8. $(|\cdot|, \operatorname{Sing})$ *is a Quillen adjunction and thus induces an equivalence of categories*

$$\mathcal{H}(\mathsf{sSet}) \cong \mathcal{H}(\mathsf{Top})$$

Proof. If $S \in$ sSet then it turns out that $\pi_n(S, s) = \pi_n(|S|, |s|)$ for all n, so weak equivalences are sent to weak equivalences. As mentioned before, Kan fibrations are sent to Serre fibrations. ☐

4. Pro-Categories

So far I've been talking about topological spaces and their fundamental groups/higher homotopy groups. In this study group, we want to study schemes and their étale fundamental groups/higher étale homotopy groups. You might expect that there's a topological space underlying the higher étale homotopy groups, and in fact Artin-Mazur and Friedlander made this precise. However, since the étale fundamental group is a profinite group, you need to consider a pro-topological space instead. But we don't care about topological spaces, we only care about their homotopy types. So we only care about $\mathcal{H}(\mathsf{Top})$. But this is the same as $\mathcal{H}(\mathsf{sSet})$, and we like simplicial sets more. So we study pro-$\mathcal{H}(\mathsf{sSet})$.

Informally, a pro-category should be thought of as inverse systems of objects in your category. The only question is what the indexing set of the system is.

Definition 4.0.1. A category I is **cofiltering** if it satisfies the following conditions:

(1) For all $i, j \in \mathsf{I}$, there is a diagram
$$\begin{array}{ccc} & & i \\ & \nearrow & \\ k & & \\ & \searrow & \\ & & j \end{array}$$

(2) For $\alpha, \beta : i \to j$ there is a morphism $\gamma : k \to i$ such that $\alpha\gamma = \beta\gamma$.

The natural numbers with $\geq$ relation, the positive natural numbers with the divisibility relation, or connected pointed étale coverings of a pointed scheme all form cofiltering categories.

Definition 4.0.2. Given a category C, the **pro-category** pro-C is the category of functors $F : \mathsf{I} \to \mathsf{C}$ from a cofiltering indexing category I with morphisms given by

$$\mathrm{Hom}_{\mathrm{pro-}\mathsf{C}}(\{c_i\}_{\mathsf{I}}, (c_j)_{\mathsf{J}}) := \varprojlim_{\mathsf{J}} \varinjlim_{\mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(c_i, c_j)$$

One can define a pointed variant of sSet where you include a basepoint $* \in S_0$ everywhere, and then pro-sSet_* lets you define homotopy pro-groups by taking

$$\pi_n(\{S_i\}_{\mathsf{I}}, *) := \varprojlim_{i \in \mathsf{I}} \pi_n(S_i, *)$$

equipped with the profinite topology.