THE SPECTRAL ACTION, PART I

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This talk will discuss Section 6 of XZ. The plan is to

(1) Discuss function field Langlands, where there are *excursion operators*
(2) Discuss how XZ apply the idea of these operators.

1. Motivation from the function field case

Let $X/\mathbb{F}_q$ be a smooth projective geometrically connected curve. Let F be its function field. Let $G_F := \mathrm{Gal}(\overline{F}/F)$. Let G be a split connected reductive group over F, and let $\mathbb{A}_F$ denote the F-adèles.

We want to understand

$$\mathrm{Bun}_G(\mathbb{F}_q) = G(F)\backslash G(\mathbb{A}_F)/G(\mathbb{O}),$$

where $\mathbb{O} := \prod_v \mathcal{O}_v$ and v varies over all places of F. We want to decompose

$$\mathrm{Aut}_G := \mathcal{C}_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell),$$

with respect to the Hecke action: given any dominant coweight λ of G and a place v of F, there is a Hecke operator

$$T_{\lambda,v}$$

acting on Aut_G.

In the $G = \mathrm{GL}_n$ case, the decomposition gives the following result.

Theorem 1.1 (Drinfeld for $G = \mathrm{GL}_2$, L. Lafforgue for GL_n)**.** *To each cuspidal automorphic representation π (whose central character has finite order), there is a unique Galois representation (whose determinant has finite order)*

$$\sigma_\pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$$

which is unramified at each place $x \in |X|$ where π is unramified, and such that the eigenvalues of $\sigma_\pi(\mathrm{Frob}_x)$ are the same as the Hecke eigenvalues of π_x for all unramified places x.

The point is that by Chebotarev density, specifying the eigenvalues of the Frobenii uniquely determines $\sigma(\pi)$. But this doesn't hold if we replace GL_n with a more general reductive group G: $\sigma(\pi)$ may no longer be uniquely determined by the Hecke action.

The solution of Vincent Lafforgue is to introduce *excursion operators*, which include the $T_{\lambda,v}$ as a subalgebra. But the algebra of excursion operators will be big enough to uniquely determine the associated Galois representation.

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We study these excursion operators on the ℓ-adic cohomology of some stacks of shtukas. Given a finite set I (thought of as the number of "legs"), and an irreducible representation W of $\hat{G}^I$, Lafforgue defines a moduli space of shtukas

$$\mathrm{Sht}_{I,W},$$

and cuts out a $\mathbb{Q}_\ell$-v.s.

$$H_{I,W}$$

from the ℓ-adic cohomology of $\mathrm{Sht}_{I,W}$.

This has some nice properties.

(1) It is functorial in W
(2) For any map $I \to J$ of index sets, there exists

$$H_{I,W} \longrightarrow H_{J,W^J}$$

(which turns out to be an isomorphism) where W^J is a $\hat{G}^J$-representation via

$$\hat{G}^J \to \hat{G}^I$$

via $(g_j)_j \mapsto (g_{\phi(i)})_i$.
(3) $H_{I,W}$ has a $(G_F)^I$-action.

For example, for $\emptyset \to \{0\}$, there is

$$H_{\emptyset,1} \xrightarrow{\sim} H_{\{0\},1}.$$

And we are interested in $H_{\emptyset,1}$ because

Proposition 1.2. $H_{\emptyset,1} \cong \mathrm{Aut}_G$.

An excursion is the following. Given a function

$$f : \hat{G}(\mathbb{Q}_\ell)\backslash\hat{G}(\mathbb{Q}_\ell)^I/\hat{G}(\mathbb{Q}_\ell) \longrightarrow \overline{\mathbb{Q}}_\ell,$$

(quotient is taken with respect to the diagonal action) and elements $(\gamma_i)_I \in G_F$,

(1) pick an irreducible representation W of $\hat{G}^I$ and $x \in W$, $\xi \in W^*$ such that

$$f((g_i)_I) = \langle \xi, (\gamma_i) \cdot x\rangle.$$

(2) Put together

$$H_{\emptyset,1} \cong H_{\{0\},1} \xrightarrow{x} H_{\{0\},W^{\{0\}}} \xrightarrow{\sim} H_{I,W} \xrightarrow{(\gamma_i)_I} H_{I,W} \xrightarrow{\sim} H_{\{0\},W^{\{0\}}} \xrightarrow{\xi} H_{\{0\},1} \cong H_{\emptyset,1}.$$

To see that this is a suitable generalization of a Hecke operator, note the following Proposition.

