Introduction to Topology

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1 Week 1

1.1 Sets

We will use naive set theory, and take the notion of a *set* as a primitive; for us, this is any definable collection. We will use curly braces {} to describe a set, and will use : to separate an object from the conditions it satisfies. So if X is a set, we can say

$$\{x \in X : x \text{ satisfies some condition}\}$$

We have the notion of a *subset* $Y \subset X$. This is a collection of elements of X.

Definition 1.1.1. We let $\varnothing$ denote the empty set, which contains no elements. If X is a set, we let $\mathcal{P}(X)$ denote its *powerset*, i.e.

$$\mathcal{P}(X) = \{Y \subset X\},$$

the set of all subsets of X. Note $X \in \mathcal{P}(X)$ and $\varnothing \in \mathcal{P}(X)$. □

Definition 1.1.2. If X is a finite set, let $|X|$ denote its size. □

- $\mathbb{Z}$ denotes the integers
- $\mathbb{N}$ denotes the natural numbers (for me, $0 \notin \mathbb{N}$)
- $\mathbb{Q}$ denotes the rational numbers
- $\mathbb{R}$ denotes the real numbers

Definition 1.1.3. If $X \subset Z$ and $Y \subset Z$ then we define:

- the *union* of X and Y is
$$X \cup Y = \{z \in Z : z \in X \text{ or } z \in Y\}$$
- the *intersection* of X and Y is
$$X \cap Y = \{z \in Z : z \in X \text{ and } z \in Y\}.$$
- If $X \cap Y = \varnothing$ then we sometimes write $X \sqcup Y := X \cup Y$, and we refer to this as a *disjoint union* [1].
- The *complement* of X in Z is
$$X^c = \{z \in Z : z \notin X\}$$
- The *difference* between X and Y denoted $X - Y$ or $X \setminus Y$ is
$$X \cap Y^c = \{z \in Z : z \in X \text{ and } z \notin Y\}$$

Definition 1.1.4. If X and Y are sets, then a *function* or *map* $f : X \to Y$ is an assignment $f(x) \in Y$ to every $x \in X$.

[1] if X and Y are not common subsets of a bigger set, we may also define $X \sqcup Y$ to be some set whose elements are the elements of X and the elements of Y

- $f : X \hookrightarrow Y$ is *injective* or *an injection* if $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

  Note that if $f : X \to Y$ is injective and X and Y are both finite, then $|X| \leq |Y|$.
- $f : X \twoheadrightarrow Y$ is *surjective* or *a surjection* if for every $y \in Y$ there exists some $x \in X$ such that $f(x) = y$.
- $f : X \xrightarrow{\sim} Y$ is *bijective* or *a bijection* if it is both injective and surjective.
- The *image* of f, denoted $\mathrm{im}(f)$ or $f(X)$ is

$$f(X) = \{y \in Y : \text{there exists } x \in X \text{ such that } f(x) = y\}$$

- If $V \subset Y$ is a subset, the *preimage* of V under f, denoted $f^{-1}(V)$, is

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

- We let $\mathrm{Fun}(X, Y)$ denote the set of all functions $f : X \to Y$.
- If $f : X \to Y$ and $g : Y \to Z$ are two functions, we let $g \circ f : X \to Z$ denote their composition.

□

Definition 1.1.5. If X and Y are two sets, the *cartesian product* $X \times Y$ is the set of pairs of elements (x, y) with $x \in X$ and $y \in Y$.

A subset $R \subset X \times X$ is called an *equivalence relation* if (we write $x_1 \sim x_2$ if $(x_1, x_2) \in R$)

- (reflexive) $x \sim x$ for all $x \in X$
- (symmetric) $x_1 \sim x_2$ if and only if $x_2 \sim x_1$
- (transitive) $x_1 \sim x_2$ and $x_2 \sim x_3$ implies $x_1 \sim x_3$

An equivalence relation is the same as a *partition* of X, which is a decomposition of X into disjoint subsets. We write this as

$$X = \bigsqcup_{i \in I} X_i$$

where I is some (possibly infinite) indexing set. □

Definition 1.1.6. An infinite set X is *countable* if there exists a bijection

$$X \xrightarrow{\sim} \mathbb{N}$$

and *uncountable* if not. Note $\mathbb{R}$ is uncountable. □

2 Week 2

This week we will study *metric spaces*. Our eventual goal is to study *topological spaces*, but as with anything in math, it's best to study simpler examples first before trying to understand general theory.

2.1 Metric spaces

You're probably used to the real numbers $\mathbb{R}$, and n-dimensional Euclidean space $\mathbb{R}^n$ from your calculus or real analysis course. You might have talked about length, area, volume, measure, convergence, continuity, differentiation and integration, etc. If you look at the definitions, you'll realize that the thing underlying all of these notions is the *absolute value*. This is a function

$$|\cdot| : \mathbb{R} \to \mathbb{R}_{\geq 0}$$

and it measures *size*. Here $\mathbb{R}_{\geq 0}$ is the set of nonnegative real numbers. In higher dimension, we take

$$|\cdot| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$$

sending $|(x_1, \ldots, x_n)| = \sqrt{\sum_{i=1}^n x_i^2}$.

From a certain perspective, a more fundamental notion is that of *distance*. The usual Euclidean distance is

$$\begin{aligned} d : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto d(x, y) = |x - y| \end{aligned}$$

where $|x - y|$ is the Euclidean distance.

Let's generalize this definition and see what happens.

Definition 2.1.1. Fix a set M. A *metric* on M is a function

$$d : M \times M \to \mathbb{R}_{\geq 0}$$

satisfying

1. (positivity) $d(x, y) = 0$ if and only if $x = y$
2. (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in M$
3. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$

The pair (M, d) is called a *metric space*. If d is clear from context we often say M is a metric space. □

So a metric space is "a set with a good notion of distance".

Example 2.1.2.

1. You can check that the Euclidean metrics satisfy the 3 defining properties.
2. Taxicab metric.
3. If M is any set, then consider the *discrete metric*

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

4. Fix (M, d) a metric space and fix $[a, b] \subseteq \mathbb{R}$ and consider $C([a, b], M)$, the space of continuous functions. Define a function $d_{\max}$ by taking

$$d_{\max}(f, g) = \max_{x \in [a,b]} d(f(x), g(x)) \text{ for all } f, g : [a, b] \to M$$

This is well-defined; it turns out that the maximum will always be attained, since f and g are assumed to be continuous (we will see why later). On the homework you will show that this is a metric when $M = \mathbb{R}$. Note that if $a = b$, this space is the same as M.

2.2 Convergence

One of the reasons we define metric spaces is to talk about *convergence*. Convergence is defined in basically the same way as Euclidean space, but let's spell it out and try some examples.

Definition 2.2.1. Suppose (M, d) is a metric space, and suppose $(x_i) = (x_1, x_2, \dots)$ is a sequence in M. We say that (x_i) *converges* to $x \in M$ if for every $\epsilon > 0$ there exists a $N_\epsilon > 0$ such that if $n > N_\epsilon$ then $d(x, x_n) < \epsilon$. In this case we say that x is the *limit* of $x_1, \dots, x_n$, and we write $x_i \to 0$. □

The idea of course is that as you go further down the sequence, you get closer and closer to x.

Example 2.2.2.

1. If $M = \mathbb{R}$ and $x_i = 2^{-i}$, then $x_i \to 0$.
2. Not every sequence converges, e.g. $x_i = i$.
3. If M is a set and d_{disc} is the discrete metric, then can you characterize the convergent sequences?

Before moving on let's slightly rephrase this definition.

Definition 2.2.3. If (M, d) is a metric space, then the *open ball of radius* $r > 0$ around a point $x \in M$ is

$$B_r(x) = \{y \in M : d(x, y) < r\}.$$

Similarly, the *closed ball of radius* $r > 0$ around a point $x \in M$ is

$$B_r^\bullet(x) = \{y \in M : d(x, y) \leq r\}.$$

So convergence can be characterized as saying that for every $\epsilon > 0$ there exists N_ϵ such that if $n > N_\epsilon$ then $x_n \in B_\epsilon(x)$.

2.3 Continuity

Sets are interesting, but functions between sets are more interesting. So what should a function between metric spaces be?

If (M_1, d_1) and (M_2, d_2) are two metric spaces, then we want a notion of a "function"

$$f : (M_1, d_1) \to (M_2, d_2)$$

A metric space is "a set with extra structure". A general paradigm says that a map between them should be a map between the sets respecting the structure. In this case:

Definition 2.3.1. If $f : M_1 \to M_2$ is a function, then we say it is *continuous* if for every $x \in M_1$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $d_1(x, y) < \delta$ then $d_2(f(x), f(y)) < \epsilon$. In other words, if $y \in B_\delta(x)$ then $f(y) \in B_\epsilon(x)$. In other words still,

$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x))).$$

□

In the homework, you will show that this is the same as saying that if

$$x_i \to x \text{ then } f(x_i) \to x$$

for every converging sequence $x_i \in M_1$.

Example 2.3.2.

1. For functions $\mathbb{R} \to \mathbb{R}$ with the Euclidean metric, this recovers the usual definition of continuous function you're used to.
2. If M_1 has the discrete metric, then what are the continuous functions $f : M_1 \to M_2$?
3. Is the composition of two continuous functions continuous? Can we see this both using the ϵ-δ definition and the sequence definition?

3 Week 3

Today we'll finish up continuity and then discuss *compactness*, an important property of a metric space which admits an important generalization to the topological context.

