STARK'S CONJECTURE FOR IMAGINARY QUADRATIC FIELDS

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1. Introduction

Today we will discuss the proof of Stark's conjecture in the case where the base field is an imaginary quadratic field. First we will discuss the statement of Stark's conjecture. We're following [Tat84] and [Sta80].

Fix an abelian extension K/F of number fields with Galois group G. Fix a set of places S of F such that

(1) S contains the archimedean places of F,

(2) S contains the places which ramify in K,

(3) S contains at least one place which splits completely in K, and

(4) $|S| \geq 3$ (in fact Stark treats $|S| \geq 2$, but for simplicity, we omit the $|S| = 2$ case, which isn't too much harder).

Last time we discussed the partial L-functions, but we can equivalently talk about the partial zeta functions.

Definition 1.0.1 (Partial Zeta Function). For $\sigma \in G$ define

$$\zeta_S(s, \sigma) := \sum_{(\mathfrak{n}, S)=1, (\mathfrak{n}, K/F)=\sigma} \mathrm{Nm}_{F/\mathbf{Q}}(\mathfrak{n})^{-s}$$

where $(\mathfrak{n}, K/F)$ denotes the Artin symbol.

Let $e = |\mu(K)|$ denote the number of roots of unity in K.

Conjecture 1.0.2 (Stark). *Fix a place $v \in S$ which splits completely in F and choose a place $w \mid v$ in L. There exists a unit $u \in K^\times$ such that for all $\sigma \in G$,*

(1) $|u|_{w'} = 1$ for all places $w' \nmid v$ in K.

(2) $K(u^{1/e})/F$ is abelian.

(3) $\zeta'(0, \sigma) = -\frac{1}{e} \log |\sigma(u)|_w$

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Call the conjecture $\mathrm{St}(K/F, S)$.

Stark originally stated this conjecture in [Sta80]. In [Sta80] he gave a proof of the theorem for $F = \mathbf{Q}$ and $F/\mathbf{Q}$ an imaginary quadratic field. Today we present his proof of the latter (as written in [Tat84]).

To do this, we need Kronecker's second limit formula, a reciprocity law from the theory of complex multiplication for elliptic curves and modular functions, and a few auxiliary reduction steps and ingredients.

2. Proof

So we wish to prove the following.

Theorem 2.0.1. *If F is an imaginary quadratic field, then $\mathrm{St}(K/F, S)$ is true for any abelian K/F.*

Proof. We proceed in the following steps.

Step 1: First, we reduce to the case where K is a ray class field. In general one first proves ([Tat84, Proposition IV.3.5]) that if $F \subseteq L' \subseteq L$, then

$$\mathrm{St}(L/F, S) \implies \mathrm{St}(L'/F, S)$$

Then if we let

$$\mathfrak{m} = \left(\prod_{\mathfrak{p} \in S_{\mathrm{fin}}} \mathfrak{p} \right)^n$$

for large enough n, we will have $K \subseteq F(\mathfrak{m})$, where $F(\mathfrak{m})$ is the ray class field associated to $\mathfrak{m}$. Furthermore, taking n large enough also shows that

$$w(\mathfrak{m}) := |\{\zeta \in \mu(F) \mid \zeta \equiv 1 \mod \mathfrak{m}\}| = 1$$

One can check that S still satisfies the conditions of the conjecture, so it suffices to prove the conjecture when $K = F(\mathfrak{m})$ and $w(\mathfrak{m}) = 1$.

Step 2: We want to use Kronecker's second limit formula. For that we need some auxiliary functions. Let L be a $\mathbf{Z}$-lattice $L \subseteq \mathbf{C}$. Then let

$$\sigma(z, L) = z \prod_{w \in L \setminus \{0\}} \left(1 - \frac{z}{w}\right) e^{z/w + \frac{1}{2}(z/w)^2}$$

be the Weierstrass σ function (it's entire with simple poles at the lattice points) and let

$$\zeta(z, L) = \frac{\sigma'(z, L)}{\sigma(z, L)}$$

be its logarithmetic derivative, the Weierstrass ζ function: its derivative is $-\wp$. Note if we write

$$\zeta(z + w, L) = \zeta(z, L) + \eta(w, L)$$

then η does not depend on z, and defines an $\mathbf{R}$-linear function on $\mathbf{C}$. Finally we let

$$G(z, L) = e^{-6\eta(z,L)} \sigma(z, L)^{12} \Delta(L)$$

where Δ is the discriminant of the elliptic curve associated to L.