Proposition 1.3. *Fix a highest weight representation V_λ, and let $I = \{1,2\}$. Let $f : (g_1, g_2) \mapsto \chi_{V_\lambda}(g_1 g_2^{-1})$. Fix a place v of F. For $(\gamma_1, \gamma_2) = (\mathrm{Frob}_v, 1)$, we have*

$$S_{\{1,2\},f,(\mathrm{Frob}_v,1)} = T_{\lambda,v}.$$

2. Application of these ideas in XZ

We recall the setting.

- Local field $F \supset \mathcal{O}$.
- $G/\mathcal{O}$ is unramified.
- $\hat{G}/\overline{\mathbb{Q}}_\ell$, the Langlands dual.
- The action of G_F on $\hat{G}$ reduces to an action of Frob.
- We can define a L-group
$$^LG := \hat{G} \rtimes \langle \sigma \rangle.$$
Looking at $\hat{G}\sigma \subset {}^LG$, this has an $\hat{G}$-action
$$g \cdot h = gh\sigma(g^{-1}).$$

Given any $V \in \mathrm{Rep}(G)$, we can define a trivial vector bundle
$$V \times \hat{G}\sigma \longrightarrow \hat{G}\sigma,$$
and endow it with a $\hat{G}$-action. In particular, this takes
$$g \cdot (v, h\sigma) = (g \cdot v, gh\sigma^{-1}(g)\sigma).$$
This is a $\hat{G}$-equivariant vector bundle on $\hat{G}\sigma$,
$$\begin{tikzcd} \hat{G} \times V \times \hat{G}\sigma \arrow[r] \arrow[d] & V \times \hat{G}\sigma \arrow[d] \\ \hat{G} \times \hat{G}\sigma \arrow[r] & \hat{G}\sigma \end{tikzcd}$$
This construction defines a functor
$$\mathrm{Rep}(\hat{G}) \longrightarrow \mathrm{Coh}^{\hat{G}}(\hat{G}\sigma).$$
The upshot is that we can make this definition.

Definition 2.1. We let $\mathrm{Coh}^{\hat{G}}_{\mathrm{fr}}(\hat{G}\sigma)$ be the essential image of this functor. Given $V \in \mathrm{Rep}(\hat{G})$, its image in this category is denoted $\tilde{V}$.

Theorem 2.2 (Thm. 6.0.1 of XZ)**.** *(1) There exists a functor*
$$S : \mathrm{Coh}^{\hat{G}}_{\mathrm{fr}}(\hat{G}\sigma) \longrightarrow P^{\mathrm{Corr}}(\mathrm{Sht}_{\bar{k}})$$
such that the following diagram commutes
$$\begin{tikzcd} \mathrm{Rep}(\hat{G}) \arrow[r, "\mathrm{Sat}"] \arrow[d, "V \mapsto \tilde{V}"'] & P(\mathrm{Hecke}_{\bar{k}}) \arrow[d, "\Phi"] \\ \mathrm{Coh}^{\hat{G}}_{\mathrm{fr}}(\hat{G}\sigma) \arrow[r, "S"'] & P^{\mathrm{Corr}}(\mathrm{Sht}_{\bar{k}}) \end{tikzcd}$$

(2) We have $S(\tilde{1}) = \delta_1$ (which is defined to be the image of IC_0 in $P(\mathrm{Sht}_{\bar{k}})$) and
$$(\mathcal{O}_{\hat{G}\sigma})^{\hat{G}} = \mathrm{End}_{\mathrm{Coh}^{\hat{G}}_{\mathrm{fr}}(\hat{G}\sigma)}(\tilde{1}) \xrightarrow{S} \mathrm{End}(\delta_1) = H_G \otimes \overline{\mathbb{Q}}_\ell$$
(where the right equality was in Rebecca's talk last time) concides with the classical Satake isomorphism.

Here "Sat" is geometric Satake, the left downward arrow is the $V \mapsto \tilde{V}$ we just discussed. Φ is the map from the Hecke stack to the stack of shtukas corresponding to the forgetful functor.

Then it is natural to ask about what morphisms in $\mathrm{Coh}^{\hat{G}}_{\mathrm{fr}}(\hat{G}\sigma)$ are not in $\mathrm{Rep}(\hat{G})$.

- S on objects is $S(\tilde{V}) = \Phi(\mathrm{Sat}(V))$.
- S on morphisms is the question.