3.1 Subspaces

One thing I didn't mention but probably should have is that if (M, d) is a metric space and $N \subsetneq M$ is a subset, then N is still a metric space; you just define the metric to be the restriction of d to N. So subsets of metric spaces are naturally metric spaces.

3.2 More on Continuity

Recall from last week that we saw that a map $f : M_1 \to M_2$ of metric spaces is continuous if for all $x \in M_1$ and all $\epsilon > 0$, there exists δ such that

$$B_\delta(x) \subset f^{-1}(B_\epsilon(f(x))).$$

Recall also from last week that we defined the notion of an open set. We said that if M is a metric space then $U \subseteq M$ is open if for every $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.

Next week we're going to consider spaces with notions of openness which have nothing to do with balls of a certain radius. These spaces should still admit continuous functions between them though, so we need a way to say this without the words (and letters) "ball", "radius", "ϵ", "δ", etc. As you will see on the homework, it turns out we can combine these two statements in the most natural way: f is continuous if and only if

$$U \text{ is open } \implies f^{-1}(U) \text{ is open }.$$

Notice how succinct this statement is; it shows us that any time we have a good notion of "open-ness", we have a good notion of continuity. Stay tuned.

3.3 Compactness

Recall from calculus that a continuous function $f : [a, b] \to \mathbb{R}$ always attains its maximum and minimum; in other words, there exists $x_{\min}, x_{\max} \in [a, b]$ such that $f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for all $x \in [a, b]$.

But why is this true? Let's try to generalize this a bit and see. First of all, note that $[a, b]$ is closed because its complement is $(-\infty, a) \sqcup (b, \infty)$. So you might ask, it this true for any closed set?

The answer is no; e.g. $\mathbb{R}$ itself is closed, but a function $\mathbb{R} \to \mathbb{R}$ does not necessarily attain its maximum or minimum. It turns out that we need the further condition that the closed set is *bounded*.

Definition 3.3.1. If (M, d) is a metric space, then a subset $N \subseteq M$ is *bounded* if there exists $r > 0$ such that $d(x, y) < r$ for all $x, y \in N$.

Remark 3.3.2. What does a bounded closed set in $\mathbb{R}$ look like, anyway? In the Week 2 group problems, one question was to describe all of the open sets in $\mathbb{R}$. The answer (I'll leave you to think about why this is true) is that it's the disjoint union of countably many open intervals, i.e. things of the form $(-\infty, a)$, (a, b) for $a < b$, (b, ∞), or $\mathbb{R}$ itself. The complements of such disjoint unions are therefore (certain, not all) disjoint unions of closed sets of the form $(-\infty, a]$, $[a, b]$, $[b, \infty)$, where now we're allowing $a \leq b$. So clearly the closed and bounded things are the closed sets which are contained within some big enough open ball.

But these can look weird; e.g. the Cantor set is closed and bounded! Why? It's clear that it's bounded, and the point is that its complement is the countable disjoint union of open intervals (remember you construct it by removing middle thirds). □

Theorem 3.3.3. *If $Z \subseteq \mathbb{R}$ is closed and bounded, Z is nonempty, and $f : Z \to \mathbb{R}$ is continuous then f attains its maximum and minimum.*

On the other hand, this is somewhat special to $\mathbb{R}$, as evidenced by the following example.

Example 3.3.4. Every subset of a discrete metric space is closed. Which are the bounded ones? All of them! But if M is a discrete metric space, a (continuous) function $M \to \mathbb{R}$ only necessarily attains its maximum and minimum if M is finite.

So if we want to generalize this notion of "closed and bounded", we need a different way of seeing it. The answer goes as follows.

Definition 3.3.5. A metric space (M, d) is *sequentially compact* if every sequence $x_1, x_2, \dots$ has a convergent subsequence.

Theorem 3.3.6 (Bolzano–Weierstrass)**.** *A subset of* $\mathbb{R}$ *(with the Euclidean metric) is sequentially compact if and only if it is closed and bounded.*

Proof. This is a topology course and not a real analysis course, so I'm not going to give a detailed proof of this, but I'll sketch it.

1. First we show that a subset Z inside a metric space M is closed if and only if the limit of every sequence $(x_i) \subseteq Z$ is contained in Z (you will show this on your homework)
2. Use (1) to show that the statement of the Theorem is equivalent to showing that every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
3. Show that every sequence in $\mathbb{R}$ has a monotone subsequence, i.e. one of the form

$$x_1 \geq x_2 \geq \cdots \text{ or } x_1 \leq x_2 \leq \cdots$$

4. Conclude by using the *monotone convergence theorem*: every bounded monotone sequence converges.

□

Theorem 3.3.7. *If* $f : M_1 \to M_2$ *is a continuous map of metric spaces and* M_1 *is sequentially compact, then* $f(M_1)$ *is sequentially compact.*

Proof. If $(y_i) \subset M_2$ is a sequence in $f(M_1)$, then we can write $y_i = f(x_i)$ for some $x_i \in M_1$ for each $i = 1, 2, \dots$. But M_1 is sequentially compact, so the sequence $(x_i) \subset M_1$ has a convergent subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$. But f is continuous, so $f(x_{n_1}), f(x_{n_2}), f(x_{n_3}), \dots$ converges, and thus $f(M_1)$ is also sequentially compact. □

Corollary 3.3.8. *If* $f : M \to \mathbb{R}$ *is a map of metric spaces and* M *is sequentially compact and nonempty, then* f *has a maximum and minimum, i.e. there exists* $x_{\min}$ *and* $x_{\max}$ *such that* $f(x_{\min}) \leq f(x) \leq f(x_{\max})$ *for all* $x \in M$.

Proof. Just take $M_1 = M$ and $M_2 = \mathbb{R}$ and then apply the Bolzano–Weierstrass theorem. □

So this is all well and good for metric spaces, but we're going to be interested in a more general class of spaces (topological spaces) shortly. So we should reformulate this notion in a way that doesn't mention convergence (which implicitly involves ϵ and δ). This will be the topic of your group work this week, where you will explore *compactness*.

4 Week 4

So far we've been busy studying metric spaces. We defined them, gave some examples, stated some basic notions related to them like continuity, (sequential) compactness, etc. We also spent time carefully re-working these notions so that they don't look like they have anything to do with metric spaces.

So let's take this to its logical end. What's the minimal amount of structure needed to capture all this?

4.1 Topological spaces

Definition 4.1.1. If X is a set, a subset $\mathcal{T} \subset \mathcal{P}(X)$ is called a *topology on* X (or just a *topology* if X is clear from context) if

1. $\varnothing, X \in \mathcal{T}$,
2. $\mathcal{T}$ is closed under arbitrary unions, i.e. if $\{U_i\}_{i \in I} \subset \mathcal{T}$ is a collection of subsets of X then

$$\bigcup_{i \in I} U_i \in \mathcal{T}$$

   and

3. $\mathcal{T}$ is closed under finite intersections, i.e. for $U_1, \ldots, U_n \in \mathcal{T}$,

$$U_1 \cap \cdots \cap U_n \in \mathcal{T}.$$

We say that $(X, \mathcal{T})$ is a *topological space*. The elements of $\mathcal{T}$ are called *open subsets of* X. A subset $Z \subseteq X$ is called *closed* (with respect to $\mathcal{T}$) if $X^c \in \mathcal{T}$. □

Remark 4.1.2. Strictly speaking, the fact that $\varnothing, X \in \mathcal{T}$ follows from (2) and (3), if we accept that the intersection of an empty collection of subsets of X is X, and the union of an empty collection of subsets of X is $\varnothing$; if this is confusing, just ignore it. □

Remark 4.1.3. de Morgan's laws imply that you could also define a topology by specifying the closed sets; i.e. some collection of subsets closed under finite union and arbitrary intersection. □

If (M, d) is a metric space, recall that we said that $U \subset M$ is *open* if for every $x \in U$ there exists an open ball $B_r(x)$ contained in U.

Proposition 4.1.4. *The set $\mathcal{T}_d \subset \mathcal{P}(M)$ of open sets in M is a topology on M.*[2]

Proof. You proved this in the group work from Week 2! □

Example 4.1.5. Here are some basic examples of topological spaces.

1. If X is a set, take $\mathcal{T} = \mathcal{P}(X)$. This is called the *discrete topology*. You can check that if (M, d) is a discrete metric space then $\mathcal{T}_d$ (see Proposition 4.1.4) is the discrete topology.
2. If X is a set, take the set $\mathcal{T} = \{\varnothing, X\}$. This is the spiritual opposite of the discrete topology, so we call it the *indiscrete topology on* X.
3. The topology on $\mathbb{R}^n$ generated by the Euclidean metric is called the *Euclidean topology*.
4. If $S = \{0, 1\}$ then define $\mathcal{T}_S = \{\varnothing, \{0\}, \{0, 1\}\}$. This topological space is called the Sierpiński space, and you will study it on Homework 4.

Definition 4.1.6. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X with $\mathcal{T}_1 \subset \mathcal{T}_2$, we say that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$, and that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$. Not every pair of topologies is comparable, but on every set X the discrete topology is always the finest topology, and the indiscrete topology is always the coarsest topology.

[2] The subscript d in $\mathcal{T}_d$ refers to the metric d on M.

Remember that when we defined the topology for a metric space we started with $B_r(x)$ and then used them to define a notion of open set. This can be done more generally, as follows.