Finally, for each $c \in G$ (now $G = \mathrm{Cl}(\mathfrak{m})$ is the ray class group associated with $\mathfrak{m}$) we define

$$g_{\mathfrak{m}}(c) := G(1, \mathfrak{m}\mathfrak{o}^{-1})^f$$

where f is the smallest positive integer in $\mathbf{Z} \cap \mathfrak{m}$, and where $\mathfrak{o}$ is a representative integral ideal for c. This (called the Siegel-Ramachandra invariant) is the number we will need for the second Kronecker limit formula.

Step 3: Now we note the following facts about $g_{\mathfrak{m}}(c)$, which we will discuss a bit more later when we discuss the theory of complex multiplication.

(a) $g_{\mathfrak{m}}(c)$ does not depend on the choice of $\mathfrak{o}$.

(b) $g_{\mathfrak{m}}(c) \in F(\mathfrak{m})$.

(c) If $\sigma_c \in \mathrm{Gal}(F(\mathfrak{m})/F)$ is the element associated to c under the Artin reciprocity map, then
$$g_{\mathfrak{m}}(c) = \sigma_c(g_{\mathfrak{m}}(1))$$

(d) $|g_{\mathfrak{m}}(c)|_{w'} = 1$ for $w' \nmid v$ if $\mathfrak{m}$ has at least 2 distinct prime divisors: since we assumed $|S| > 2$, this is true.

(e) The extension $F(g_{\mathfrak{m}}(c)^{1/12f})/F$ is abelian.

Step 4: With these properties in hand, we finally state Kronecker's second limit formula:
$$\zeta_S'(0,\sigma) = -\frac{1}{12fw(\mathfrak{m})}\log|g_{\mathfrak{m}}(c_\sigma)| = -\frac{1}{12f}\log|\sigma(g_{\mathfrak{m}}(1))|$$
Then Stark proves ([Sta80, Lemma 6]) a short lemma which says that $g_{\mathfrak{m}}(1)^e$ admits a $(12f)$th root of unity which is actually contained in $F(\mathfrak{m})$: call this u. Finally we see that
$$\zeta_S'(0,\sigma) = -\frac{1}{12fe}\log|\sigma(g_{\mathfrak{m}}(1)^e)| = -\frac{1}{e}\log|\sigma(u)|$$
as desired. Note $F(\mathfrak{m})(u^{1/e}) = F(\mathfrak{m})(g_{\mathfrak{m}}(c)^{1/12f})$ which by (e) is an abelian extension of F, and (d) gives us the desired norm property away from v.

□

So the only thing we need to address is Step 3: how do we prove these things? In this talk, we will only address (b) and (e), via the theory of complex multiplication.

3. Complex Multiplication

Theorem 3.0.1. *Let E be an elliptic curve over $\mathbf{C}$ which has CM by F. Fix $s \in \mathbf{A}_F^\times$ an idele and $\sigma \in \mathrm{Aut}(\mathbf{C}/F)$, such that $\mathrm{rec}(s) = \sigma|_{F^{\mathrm{ab}}}$, where rec denotes the global Artin reciprocity map. Find a lattice $\Lambda_E \subseteq F$ and an isomorphism $\theta : \mathbf{C}/\Lambda_E \xrightarrow{\sim} E$. Then there is an isomorphism $\theta_s : \mathbf{C}/s^{-1}\Lambda_E \xrightarrow{\sim} \sigma(E)$ and the following diagram commutes:*

$$\begin{array}{ccc} F/\Lambda_E & \xrightarrow{\theta} & E \\ \downarrow{\scriptstyle s^{-1}} & & \downarrow{\scriptstyle \sigma} \\ F/s^{-1}\Lambda_E & \xrightarrow{\theta_s} & \sigma(E) \end{array}$$

Here $s^{-1}\Lambda_E$ denotes the unique lattice such that $s_p^{-1}(\Lambda_E \otimes_{\mathbf{Z}} \mathbf{Z}_p) = (s^{-1}\Lambda_E) \otimes_{\mathbf{Z}} \mathbf{Z}_p$ for all finite primes p.