For example, what is $\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}}(\tilde{V}_1, \tilde{V}_2)$? Well,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}}(\tilde{V}_1, \tilde{V}_2) &= \mathrm{Hom}_{\mathrm{Coh}}(\tilde{V}_1, \tilde{V}_2)^{\hat{G}} \\ &= \mathrm{Hom}_{\mathcal{O}_{\hat{G}\sigma}}(\mathcal{O}_{\hat{G}\sigma} \otimes V_1, \mathcal{O}_{\hat{G}\sigma} \otimes V_2)^{\hat{G}} \\ &= \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(V_1, \mathcal{O}_{\hat{G}\sigma} \otimes V_2)^{\hat{G}} \\ &= (\mathcal{O}_{\hat{G}\sigma} \otimes V_1^* \otimes V_2)^{\hat{G}}.\end{aligned}$$

Note that the structure sheaf has a Frobenius twist, which thereby gives rise to a Frobenius action on some objects later, which is then important for making an excursion operator.

Remark 2.3. The analogy with the function field case of Lafforgue is that Lafforgue's $f, (\gamma_i)_I$ corresponds to $a \in \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}}(\tilde{V}_1, \tilde{V}_2), \sigma$. This is why we introduced Lafforgue's construction of excursion operators. Maybe one interesting thing to note is that the interesting choices of excursion operators in this context are really the elements of $\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}}(\tilde{V}_1, \tilde{V}_2)$, rather than the Galois element, since we always choose Frobenius.

Next, we need a few definitions in order to move forward.

Definition 2.4.
- If $V \in \mathrm{Rep}(\hat{G})$, then $V \cong \bigoplus V_{\lambda_i}$, and the image under the geometric Satake isomorphism $\mathrm{Sat}(V)$ is supported on $\mathrm{Gr}_V := \bigcup_i \mathrm{Gr}_{\lambda_i}$.
- As in James's talk, we also have
$$\mathrm{Gr}_{V_1|V_2} = \mathrm{Gr}_{V_1} \times_{\mathrm{Gr}} \mathrm{Gr}_{V_2}.$$
- $\mathrm{Hk}_V = [L^+G \backslash \mathrm{Gr}_v]$
- $\mathrm{Hk}_{V_1|V_2} = [L^+G \backslash \mathrm{Gr}_{V_1|V_2}]$

This is a computation I didn't have time to unpack.

Proposition 2.5 (Prop 3.4.4 of XZ)**.** *Geometric Satake descends to an isomorphism of morphisms*

$$\mathrm{Hom}_{\hat{G}}(V_1, V_2) \xrightarrow{\sim} \mathrm{Corr}_{\mathrm{Hk}^0_{V_1|V_2}}(\mathrm{Sat}(V_1), \mathrm{Sat}(V_2))$$

which in turn is isomorphic via Φ *to* $\mathrm{Corr}_{\mathrm{Sht}_{V_1|V_2}}(S(\tilde{V}_1, \tilde{V}_2))$.

Recall from James's talk the "partial Frobenius" map

$$F : \mathrm{Sht}_{\mu_1,\ldots,\mu_t} \xrightarrow{F} \mathrm{Sht}_{\sigma(\mu_t),\mu_1,\ldots,\mu_{t-1}}.$$

given by cyclically permuting once, and then apply Frobenius to one coordinate. On bundles, this amounts to

$$(\mathcal{E}_t \overset{\beta_t}{\dashrightarrow} \cdots \dashrightarrow \mathcal{E}_1 \dashrightarrow {}^\sigma\mathcal{E}_t) \mapsto (\mathcal{E}_{t-1} \dashrightarrow \cdots \dashrightarrow \mathcal{E}_1 \dashrightarrow {}^\sigma\mathcal{E}_t \dashrightarrow {}^\sigma\mathcal{E}_{t-1}).$$

This operation is obviously invertible. So

Proposition 2.6. *This induces a correspondence*

$$\mathbb{D}\Gamma_{F^{-1}} : (\mathrm{Sht}_{\sigma V_2 \boxtimes V_1}, S(\widetilde{\sigma V_2} \boxtimes \widetilde{V_1}) \longrightarrow (\mathrm{Sht}_{V_1 \boxtimes V_2}, S(\widetilde{V_1} \boxtimes \widetilde{V_2}))$$

where if we have $W \in \mathrm{Rep}(\hat{G})$*,*

$$\sigma W \text{ is given by } \hat{G} \xrightarrow{\sigma^{-1}} \hat{G} \longrightarrow \mathrm{GL}(W).$$

To conclude the talk, I'll discuss the construction of the excursion operator. It will be not exactly correct, but then I'll indicate how to fix it.

We use the fact that if a group scheme $\mathcal{G}$ acts on a scheme X, then there is a representation of $\mathcal{G}$ on $\mathcal{O}_X$. This is in general infinite dimensional.

Now recall that

$$\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}}(\tilde{V}_1, \tilde{V}_2) \cong (\mathcal{O}_{\hat{G}\sigma} \otimes V_1^* \otimes V_2)^{\hat{G}}.$$

To fix this, you have to make this infinite-dimensional thing into the right colimit. One picks some finite-dimensional subs, and then proves that it is OK to do that. For now, we'll proceed as if $\mathcal{O}_{\hat{G}\sigma}$ is finite-dimensional.