Definition 4.1.7. If X is a set, then a *base of open sets on* X is a subset $\mathcal{B} \subset \mathcal{P}(X)$ such that:

1. for all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$, and
2. if $B_1, B_2 \in \mathcal{B}$ and $x \cap B_1, B_2$ then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and

$$B_3 \subseteq B_1 \cap B_2$$

□

Example 4.1.8.

1. For example, if M is a metric space, then the set $\mathcal{B}_d$ of all open balls in M is a base for the metric topology $\mathcal{T}_d$.
2. If X has the discrete topology, then the set $\mathcal{B} = \{\{x\} : x \in X\}$ is a base.

Definition 4.1.9. If $\mathcal{B}$ is a base of open sets in X, then it *generates* a topology $\mathcal{T}_{\mathcal{B}}$ on X by taking the collection of unions of open sets in $\mathcal{B}$ (along with the empty set $\varnothing$). You will explain this more thoroughly on the homework.

If $(X, \mathcal{T})$ is a topological space and $\mathcal{B} \subseteq \mathcal{P}(X)$, we say that $\mathcal{B}$ is a *base for* $\mathcal{T}$ if $\mathcal{B}$ generates $\mathcal{T}$. □

Often we will *start* with a topology and ask for a basis of open sets which generates it.

4.2 Continuity

Recall that we proved that a function $f : M_1 \to M_2$ between two metric spaces is continuous if and only if $f^{-1}(U)$ is open whenever U is open. This works in general.

Definition 4.2.1. A function $X \to Y$ between two topological spaces is *continuous* if

$$V \subseteq Y \text{ open} \implies f^{-1}(V) \subseteq X \text{ open.}$$

Example 4.2.2.

1. Just as for metric spaces, if X has the discrete topology and Y is any topological space, any function $f : X \to Y$ is continuous. If Y has the indiscrete topology and X is any topological space, then any function $f : X \to Y$ is continuous.
2. If X and Y are metric spaces, this is equivalent to metric space continuity.
3. If $\mathcal{B}$ is a base of open sets for a topology on X then you can check that $f : X \to Y$ is continuous iff $f^{-1}(B)$ is open for all $B \in \mathcal{B}$. This is because preimages preserve unions, as you showed on Homework 3.

For sets, we have the notion of bijective map. If $f : A \to B$ is a bijection of sets, then for all intents and purposes, A and B are the "same"; in other words, one is a relabelling of the other. For topological spaces, we have the following.

Definition 4.2.3. If $f : X \to Y$ is a continuous function between topological spaces, then we say f is a *homeomorphism* if

1. f is bijective (so f^{-1} is uniquely defined) and
2. f^{-1} is continuous.

Example 4.2.4.

1. As a dummy example, take any set X with at least two elements and take the identity map $X \to X$, but consider the first X to have the discrete topology and the second one to have the indiscrete topology. Then the identity is a continuous bijection, but is not a homeomorphism.
2. There are a number of continuous bijections $\mathbb{R} \to \mathbb{R}^n$ called "space filling-curves". But these are not homeomorphisms. We won't discuss them in this class.
3. The logistic function $f : \mathbb{R} \to (0, 1)$ given by $f(x) = (1 + e^{-x})^{-1}$ is a homeomorphism. In fact, any two open intervals are homeomorphic, as are any two closed intervals.
4. Let S^1 denote the unit circle in the complex plane with the Euclidean topology (i.e., view $S^1 \subseteq \mathbb{C}$ as a metric space with the Euclidean metric). The map

$$\begin{aligned} [0,1) &\to S^1 \\ x &\mapsto e^{2\pi i x} \end{aligned}$$

is continuous and bijective, but is not a homeomorphism.

5 Week 5

This week we will discuss how to construct new topologies from old ones.

5.1 Set-theoretic constructions

Remember that we have some constructions for sets.

1. For instance, if X is a subset, there are subsets $Y \subseteq X$.
2. If $(X_i)_{i \in I}$ is a collection of sets, we can form the Cartesian product $\prod_{i \in I} X_i$ and the disjoint union $\bigsqcup_{i \in I} X_i$.
3. If X is a set and $\sim$ is an equivalence relation on X then we can define the "quotient space" $Y = X/\sim$, where we identify equivalent objects.

We understand these operations. Great. But now what if the sets we started with above also carry topologies? Then our constructions should also naturally carry a topology somehow. The problem is that there are many choices; for instance, we could give all of our constructions the discrete or indiscrete topology, but this is a sort of unnatural thing to do. So what do we do?

5.2 Natural maps

The solution is to notice that in all of our constructions, there's an extra piece of information we get for free that we haven't taken into account; natural *maps*.

1. If $Y \subseteq X$, then there is a natural inclusion map $Y \to X$; this is pretty straightforward.
2. If $(X_i)_{i \in I}$ is a collection of sets, an element of $\prod_{i \in I} X_i$ is a choice of one element $x_i \in X_i$ for each $i \in I$. So for every $j \in I$, we get a natural *projection* map
$$\pi_j : \prod_{i \in I} X_i \to X_j$$
sending $(x_i)_{i \in I}$ to x_j.
3. In the same vein, an element of $\bigsqcup_{i \in I} X_i$ is a choice of an element $x_j \in X_j$ for a *single* $j \in I$. So in this case we don't have projections, but instead we have natural *inclusion* maps going in the opposite direction:
$$\iota_j : X_j \to \bigsqcup_{i \in I} X_i$$
4. Finally, given a set X and an equivalence relation, we get a natural *quotient map*
$$q : X \to X/\sim$$
sending x to its equivalence class. Note q is surjective, and in fact given any surjective map $X \twoheadrightarrow Y$ there is an equivalence relation $\sim$ on X making $Y = X/\sim$ (just say that $x \sim x'$ if $f(x) = f(x')$).

So for example, in this sense the product of two sets X and Y should be thought of as the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_X} & X \\ \big\downarrow{\scriptstyle \pi_Y} & & \\ Y & & \end{array}$$

or in other words, the set $X \times Y$ equipped with these two maps.

Now if X and Y have topologies, then from this perspective it becomes a bit clearer how to define a topology on $X \times Y$. Why? Well, the projection maps π_X and π_Y should "respect the topology"; or in other words, they should be continuous maps. But this is all we will impose! So if $U \subseteq X$ and $V \subseteq Y$ are open sets, we want $\pi_X^{-1}(U) = U \times Y$ and $\pi_Y^{-1}(V) = X \times V$ to be open. Note that if $X \times Y$ is very coarse, then there's less of a chance that π_X, π_Y will be continuous. But if $X \times Y$ is very fine, then there are a lot of open sets that don't need to be open.

So in conclusion we define the topology on $X \times Y$ to be the *coarsest* topology making π_X, π_Y continuous. More concretely, we view

$$\left\{\pi_X^{-1}(U) : U \subseteq X \text{ open}\right\} \cup \left\{\pi_Y^{-1}(V) : V \subseteq Y \text{ open}\right\}$$

as a *subbase*, and then generate a topology by first closing under finite intersection and then arbitrary union. That's the product topology (for two sets)!

For instance, you can think about the topology on $\mathbb{R} \times \mathbb{R}$; it's the same as the Euclidean topology.

5.3 Topologies

Let's propagate this idea to the other examples.

1. If X is a topological space and $Y \subseteq X$ is a subset, then we want $Y \hookrightarrow X$ to be continuous, i.e. if U is open in X then $U \cap Y$ should be open in Y. You can check that

$$\{U \cap Y : U \subseteq X \text{ open}\} \subseteq \mathcal{P}(Y)$$

is a topology, called the *subspace topology.*

2. As discussed previously, if $(X_i)_{i \in I}$ is a collection of sets, then the *product topology* is the coarsest topology on $\prod_{i \in I} X_i$ such that every projection map $\pi_j : \prod_{i \in I} X_i \to X_j$ is continuous; in other words, the topology generated (as a subbase) by sets of the form $\pi_j^{-1}(U)$ where $U \subseteq X_j$ is open.

3. For disjoint unions, we have the opposite description; the *disjoint union topology* on $\bigsqcup_{i \in I} X_i$ is the *finest* topology making the inclusion maps ι_j continuous for all j. A base of open sets for this topology is just

$$\bigsqcup_{i \in I} \mathcal{T}_{X_i}$$

where $\mathcal{T}_{X_i}$ is the topology on X_i.

4. Similarly, in the case of quotients we are also trying to define a topology on a *target* of a natural map. So we define the *quotient topology* on $X/\sim$ be the *finest* topology making $X \to X/\sim$ continuous. So a set of equivalence classes in X is open if the union of all of those equivalence classes is open in X.

Let's see a few examples.

1. If X is a metric space and $Y \subseteq X$ is a subset, then it's not so hard to check that the metric topology on Y is the subspace topology. So for instance the subspace topology on $S^1 \subseteq \mathbb{R}^2$ is just the metric topology.

2. The subspace $\mathbb{Q} \subseteq \mathbb{R}$ is not discrete, but it is *totally disconnected*; every open is a disjoint union of two opens.

3. If $X_1, \dots, X_n$ are discrete spaces, then $X_1 \times \cdots \times X_n$ is again discrete. But things don't work so well for infinite products. For instance,

$$\prod_{\mathbb{N}} \{0, 1\}$$

is not discrete, and turns out to be homeomorphic to the Cantor set!

4. Here's an example of a quotient topology. Take $X = \mathbb{R}$ with the Euclidean topology and define an equivalence relation where $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Then as a set, the quotient space has equivalence class representatives given by $[0, 1)$. But on the other hand, $1 \sim 0$, so you're effectively gluing 0 to 1, and you get a circle. The topology is exactly the metric topology again on the circle.