Since F/Λ_E is the subgroup of torsion points on E, this gives us a way to do explicit class field theory using torsion points on elliptic curves. In particular this allows us to prove

Theorem 3.0.2. *If F is imaginary quadratic and E is an elliptic curve such that $\mathrm{End}(E) \cong \mathscr{O}_F$ and $I \subseteq \mathscr{O}_F$ is an ideal, then*
$$F(j(E), h(E[I]))$$
is the ray class field for F associated to I, where h is a certain function $E \to E/\mathrm{Aut}(E) \cong \mathbf{P}^1$ (if we consider E as $y^2 = x^3 + Ax + B$ then h is just projection onto the x-coordinate, as long as $j(E) \neq 0, 1728$). Here $E[I]$ is the I-torsion of C, which makes sense since $I \to \mathscr{O}_F \xrightarrow{\sim} \mathrm{End}(E)$.

In fact, $j(E)$ and $h(t)$ for each torsion point $t \in E$ generate F^{ab} over F.

So this is the classical theory for elliptic curves, but for our purposes we need the following fact:

Proposition 3.0.3. *Let $\mathcal{F}_{\mathbf{C}}$ be the modular function field, i.e. the field of meromorphic complex functions on the upper half plane which are invariant under the action of the principal congruence subgroup Γ_N for some N. Then $\mathcal{F}_{\mathbf{C}} = \mathbf{C}(j, f_{r,s})_{r,s\in\mathbf{Z}}$ for some particular functions $f_{r,s}$, and where j is the j-invariant. Let*

$$\mathcal{F} = \mathbf{Q}(j, f_{r,s})_{r,s\in\mathbf{Z}}$$

Then if we fix $z_0 \in F \cap \mathcal{H}$ (think of this as giving E), we get

$$F^{\mathrm{ab}} = \{f(z_0) \mid f \in \mathcal{F}\}$$

The idea is that $f_{0,0}(z_0)$ gives the x-coordinate of a torsion point on E and r, s shift the torsion point around the lattice.

Finally we need the following theorem, which is a corollary of a reciprocity law in this modular function setting. The level of the modular function is analogous to the conductor of the ray class field.

Theorem 3.0.4 ([Sta80] Theorem 3)**.** *If $f \in \mathcal{F}$ is a modular function of level Γ_N and $\theta \in F \cap \mathcal{H}$, then*

$$f(\theta) \in F(N\mathscr{O}_F)$$

4. Properties of $g_{\mathfrak{m}}(c)$

Finally, the point is that $g_{\mathfrak{m}}(c)$ and $g_{\mathfrak{m}}(c)^{1/12f}$ are both values of certain modular functions at certain $\theta \in F \cap \mathcal{H}$.

Lemma 4.0.1. *$G(1, L)^f$ is modular of level Γ_f.*

So $g_{\mathfrak{m}}(c)$ lands in $F(\mathfrak{m})$ (take $L = \mathfrak{m}\mathfrak{o}^{-1}$ and use the previous theorem), proving property (b) of Step 3 in the proof.

Furthermore, Stark shows that $g_{\mathfrak{m}}(c)^{1/12f}$ is also a special value of a modular function (this follows from an alternative definition of $g_{\mathfrak{m}}(c)$ as a $12f$-th power of a modular function of level $12f^2$, which I won't write down), and thus lands in an abelian extension of F, proving property (e) of Step 3 in the proof. Actually $g_{\mathfrak{m}}(c)$ is a modular function of level f, and is thus contained in $F(\mathfrak{m})$, as desired. We omit the proofs of the other properties.

References

[Sta80] Harold M. Stark. L-functions at $s = 1$. IV. First derivatives at $s = 0$. *Adv. in Math.*, 35(3):197–235, 1980.

[Tat84] John Tate. *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$*, volume 47 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1984. Lecture notes edited by Dominique Bernardi and Norbert Schappacher.