Firstly, define a $\hat{G}$-equivariant map

$$\hat{G} \times \hat{G} \longrightarrow \hat{G}\sigma, \quad (h_1, h_2) \mapsto \sigma(h_1)^{-1}\sigma(h_2)\sigma,$$

where we have the usual action on the RHS and the LHS action is

$$g \cdot (h_1, h_2) = (h_1\sigma^{-1}(g^{-1}), h_2 g^{-1}).$$

Thuse we have

$$\mathcal{O}_{\hat{G}\sigma} \longrightarrow \sigma\mathcal{O}_{\hat{G}} \otimes \mathcal{O}_{\hat{G}}.$$

This induces

$$d_\sigma : (\mathcal{O}_{\hat{G}\sigma} \otimes V_1^* \otimes V_2)^{\hat{G}} \longrightarrow (\sigma\mathcal{O}_{\hat{G}} \otimes \mathcal{O}_{\hat{G}} \otimes V_1^* \otimes V_2)^{\hat{G}} \cong \mathrm{Hom}_{\hat{G}}(V_1, \sigma\mathcal{O}_{\hat{G}} \otimes \mathcal{O}_{\hat{G}} \otimes V_2).$$

There's a map from point to $\hat{G} \times \hat{G}$ given by the identity element, so we have

$$\mathrm{ev}_{1,1} : \mathcal{O}_{\hat{G}} \times \mathcal{O}_{\hat{G}} \longrightarrow 1.$$

Given $a \in (\mathcal{O}_{\hat{G}\sigma} \otimes V_1^* \otimes V_2)^{\hat{G}}$, we get

$$S_a : S(\tilde{V}_1) \xrightarrow{\mathcal{C}(d_\sigma(a))} S(\widetilde{\sigma\mathcal{O}_{\hat{G}}} \boxtimes (\tilde{V}_2 \otimes \widetilde{\mathcal{O}_{\hat{G}}})) \xrightarrow{\mathbb{D}\Gamma_{F^{-1}}} S((\tilde{V}_2 \otimes \mathcal{O}_{\hat{G}}) \boxtimes \mathcal{O}_{\hat{G}}) \xrightarrow{\mathcal{C}(\mathrm{ev}_{1,1})} S(\tilde{V}_2).$$

where $\mathcal{C} = \Phi \circ \mathrm{Sat}$. Here $\mathcal{C}(d_\sigma(a))$ is to be compared with the creation operator, $\mathbb{D}\Gamma_{F^{-1}}$ with the Galois action, and $\mathcal{C}(\mathrm{ev}_{1,1})$ with the annihilation operator.

Note what we really do is find a W such that a is in the image of the right map in the following diagram, then similarly define a map Ξ_a in the following diagram:

$$\begin{tikzcd}
& \mathrm{Hom}_{\hat{G}}(\sigma W \otimes V_1 \otimes W, V_2) \arrow[dl, "\Xi_a"'] \arrow[dr] & \\
\mathrm{Mor}_{P\mathrm{Corr}}(S(\tilde{V}_1, \tilde{V}_2)) & & (\mathcal{O}_{\hat{G}\sigma} \otimes V_1^* \otimes V_2)^{\hat{G}}.
\end{tikzcd}$$

and then show it's independent of choices.

Proposition 2.7. *This turns S into a functor.*

See the proof in XZ, Lemma 6.2.7.

Next, we want to show that the diagram of Theorem 2.2 commutes. We reproduce it here:

$$\begin{tikzcd} \mathrm{Rep}(\hat{G}) \arrow[r, "\mathrm{Sat}"] \arrow[d, "V \mapsto \tilde{V}"'] & P(\mathrm{Hecke}_{\bar{k}}) \arrow[d, "\Phi"] \\ \mathrm{Coh}^{\hat{G}}_{\mathrm{fr}}(\hat{G}\sigma) \arrow[r, "S"'] & P^{\mathrm{Corr}}(\mathrm{Sht}_{\bar{k}}) \end{tikzcd}$$

The diagram commutes on objects by definition. On morphisms, note that if we choose $W = 1$ in the diagram before Prop 2.7, then we get the down-right direction in the diagram on morphisms. But in this case, note that $\mathrm{ev}_{1,1}$ is the identity, d_σ is the identity, and the partial Frobenius is as well, because they all depend on the representation W being nontrivial. Therefore,

$$S_b = \mathcal{C}(a) = \Phi(\mathrm{Sat}(a))$$