6 Week 6

Remember in week 3 that there was a problem which asked you to try to cover $B_1((0,0)) \subseteq \mathbb{R}^2$ with disjoint open balls, but I then said that you couldn't do this because it was "connected". This week will study connectedness in more detail, and discuss some of the subtleties that come along with it.

6.1 Connectedness

Definition 6.1.1. If X is a topological space, we say that X is *connected* if $\varnothing, X$ are the only two subsets of X which are both open and closed. Equivalently, you cannot write $X = U \sqcup V$ where $U, V \subseteq X$ are disjoint nonempty open subsets. □

Connectedness is a homeomorphism invariant; in other words, if $X \cong Y$ then X is connected if and only if Y is connected.

Example 6.1.2. Here are some examples to consider.

1. A discrete space is connected if and only if it consists of one point, while all indiscrete spaces are automatically connected.
2. $\mathbb{R}$ is connected. To see this, note that if $U \sqcup V = \mathbb{R}$ both nonempty then pick $x \in U$ and $y \in V$ and consider the interval $[x, y]$. Let $z = \inf\{a \in [x, y] : a \in V\}$. Then z is a limit point for V, so $z \in V$ since V is closed. But since $z \in V$ we have $x < z$ and thus z must be a limit for point U as well, so $z \in U$ since U is closed as well. Contradiction. This implies that (a, b) is connected, and in fact also that $[a, b]$ is connected.
3. $\mathbb{Q}$ is not connected (as a subspace of $\mathbb{R}$); this is because $(-\infty, r) \sqcup (r, \infty)$ disconnects $\mathbb{Q}$ for any irrational $r \in \mathbb{R}$.
4. The union of the line $y = 0$ and $y = 1/x$ (for $x > 0$) is not connected.

Lemma 6.1.3. *If $X = U \sqcup V$ for U, V nonempty disjoint opens and $A \subseteq X$ is a connected subset, then either $A \subseteq U$ or $A \subseteq V$.*

Proof. Since U and V are open in X we have $U \cap A$ and $V \cap A$ open in A. Since A is connected and $A = (U \cap A) \cup (V \cap A)$ one of $U \cap A$ or $U \cap V$ is empty. □

Proposition 6.1.4. *If X is a topological space and Y_i is a collection of connected subspaces of X which contain a common point, then $Y = \bigcup_i Y_i$ is connected.*

Proof. Suppose $Y = U \cup V$ nonempty disjoint opens. By assumption there is a point $p \in Y$ which is contained in each Y_i, so wlog suppose $p \in U$. Since Y_i is connected, it is either contained in U or V, but $p \in U$, so $Y_i \subseteq U$ for all i, so V is empty, contradiction. □

In view of this proposition, we can make the following definition.

Definition 6.1.5. If X is a topological space, then a *connected component* of X is a maximal connected subset of X.

Note that if $x \in X$ then the "connected component of X containing x" is well-defined. Why? Well, just take the union of all connected subsets of X containing x; this will still be connected.

On the homework you will show that if $C \subseteq X$ is connected, then $\overline{C}$ is connected as well. Therefore, every connected set is closed. If a space has only finitely many connected components then they're all open as well. But in general this is not true; take $\mathbb{Q}$ for instance. The connected components are singleton sets, but these are not open, because every open in $\mathbb{Q}$ contains infinitely many rational numbers! This is an example of a more general phenomenon:

Definition 6.1.6. A topological space X is called *totally disconnected* if the connected components are the singletons $\{x\}$.

So for instance, a discrete space is totally disconnected, but so is something like $\mathbb{Q} \subseteq \mathbb{R}$. Also, the Cantor set turns out to be totally disconnected; this is not so hard to see from the description of "removing thirds".

Let's talk about one related notion.

Definition 6.1.7. A topological space X is called *path-connected* if for every $x, y \in X$ there exists a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. This is another way of saying that "any two points in X can be connected by a path".

Proposition 6.1.8. *A path-connected space is connected.*

Proof. This is easy: if X is path-connected and $X = U \sqcup V$, then pick $x \in U$ and $y \in V$, and pick a path $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. But then $f^{-1}(U) \cup f^{-1}(V) = [a, b]$ and we've therefore disconnected the interval, contradiction. □

Path-connectedness for subspaces of Euclidean space is usually easy to detect. On the other hand, there are some pathological examples.

Example 6.1.9 (Topologist's sine curve)**.** Let $X = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, y) : y \in \mathbb{R}\}$. Each piece in the union is connected and even path-connected. The union turns out to be connected as well. But this is not path-connected! □

7 Week 7

In Week 3 we talked about *sequential compactness* of a metric space. Recall that a metric space is sequentially compact if every sequence admits a convergent subsequence. Today we'll generalize this notion to an arbitrary topological space and discuss some properties.

7.1 Compactness

Definition 7.1.1. A topological space X is *compact* if every open cover of X has a finite subcover. In other words, for any collection $\{U_i\}_{i\in I}$ of open subsets of X such that

$$X = \bigcup_{i\in I} U_i$$

there exist $i_1, \ldots, i_n \in I$ such that

$$X = U_{i_1} \cup \cdots \cup U_{i_n}.$$

□

Remark 7.1.2. In the group work in Week 3 you showed that if M is a compact metric space then it is sequentially compact; in fact for a metric space these turn out to be equivalent, but since this is not a real analysis class, we won't go into the (somewhat complicated) proof of this fact.

Example 7.1.3. Let's do some examples.

1. If $Z \subseteq \mathbb{R}^n$ is a subset of Euclidean space, then it is a metric space so compactness is the same as sequential compactness. But we saw that for Euclidean space this is the same as being closed and *bounded*; this was the Bolzano–Weierstrass theorem, which we sketched a proof of.
2. If X is a discrete topological space, then $\{\{x\}\}_{x\in X}$ is an open cover. But if you remove any sets from this cover then it's no longer a cover. Therefore, a discrete topological space is compact if and only if it contains finitely many points.
3. Indiscrete spaces are always compact.
4. Familiar topological spaces are compact; e.g. the circle S^1, the torus $T = S^1 \times S^1$, the interval $[0, 1]$, a closed ball $B_r^\bullet(x)$ in Euclidean space.
5. A *cofinite topology* is always compact. Recall that is X is a set then we define this topology by specifying that all finite subsets of X are closed (so their complements are open). Why? If $\{U_i\}_{i\in I}$ is an open cover and pick a random member U_i in the collection, then U_i^c is finite, so if we write $U_i^c = \{x_1, \ldots, x_n\}$, you can find $i_1, \ldots, i_n \in I$ such that $x_k \in U_{i_k}$ for $k = 1, \ldots, n$. Then $U_i \cup U_{i_1} \cup \cdots U_{i_n} = X$.

Compact sets satisfy some useful properties. One such property is the following:

Proposition 7.1.4. *If $f : X \to Y$ is a continuous map of topological spaces and X is compact, then $f(X)$ is compact (as a subspace of Y with the subspace topology).*

Proof. Since $f(X)$ has the subspace topology from Y, without loss of generality we may assume that f is surjective, i.e. $f(X) = Y$. Then if $\{V_i\}_{i\in I}$ is an open cover of Y, we immediately see that $\left\{f^{-1}(V_i)\right\}_{i\in I}$ is an open cover of X. Since X is compact is must have a finite subcover. So we can write

$$X = f^{-1}(V_{i_1}) \cup \cdots \cup f^{-1}(V_{i_n})$$

for some $i_1, \ldots, i_n \in I$. So we want to show that

$$Y = V_{i_1} \cup \cdots \cup V_{i_n}$$

But if $y \in Y$, then f is surjective so there is some $x \in X$ such that $f(x) = y$. But then $x \in f^{-1}(V_{i_k})$ for some k, so we conclude because

$$y = f(x) \in V_{i_k}.$$

□

This is great, because for instance this implies that every function $f : X \to \mathbb{R}$ from a nonempty compact space X achieves its maximum and minimum. While the condition of compactness is somewhat opaque on first glance, we'll see that one can construct new compact spaces from old ones, so it's actually a fairly flexible notion.

For this, we need to take a brief digression and talk about Hausdorff spaces.

7.2 Hausdorff spaces

Definition 7.2.1. A topological space X is *Hausdorff* if points in X can be separated by disjoint opens. In other words, if $x_1, x_2 \in X$ are two distinct points, then there exist open subsets $U_1, U_2 \subseteq X$ such that $x_1 \in U_1$, $x_2 \in U_2$, and $U_1 \cap U_2 = \varnothing$. □

Example 7.2.2. Here are some examples.

1. Any metric space is Hausdorff: if $x \neq y$ then $d(x, y) > 0$ and the triangle inequality implies that
$$B_{d(x,y)}(x) \cap B_{d(x,y)}(y) = \varnothing.$$
2. Any discrete space is clearly Hausdorff. Indiscrete spaces are clearly not (if they have more than one point).
3. If X is an infinite set, then the cofinite topology is very much *not* Hausdorff, because the intersection of any two nonempty open sets is infinite.

On the homework, you will see that these play very nicely with respect to compactness:

Proposition 7.2.3. *Every closed subspace of a compact space X is compact. If X is Hausdorff, every compact subspace of X is closed.*

Proof. Homework. □

For instance, this implies:

Proposition 7.2.4. *If $f : X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Proof. It suffices to show that f takes open sets to open sets, or equivalently that f takes closed sets to closed sets. But if $Z \subseteq X$ is closed then it is compact by Proposition 7.2.3, so $f(Z)$ is compact in the Hausdorff space Y and is therefore closed, again by Proposition 7.2.3. □

8 Week 8

We will start talking about homotopy theory this week.

So far in the course we have studied topological spaces, and declared that two of them are the same if they are *homeomorphic*. Although homeomorphic spaces may look slightly different from one another (for example, $\mathbb{R}$ is homeomorphic to $(0,1)$), being homeomorphic means that any topological property of one space is the same as the same property for the other space.

8.1 Homotopy

Homotopy is a weaker notion of "sameness", where two topological spaces are not the same if they are in bicontinuous bijection with each other, but instead if one can be continuously deformed into another without "tearing" or "breaking" the space. For example $\mathbb{R}^n \cong^*$; this sentence is read "$\mathbb{R}^n$ is homotopy equivalent to a point". Intuitively you can imagine taking Euclidean space and contracting it inwards to a point.

Another example: the letter Q and the letter O in the previous group work were not homeomorphic. But they *are* homotopy equivalent, because you can take the tail of the Q and just continuously shrink it until it becomes part of the O.

A non-example: the circle S^1 is not homotopy equivalent to $*$. As we will see, this is because if you want to shrink the circle to a point, you have to "break it".

Definition 8.1.1. If X and Y are two topological spaces and $f, g : X \to Y$ are two continuous maps, then a *homotopy* from f to g is a continuous map

$$H : X \times [0,1] \to Y$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. If there exists a homotopy from f to g, then we say that f and g are *homotopic*, and we write $f \simeq g$. □

We will justify the apparent symmetry in the definition in a moment, but for now let's do some examples.

Example 8.1.2. If X is any topological space, then any two maps $f, g : X \to \mathbb{R}^n$ are homotopic. To see why, define the function

$$\begin{aligned} H : X \times [0,1] &\to \mathbb{R}^n \\ (x,t) &\mapsto (1-t)f(x) + tg(x) \end{aligned}$$

On the homework you will show that this is continuous, and therefore defines a homotopy.

Example 8.1.3. If Y is path-connected, then any two constant maps are homotopic. To see this, note that if $f_1(x) = y_1$ and $f_2(x) = y_2$, then simply pick a path $p : [0,1] \to Y$ such that $p(0) = y_1$ and $p(1) = y_2$ and then take

$$H(x,t) = p(t).$$

This is continuous because it is the composition of the projection $X \times [0,1] \to [0,1]$ with $p : [0,1] \to Y$.

A map which is homotopic to a constant map is called *null-homotopic*.

Proposition 8.1.4. *Homotopy from f to g is an equivalence relation.*

Proof. First of all, $f \simeq f$. To see this, note that you can just take $H(x,t) = f(x)$ for all x and t.

Secondly, if $f \simeq g$ then there exists $H : X \times [0,1] \to Y$ such that $H(\cdot,0) = f$ and $H(\cdot,1) = g$. Note that the map $[0,1] \to [0,1]$ defined by $t \mapsto 1-t$ is continuous, so we can define

$$H_s : X \times [0,1] \xrightarrow{(x,t)\mapsto(x,1-t)} X \times [0,1] \xrightarrow{H} Y.$$

You can then easily check that $H_s(\cdot,0) = g$ and $H_s(\cdot,1) = f$.

Finally, if $f \simeq g$ and $g \simeq h$, we have $H_1 : X \times [0,1] \to Y$ and $H_2 : X \times [0,1] \to Y$ such that $H_1(\cdot, 0) = f$, $H_1(\cdot, 1) = g$, $H_2(\cdot, 0) = g$ and $H_2(\cdot, 1) = h$. But then

$$H_3(x,t) = \begin{cases} H_1(x, 2t) & 0 \le t \le 1/2 \\ H_2(x, 2t-1) & 1/2 \le t \le 1 \end{cases}$$

is a homotopy between H_1 and H_2; on the homework you will check that H_3 is still continuous. □

In general, you say that $f : X \to Y$ is an *isomorphism* whenever there exists a left-and-right inverse. This applies to basically any mathematical object.

But here is a weaker version of this.

Definition 8.1.5. A (continuous) map $f : X \to Y$ of topological spaces is a *homotopy equivalence* if there exists $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X \text{ and } f \circ g \simeq \mathrm{id}_Y .$$

If there exists a homotopy equivalence between X and Y, then we say they are *homotopy equivalent* and write $X \cong^{\mathrm{h}} Y$.

Remark 8.1.6. This is an equivalence relation (check this for yourself).

Example 8.1.7. Let's go back to the Euclidean space example. There is a unique continuous map $\mathbb{R}^n \to *$, and we can define $* \to \mathbb{R}^n$ by just sending $*$ to 0. The compositions are $\mathbb{R}^n \xrightarrow{x \mapsto 0} \mathbb{R}^n$ and the unique map $* \to *$. Note $* \to *$ is just id_*, so in particular it is homotopic to itself. But we showed that any two maps into $\mathbb{R}^n$ are homotopic, so $\mathbb{R}^n \xrightarrow{x \mapsto 0}$ is homotopic to the identity. Therefore, $\mathbb{R}^n \cong^{\mathrm{h}} *$.

Example 8.1.8. Here's another example. Consider $\mathbb{R}^2 - \{(0,0)\}$ and let $S^1 \subseteq \mathbb{R}^2$ denote the unit circle. Consider the map

$$\begin{aligned} f : \mathbb{R}^2 - \{(0,0)\} &\to S^1 \\ x &\mapsto \frac{x}{||x||} \end{aligned}$$

where $||x||$ denotes the distance from x to $(0,0)$. Also consider the map

$$g : S^1 \to \mathbb{R}^2 - \{(0,0)\}$$

just given by inclusion. Now, note that $f \circ g = \mathrm{id}_{S^1}$. On the other hand, what is $g \circ f$? It's "projection onto the unit circle". We want to find a homotopy $g \circ f \sim \mathrm{id}_{\mathbb{R}^2 - \{(0,0)\}}$. But we can do this by taking

$$\begin{aligned} H : \mathbb{R}^2 - \{(0,0)\} \times [0,1] &\to \mathbb{R}^2 - \{(0,0)\} \\ (x,t) &\mapsto (1-t)x + t\left(\frac{x}{||x||}\right). \end{aligned}$$

So we conclude that $\mathbb{R}^2 - \{(0,0)\} \cong^{\mathrm{h}} S^1$.

Remark 8.1.9. A similar argument shows that $\mathbb{R}^{n+1} - \{(0,0)\} \cong^{\mathrm{h}} S^n$ where S^n is the n-sphere defined by

$$S^n = \left\{x \in \mathbb{R}^{n+1} : ||x|| = 1\right\}.$$

9 Week 9

This week we will use the notion of homotopy to define an invariant of a topological space that measures the number of "distinct loops".

9.1 Path homotopy

In the previous lecture, the notion of "homotopy between maps" was defined as a fairly flexible notion. Now let's restrict it a bit, in order to study our spaces a bit more finely.

Definition 9.1.1. If $f, g : I \to X$ are two continuous maps from the interval I to a topological space X satisfying $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$, then a *path-homotopy* between f and g is a homotopy H from f to g satisfying $H(0,t) = x_0$ and $H(1,t) = x_1$. In other words, the beginning and end of the paths are preserved throughout the homotopy. □

Instead of trying to write out some examples, let's immediately specialize to the case that we care about.

Definition 9.1.2. Fix a point $x_0 \in X$. A *loop based at* x_0 in X is a continuous map $\ell : I \to X$ satisfying $f(0) = f(1) = x$. □

Recall that we showed in Proposition 8.1.4 that homotopy of maps is an equivalence relation; it's fairly straightforward to check that path homotopy between loops is also an equivalence relation, which we denote $\simeq^{\mathrm{ph}}$.

Definition 9.1.3. The *fundamental group (based at x_0)* of a topological space X (and the point x_0) is

$$\pi_1(X, x_0) = \{\ell : I \to X \text{ a loop based at } x_0\} \,/ \simeq^{\mathrm{ph}}$$

In other words "based loops taken up to path homotopy". If $\ell : I \to X$ is a loop, we write $[\ell]$ for its equivalence class in $\pi_1(X, x_0)$. □

Example 9.1.4. Here are two examples.

1. Note that $\pi_1(\mathbb{R}^n, x_0)$ is a singleton set, since every loop can be contracted to a single point because there is "enough space". In fact, you can check that the function in Example 8.1.2 is a path-homotopy.
2. We will show that $\pi_1(S^1, x_0) = \mathbb{Z}$; the point is that each equivalence class is determined by "how many times you wind around the circle".

9.2 Group structure on $\pi_1(X, x_0)$

So far we've defined $\pi_1(X, x_0)$ as a set, but we've called it a "group". So we need to understand what the group structure is. For this, we need a group operation and an identity element, and then we need to check some axioms.

Definition 9.2.1. If $f, g : I \to Y$ are two paths such that $f(1) = g(0)$, then we define

$$(f * g)(x) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1 \end{cases}$$

which we call the "product of f and g", or the "concatenation of f and g".

In particular, if f, g are both loops, then we can define $f * g$. We want to show that this induces a map

$$- * - : \pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0).$$

But for this we need to know:

Lemma 9.2.2. *Suppose $f, f' : I \to X$ are two paths with the same start and endpoint, and $g, g' : I \to X$ are two paths with the same start and endpoint, and assume $f(1) = g(0)$. If $f \simeq^{\mathrm{ph}} f'$ and $g \simeq^{\mathrm{ph}} g'$ then $f * g \simeq^{\mathrm{ph}} f' * g'$.*

Proof. If H_f is a path homotopy between f and f' and H_g is a path homotopy between g and g', then we will show that

$$H(x,t) = \begin{cases} H_f(2x,t) & 0 \leq x \leq 1/2 \\ H_g(2x-1,t) & 1/2 \leq x \leq 1 \end{cases}$$

One can then check directly that this defines a path-homotopy $f * g \simeq^{\mathrm{ph}} f' * g'$. One can check that H is continuous. □

If all the path are loops then if $[f] = [f']$ and $[g] = [g']$ the above lemma implies that $[f * g] = [f' * g']$.

Now we need an identity element: it is (the equivalence class of) the constant loop $c : I \to X$ sending $t \mapsto x_0$. You will show on the homework that $[\ell * c] = [c * \ell] = \ell$ for all loops $\ell \in \pi_1(X, x_0)$. You will also show on the homework that $*$ is associative.

Finally, the inverse map is $\ell \mapsto \ell^{-1}$, where

$$\ell^{-1}(t) = \ell(1-t).$$

You will also show on the homework that this is a valid inversion operation.

So in conclusion, $\pi_1(X, x_0)$ is actually a group! If $x_0, x_1 \in X$ can be connected by a path in X, i.e. if there exists a continuous map $f : I \to X$ sending $0 \mapsto x_0$ and $1 \mapsto x_1$, then you will also show on the homework that $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic groups; recall that this means that there exists a bijection $\varphi : \pi_1(X, x_0) \to \pi_1(X, x_1)$ such that $\varphi(x * y) = \varphi(x) * \varphi(y)$.

Definition 9.2.3. A path-connected topological space X is *simply-connected* if $\pi_1(X, x_0) = 1$ (i.e. is the trivial group).

So for instance $\mathbb{R}^n$ (or more generally, any convex subset of $\mathbb{R}^n$, as you will demonstrate on the homework) is simply-connected, while S^1 is not.

One important point to make is that π_1 should be a *functor*, meaning that any continuous map of topological spaces $f : X \to Y$ and any choice of $x_0 \in X$ with image $y_0 = f(x_0)$ should give rise to a group homomorphism

$$f_* = \pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

So how do you define this map? Simple: if $\ell : I \to X$ is a loop, just define the image under $\pi_1(f)$ to be $f \circ \ell : I \to X \to Y$. This is well-defined because of the following lemma.

Lemma 9.2.4. *If $f, g : I \to X$ are two paths satisfying $f(0) = g(0)$ and $f(1) = g(1)$, $k : X \to X'$ is another continuous map and H is a homotopy between f and g, then $k \circ H$ is a homotopy between $k \circ f$ and $k \circ g$.*

Proof. This is a homework exercise. □

Since f_* is basically just composition, it follows immediately that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two maps then $(g \circ f)_* = g_* \circ f_*$, and that if $(\mathrm{id}_X)_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is just the identity map of groups. Therefore:

Corollary 9.2.5. *If $f : X \to Y$ is a homeomorphism, then f_* is an isomorphism.*

Proof. This is just the fact that $f_* \circ f_*^{-1} = (\mathrm{id}_Y)_* = \mathrm{id}_{\pi_1(Y, y_0)}$ and $f_*^{-1} \circ f_* = (\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X, x_0)}$. □

In fact this remains true when f is a homotopy equivalence.

10 Week 10

We have noted repeatedly that $\pi_1(S^1, *) = \mathbb{Z}$. The homework from last week allows you conclude that

$$\pi_1(T, *) = \mathbb{Z}^2$$

where T is the torus. Today we'll introduce a tool that lets you prove such things.

10.1 Covering spaces

Definition 10.1.1. If $p : E \to B$ is a continuous surjective map of topological spaces, we say that p is a *covering map* (and that E is a *covering space* for B) if for every $b \in B$ there exists an open set $U \subseteq B$ containing b such that $p^{-1}(U) = \bigsqcup_i V_i$ in such a way that

1. V_i is open and
2. the restriction $p : p^{-1}(U) \to U$ maps each V_i homeomorphically onto U.

□

This is a bit wordy, but it basically means that locally around each point in the base B, the map p looks like a stack of copies of B lying over it. Note that by definition if $b \in B$ then $p^{-1}(b)$ must be discrete.

You can construct a trivial example. Take X any discrete topological space, and consider the map $X \times B \to B$. This is clearly a covering space, but it's not very interesting.

To get more interesting examples, you should assume that E is path-connected, which rules out the above sort of example. In fact, we make the following definition.

Definition 10.1.2. If E is simply-connected and $p : E \to B$ is a covering map, then we say that E is a *universal cover* of B.

Lemma 10.1.3. *The map*

$$\begin{aligned} p : \mathbb{R} &\to S^1 \\ x &\mapsto (\cos 2\pi x, \sin 2\pi x) \end{aligned}$$

is a covering map.

Proof. This follows from basic properties of trigonometric functions. We'll content ourselves with a picture to see how this works. □

On the homework you will prove some other basic properties of covering spaces.

10.2 Lifting

We are going to identify the fundamental group with the fiber over a point of a universal cover. This will be achieved by lifting paths to the universal cover, and then lifting homotopies between them.

Definition 10.2.1. If $p : E \to B$ is any map and $f : X \to B$ is any map, a *lifting* of f to E is a map $\widetilde{f} : X \to E$ such that $\widetilde{f} \circ p = f$.

Lemma 10.2.2. *If $p : E \to B$ is a covering map, fix a point $b_0 \in B$ and a lift $e_0 \mapsto b_0$. Then any map $f : [0,1] \to B$ satisfying $f(0) = b_0$ has a unique lift $\widetilde{f} : X \to E$ satisfying $\widetilde{f}(0) = e_0$.*

Proof. If U_α is an open cover of B such that $p|_{U_\alpha}$ is a stack of pancakes, then by compactness one can show that we can find $0 = s_0 < \ldots < s_n = 1$ such that for all i there exists α such that $f([s_j, s_{j+1}]) \subseteq U_\alpha$.

First define $\widetilde{f}(0) = e_0$. Note that $f([s_0, s_1]) \subseteq U_\alpha$ for some α, so if V_i is the pancake homeomorphic to U_α in E, then we can take the map

$$[s_0, s_1] \xrightarrow{f} U_\alpha \xrightarrow{\text{“}p^{-1}\text{”}} V_i \hookrightarrow E$$

This is continuous by construction.

Now we just repeat this argument to construct $f|_{[s_j,s_{j+1}]}$. By the pasting lemma, f will be continuous. Moreover, the map is defined uniquely at each step by taking connectedness into account. $\square$

In particular, suppose $\ell : [0,1] \to B$ is a loop. Then there is a unique map $\widetilde{\ell} : [0,1] \to E$ lifting ℓ but crucially, *it is no longer necessarily a loop.* For instance, think about $\mathbb{R} \to S^1$, and take the loop that just goes around the circle once.

Now if $b \in B$, then in this way we almost see that we get a map

$$\pi_1(B, b_0) \to p^{-1}(b_0)$$

by taking $[\ell] \mapsto \widetilde{\ell}(1)$. But we still need to show that this is well-defined, and it turns out to be.

Proposition 10.2.3. *If $p : E \to B$ is a covering map with $p(e_0) = b_0$ and $F : I \times I \to B$ satisfies $F(0,0) = b_0$, then there is a unique lifting $\widetilde{F}$ of F to E satisfying $\widetilde{F}(0,0) = e_0$. If F is a path-homotopy, then $\widetilde{F}$ is a path-homotopy.*

Proof. For lack of time, we will not give a full proof, but content ourselves with a sketch; first, lift the left and bottom side of the square using the interval case. Then subdivide the square into a grid of rectangles in such a way that the image of each rectangle is contained in an open set of B whose preimage under p is a stack of pancakes. Then lift each rectangle using basically the same argument as before.

If F is a path-homotopy, then the bottom of the square maps to b_0 under F. The top of the square maps to a single point as well, so call it b_1. This implies that $\widetilde{F}$ takes the bottom edge to $p^{-1}(b_0)$ and the top edge to $p^{-1}(b_1)$. But both of fibers are discrete and $[0,1]$ is connected, so $\widetilde{F}$ must be contant on top and bottom edges. $\square$

Corollary 10.2.4. *If $f, g : I \to B$ are two paths from b_0 to b_1, let $\widetilde{f}$ and $\widetilde{g}$ denote their lifts to E. If f and g are path-homotopic, then so are $\widetilde{f}$ and $\widetilde{g}$, and they must end at the same point.*

Proof. Exercise: follows from the previous proposition. $\square$

So we get a well-defined map

$$\pi_1(B, b_0) \to p^{-1}(b_0).$$

Note that this map, strictly speaking, depends on the choice of $e_0 \in E$.

Proposition 10.2.5. *If E is path-connected then above map is surjective. If E is a universal cover, it is bijective.*

Proof. If $e \in p^{-1}(b_0)$ then pick a path $\widetilde{f} : I \to E$ from e_0 to e. Then $f = p \circ \widetilde{f}$ defines a loop in B based at b_0, and by definition it maps to e.

Now, if f and g are two loops in B based at x_0, then suppose that $\widetilde{f}(1) = \widetilde{g}(1)$. If E is simply-connected, then there is a path homotopy $\widetilde{F}$ between $\widetilde{f}$ and $\widetilde{g}$. But then $p \circ \widetilde{F}$ is a path-homotopy between f and g, so we're done. $\square$

Corollary 10.2.6.

$$\pi_1(S^1, *) \cong \mathbb{Z}$$

Proof. Take $e_0 = 0 \in \mathbb{R}$ and note that $\mathbb{R} \to S^1$ is a universal cover. It is also a homomorphism (just consider what happens if you concatenate two loops). $\square$

11 Week 11

We have seen that $\pi_1(S^1, *) \cong \mathbb{Z}$. What about other spheres? For example, what about S^2? One can intuitively see that any loop on the 2-sphere can be contracted to a point. But what about in higher dimensions?

11.1 Seifert-van Kampen

One can compute examples of this phenomenon using the following theorem, which is a corollary of the Seifert-van Kampen theorem.

Theorem 11.1.1 (Seifert-van Kampen)**.** *If X is a path-connected topological space and $U, V \subseteq X$ are two open subsets such that U and V are simply-connected and $U \cap V$ is path-connected and nonempty, then X is simply connected.*

More generally, as long as $U \cap V$ is path-connected, if we pick a point $x_0 \in U \cap V$ then $\pi_1(X, x_0)$ is generated by the images of $\pi_1(U, x_0) \to \pi_1(X, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$.

Note that if $X = S^1$ then the above theorem tells us nothing, because any two proper open subsets of S^1 have non-connected intersection.

But what about larger dimensional spheres?

Corollary 11.1.2. *If $n > 1$ then $\pi_1(S^n, *) = 0$.*

Proof. Note that if you remove a point from S^n you get $\mathbb{R}^n$; one can show this by using a higher-dimensional analogue of the projection argument as for the 2-sphere. So take two points $a, b \in S^n$ which are not equal to $*$ and remove them individually to get open sets $U_a, U_b \subseteq S^n$, both homeomorphic to $\mathbb{R}^n$. Their intersection $U_a \cap U_b$ is homeomorphic to $\mathbb{R}^n \setminus \{\bullet\}$.

But now note that $\mathbb{R}^n$ is simply-connected and $\mathbb{R}^n \setminus \{\bullet\}$ is path-connected, so by Theorem 11.1.1 S^n is simply-connected. □

So the fundamental group, while very useful and extremely interesting, can't distinguish any of the spheres except for S^1! It can't even distinguish S^n from Euclidean space.

This might suggest to you that S^n and $\mathbb{R}^n$ are homotopy equivalent. This is in fact not the case, so let's try to understand why.

11.2 Higher homotopy groups

Recall that the fundamental group was defined to be the group of path-homotopy equivalence classes of loops. We can generalize this to "higher dimensions" as follows.

Definition 11.2.1. If $n > 0$, let ∂I^n denote the boundary of the n-cube.

$$\pi_n(X, x_0) = \{f : I^n \to X : f(\partial I^n) = \{x_0\}\} / \simeq^{rh}$$

where $\simeq^{rh}$ is *relative homotopy equivalence*, which is to say that two maps $f, g : I^n \to X$ satisfying $f(\partial I^n) = g(\partial I^n) = \{x_0\}$ are equivalent if there exists a *relative homotopy*, i.e.

$$H : I^n \times I \to X$$

satisfying

$$H(x, 0) = f(x), \qquad H(x, 1) = g(x), \qquad H(\partial I^n \times I) = \{x_0\}\,.$$

□

Remark 11.2.2. Note that if you take I^n and collapse ∂I^n to a point, then you obtain S^n. So we can also describe:

$$\pi_n(X, x_0) = \{f : S^n \to X : f(*) = x_0\} / \simeq^{\text{ph}}$$

where $*$ denotes a fixed point on the n-sphere. Defining it this way lets us naturally extend the definition to $n = 0$:

$$\pi_0(X, x_0) = \left\{f : S^0 \to X : f(1) = x_0\right\} / \simeq^{\text{ph}}$$

Note here we're viewing S^0 as two points labeled 1 and -1, and you're requiring 1 to be sent to x_0; but -1 can go anywhere! Note there is a path-homotopy between two maps $f, g : (S^0, 1) \to (X, x_0)$ if and only if there's a path between $f(-1)$ and $g(-1)$, so therefore $\pi_0(X, x_0)$ is in bijection with the set of path-connected components of X. □

If $n > 0$ we define a group structure on $\pi_n(X, x_0)$ by shrinking the cubes in half along one axis and then sticking them together.

It is fairly straightforward to show that this operation is commutative and associative, but there is another way that makes it also proves that it doesn't matter which axis you shrink along.

Proposition 11.2.3 (Eckmann–Hilton). *Suppose X is a set equipped with two functions $\circ, \bullet : X \times X \to X$. Suppose*

1. *$\circ$ and $\bullet$ are both unital, meaning that there exist identity elements $1_\circ, 1_\bullet \in X$ such that $1_\circ \circ x = x \circ 1_\circ = x$ and $1_\bullet \bullet x = x \bullet 1_\bullet = x$ for all $x \in X$.*
2. *$(a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d)$ for all $a, b, c, d \in X$.*

Then $\circ = \bullet$ and $1_\circ = 1_\bullet$ and both are commutative and associative.

Proof. Homework. □

Note that shrinking along an axis and adding commutes with shrinking along any other, so we get the group structure essentially for free.

11.3 Homotopy groups of spheres

So what are the homotopy groups of S^n? Perhaps surprisingly, this is a *wide* open question. But at least some facts are known.

- If $0 < i < n$ then
$$\pi_i(S^n) = 0.$$
This is because one can show that any map $f : (I^i, \partial I^i) \to (S^n, *)$ is relative homotopy equivalent to another f which is not surjective. There is really something to prove here, because there are very-difficult-to-visualize space-filling curves.

  But if the map is not surjective, then it lands in $S^n - \{\bullet\} \cong \mathbb{R}^n$ which is contractible.
- If $i = n$ it turns out that
$$\pi_n(S^n) \cong \mathbb{Z}$$
and one sees this by studying the *degree* of a map $S^n \to S^n$, which is a concept from differential topology which we won't discuss in detail. One needs to study the *homology* groups to give an explanation of where this comes from.
- If $i > n$ then things get more complicated. For instance, $\pi_3(S^2) \cong \mathbb{Z}$, and a generator is given by a map called the *Hopf fibration*, which is difficult to visualize, but there are some good YouTube videos which display it well.

Theorem 11.3.1. *If* $i < 2n - 1$ *then there is an isomorphism*

$$\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$$

These are called the stable homotopy groups of spheres.

These have not been computed in full generality, but they are known to be finite abelian groups when $i \neq n$.

12 Week 12

Recall that a covering space was defined in such a way that we could *lift* maps from the interval and homotopies between them. In fact, on the homework we saw that covering spaces satisfy a lifting property for spaces which are path-connected, locally path-connected, and simply-connected.

In topology this kind of lifting property turns out to be extremely useful and fundamental, so let's explore a more general example satisfying a more general lifting property.

12.1 Fiber bundle

Definition 12.1.1. A *fiber bundle* is a continuous surjection $p : E \to B$ of topological spaces along with a space F satisfying the following condition, called *local triviality*; B admits an open cover $\mathcal{U}$ such that for every $U \in \mathcal{U}$ there exists a homeomorphism $\varphi : p^{-1}(U) \xrightarrow{\sim} U \times F$ such that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad\varphi\quad} & U \times F \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle \pi_U} & \\ & U & \end{array}$$

commutes, where π_U is the natural projection $U \times F \to U$ sending $(u, f) \mapsto u$. □

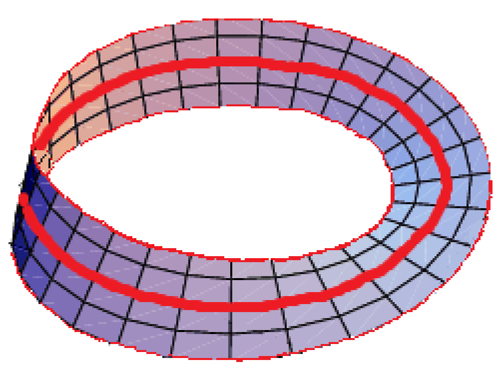
Lemma 12.1.2. *If $p : E \to B$ is a fiber bundle with fiber F, then the preimage of any point $b \in B$ is homeomorphic to F.*

Proof. Any $b \in B$ is contained in some U on which $p|_U = \pi_U$ where $\pi_U : U \times F \to U$ is the projection. The fiber over a point of π_U is clearly homeomorphic to F. □

Remark 12.1.3. Note that this definition strictly generalizes the notion of a covering spaces. You can check that a covering map is the same thing as a fiber bundle with discrete fiber F.

Example 12.1.4. Here are some fiber bundles.

1. The trivial bundle $p : B \times F \to B$ is a fiber bundle.
2. Any nontrivial covering map is a nontrivial fiber bundle.
3. Not every fiber bundle is trivial. For instance, consider the map from the Mobius strip to its central band. It is not trivial! To see this, construct the Möbius band by taking the square $I^2 = [0, 1] \times [0, 1]$ and gluing one edge to the other with a twist. The line in the center of the square maps to a circle, and so there is an induced projection map from the Möbius band to the circle with fibers $[0, 1]$.



If this were trivial then it would have to be isomorphic to the cylinder $S^1 \times [0,1]$ since $[0,1]$ is the fiber. But the “boundary” of the Möbius strip is S^1 and the boundary of the cylinder is $S^1 \sqcup S^1$, and these spaces are not homeomorphic. Here we are defining “boundary” to be the subset of points which do not admit an open neighborhood isomorphic to $\mathbb{R}^2$; this is a condition preserved under homeomorphism, so any homeomorphism must preserve the boundary, up to homeomorphism. Thus we get a contradiction.

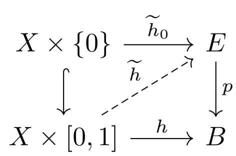
You could ask what would happen if we removed the boundaries and considered these spaces as fiber bundles with fibers $(0,1)$. This “open” Möbius band is still not trivial, but the argument is subtler and is left to part of the homework.

4. Another example is a tangent bundle to a manifold. A manifold is roughly a topological space that locally looks like Euclidean space; its tangent bundle is what you get when you take all of the tangent spaces at each point and glue them together. When doing this, one needs to be careful so that you get the right topology.

12.2 Lifting

As previously mentioned, we want to generalize the kind of lifting property that covering spaces satisfy.

Definition 12.2.1. A map $p : E \to B$ of topological spaces (not necessarily a fiber bundle) satisfies the *homotopy lifting property with respect to a topological space* X if whenever you're given a map $h : X \times [0,1] \to B$ and a lift $\widetilde{h}_0 : X \times \{0\} \to E$ of $h|_0$, you get a map $\widetilde{h} : X \times [0,1] \to E$ lifting h. In other words, the map $\widetilde{h}$ exists in the following diagram:



and the diagram commutes. Another way of saying this is that if you can lift a map into B, you can lift any homotopy starting at that map.

Definition 12.2.2. If $p : E \to B$ satisfies the homotopy lifting property with respect to every topological space X it is a *fibration*. If it satisfies the lifting property with respect to the hypercubes I^n for all $n \geq 0$ then it is a *Serre fibration*.

Proposition 12.2.3. *Every fiber bundle is a Serre fibration.*

Proof. The proof is a direct generalization of Lemma 10.2.2 and Proposition 10.2.3, except that we no longer get uniqueness of the lifts. So we reduce this to showing that a trivial fiber bundle is a Serre fibration. But given $h : I^n \times I \to B$ and a lift $\widetilde{h}_0 : I^n \times \{0\} \to B \times F$, we can extend the lift to a map $\widetilde{h} : I^n \times I \to B \times F$ by taking $(x_1, \ldots, x_{n+1}) \mapsto (h(x), (\pi_F \circ \widetilde{h}_0)(x_1, \ldots, x_n))$ where $\pi_F : B \times F \to F$ is the projection. Note this is not unique! □

Remark 12.2.4. Not every fiber bundle is a fibration, but it takes some very bizarre examples to see why. If B is *paracompact* and Hausdorff then every fiber bundle is a fibration, and most nice spaces are.

Example 12.2.5. More importantly, not every fibration is a fiber bundle. This is because in general the fibers of a fibration have to be homotopy equivalent, but not necessarily homeomorphic. So for example, take $E = \{(x,y) \in \mathbb{R}^2 : |y| \leq |x|\} \to B$ via projection onto the x-axis. This can't be a fiber bundle because the fibers are not all homeomorphic. But a homework exercise is to show that it is in fact a Serre fibration.

Example 12.2.6. Not every Serre fibration is a fibration, but this will be true whenever E and B are *CW-complexes*. These are spaces that are built by gluing disks D^n to other disks along their boundaries S^n.

Proposition 12.2.7. *If the base B is path-connected, the fibers of a fibration are homotopy equivalent.*

Proof. If $E \to B$ is the fibration, let $E_b = p^{-1}(b)$. Pick a path γ connecting b to b'. Consider the diagram

$$\begin{array}{ccc} E_b & \hookrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ E_b \times I & \xrightarrow{\gamma} & B \end{array}$$

The fibration property gives us a lift $H_\gamma : E_b \times I \to E$. But if we restrict this homotopy to $t = 1$ we get a map $L_\gamma : E_b \to E_{b'}$.

We show two facts. First, if γ and γ' have the same endpoints and are path-homotopic, then L_γ is homotopic to $L_{\gamma'}$ (and in particular doesn't depend on the lift H_γ). Second, if $\gamma * \gamma'$ is the composition of two paths, then $L_{\gamma*\gamma'}$ is homotopic to $L_{\gamma*\gamma'}$.

The theorem follows because L_γ is a homotopy equivalence with homotopy inverse $L_{\bar\gamma}$, where $\bar\gamma$ is the reverse of γ.

Now suppose $\alpha : \gamma \to \gamma'$ is a path-homotopy between two paths γ and γ'. But now note that any two lifts $H_\gamma, H_{\gamma'} : E_b \times I \to E$ are two lifts define a map $E_b \times \partial I \times I \to E$ which can be glued to the map $\iota : E_b \times I \times \{0\} \twoheadrightarrow E_b \hookrightarrow E$, and so we get a diagram

$$\begin{array}{ccc} E_b \times \partial I \times I \cup E_b \times I \times \{0\} & \xrightarrow{H_\gamma, H_{\gamma'}, \iota} & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ E_b \times I \times I & \xrightarrow{\alpha} & B \end{array}$$

But note that $\partial I \times I \cup I \times \{0\} \subseteq I \times I$ is homeomorphic to the inclusion $I \times \{0\} \subseteq I \times I$ (draw a picture), so we get a lift, and we can restrict that lift to $t = 1$ to get the desired homotopy.

The second part is left as an exercise. $\square$

13 Week 13

In this last lecture, we will explore *fixed-point theorems*, a cool application of the machinery of fundamental groups.

Definition 13.0.1. Suppose $S \subseteq X$ is a subset. A continuous map $r : X \to S$ is a *retraction* if the restriction $r|_S : S \to S$ is the identity map.

Example 13.0.2. Here are some examples.

- The identity map is a retraction.
- If $x_0 \in X$ then $X \to X$ sending every $x \mapsto x_0$ is a retraction.
- If you shrink a square to a line, that's a retraction.
- There are lots of examples.

Theorem 13.0.3 (Brouwer Fixed-Point Theorem)**.** *Any map $f : B^n \to B^n$ has a fixed point, i.e. there exists $x_0 \in B^n$ such that $f(x_0) = x_0$.*

Proof. We argue by contradiction. Let f denote a map such that $f(x) \neq x$ for all $x \in B^n$. Then we can define a map

$$F : B^n \to S^n$$

by drawing a ray from $f(x)$ to x in n-dimensional space, and then taking the intersection with S^n. Note that if $x \in S^n$ then by definition $F(x) = x$. But for F to be a retraction it needs to be continuous.

We can describe F as

$$F(x) = x + t(x - f(x))$$

where t is the positive solution to the equation

$$1 = \|x + t(x - f(x))\|^2.$$

What do I mean by "positive solution"? Note that if $x = (x_1, \cdots, x_n)$ and $f(x) = (f_1(x), \cdots, f_n(x_n))$

$$1 = \sum_{i=1}^{n} (x_i + t(x_i - f_i(x)))^2$$

If you rearrange the terms, you get some quadratic polynomial in t. By construction $t = 0$ solves this equation exactly when x is contained in S^n, but whenever $x \notin S^n$ there exist two solutions, one positive and one negative, by the quadratic formula. We only care about the positive one though. But then we can use the fact that the quadratic formula

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

is a continuous function in the coefficients to conclude that F is continuous.

But note that F is a retraction! Contradiction, by below. □

Note that if $i : S \to X$ is the inclusion and $r : X \to S$ is a retraction, then $r \circ i = \mathrm{id}_S$. But then this means that $r_* \circ i_* = \mathrm{id}_{\pi_n(S,s_0)}$ for all $n \geq 0$. In other words, we have a commuting diagram

$$\pi_n(S, s_0) \xrightarrow{i_*} \pi_n(X, s_0) \xrightarrow{r_*} \pi_n(S, s_0), \qquad r_* \circ i_* = \mathrm{id}$$

But this implies that r_* is surjective and i_* is injective. This means that you can realize $\pi_1(S, s_0)$ both as a subgroup of $\pi_1(X, s_0)$, *and* as a quotient!

Lemma 13.0.4. *There is no retraction from $B^n \to S^n$.*

Proof. Note that B^n, the closed unit ball of radius 1 in $\mathbb{R}^n$ is contractible, so $\pi_n(B^n, *) = 0$, while $\pi_n(S^n, *) = \mathbb{Z}$. We never actually showed that $\pi_m(B^n, *) = 0$, but the point is that since B^n is convex you can actually just take "straight-line homotopies" between any two maps $I^m \to B^n$. □

Note that the fixed point theorem holds for anything homeomorphic to B^n.

13.1 Borsuk-Ulam

Finally, I will briefly mention the following theorem.

Theorem 13.1.1 (Borsuk–Ulam)**.** *Given a continuous map $S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.*

The proof of this theorem is a bit more complicated, so we'll give a sketch.

If $f : S^n \to \mathbb{R}^n$ were a map satisfying $f(x) \neq f(-x)$ for all x, then we could define

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$

This is a continuous map $g : S^n \to S^{n-1}$ satisfying $g(-x) = -g(x)$. But such a thing cannot exist!

To see why, first consider $n = 1$. Then this is easy: any continuous map $S^1 \to S^0$ must have image a single point since S^1 is connected. But $g(-x) = -g(x)$ definitely does not have image a single point.

But what about in higher dimensions? Now say $n = 2$. If we restrict to the equator, we get a map $S^1 \to S^1$. But this map must be nullhomotopic because you can just swing it through the top half of the sphere to contract it to the constant map!

Then we show that any map $S^1 \to S^1$ satisfying $f(-x) = -f(x)$ cannot be nullhomotopic.