SECTION 5

ASHWIN IYENGAR

1. Review

Let G be a connected reductive group defined over \mathbb{F}_q and let $k = \overline{\mathbb{F}_q}$; as always, we implicitly identify G with its k-points. Last time we studied virtual representations constructed as follows.

Pick a maximal F-stable torus T in G. Temporarily fix a Borel B containing T. Last time, we constructed a G^F -equivariant T^F -torsor $Y_{T \subset B} \to X_{T \subset B}$

Then, after fixing a character

we defined a virtual representation

$$R_T^{\theta} = R_{T \subset H}^{\theta}$$

 $\theta: T^F \to \overline{\mathbb{Q}_\ell}$

which lives in $R(G^F)$, the Grothendieck group of representations of G^F . In particular, it was mentioned in Andy's talk that this representation does not depend on the choice of B containing T.

Today we will define an equivalence relation on the pairs (T, θ) called *geometric conjugacy*. The goal of the next talk is to show the following (somewhat vaguely stated) result.

Theorem 1.1. If (T, θ) and (T', θ') are not geometrically conjugate, then $\left\langle R_T^{\theta}, R_{T'}^{\theta'} \right\rangle = 0$. On the other hand, if (T, θ) and (T', θ') are geometrically conjugate, then $\left\langle R_T^{\theta}, R_{T'}^{\theta'} \right\rangle$ can be computed explicitly in a simple way.

2. Geometric Conjugacy

Consider an F-stable maximal torus T in G. Let $F: T \to T$ denote the q-Frobenius. This map induces a map

$$F: X_*(T) \to X_*(T)$$

where $X_*(T) = \text{Hom}(\mathbb{G}_m, T)$ denotes the cocharacter lattice of T. We have an F-equivariant isomorphism

$$X_*(T) \otimes_{\mathbb{Z}} k^{\times} \xrightarrow{h \otimes \alpha \mapsto h(\alpha)} T$$

where F acts on $X_*(T)$ via the map just defined, and trivially on k^{\times} . Thus we get an exact sequence

$$0 \to T^F \to X_*(T) \otimes k^\times \xrightarrow{F - \mathrm{id}} X_*(T) \otimes k^\times \to 0$$

But using the fact that $k^{\times} \cong (\mathbb{Q}/\mathbb{Z})_{p'}$ and a clever application of the snake lemma, one can show that we get an exact sequence

 $0 \to X_*(T) \xrightarrow{F - \mathrm{id}} X_*(T) \to T^F \to 0$

Before stating the definition, we give another definition.

Definition 2.1. For n > 0, the norm map is the map

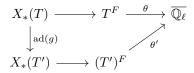
$$N = \frac{F^n - 1}{F - 1} = \sum_{i=0}^{n-1} F^i : T^{F^n} \to T^F$$

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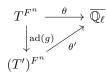
Lemma 2.2. The map $X_*(T) \to T^{F^n} \xrightarrow{N} T^F$ is the same as $X_*(T) \to T^F$.

Proposition 2.3. Let (T, θ) and (T', θ') be two pairs such that T, T' are F-stable maximal tori and θ and θ' are characters of T^F and $(T')^F$. The following are equivalent:

(1) There exists $g \in G$ such that $gTg^{-1} = T'$ and the following diagram commutes:



(2) For large enough n, there exists $g \in G^{F^n}$ such that $gTg^{-1} = T'$ and the following diagram commutes



Definition 2.4. A pair (T, θ) and (T', θ') are geometrically conjugate if the two equivalent conditions in Proposition 2.3 hold.

Now fix an absolute torus \mathbb{T} ("the torus"). Now given a pair (T, θ) as above, we get a character of $X_*(\mathbb{T})$ by taking

$$X_*(\mathbb{T}) \xrightarrow{\sim} X_*(T) \to T^F \to \overline{\mathbb{Q}_\ell}$$

and one can check that this doesn't depend on the choice of conjugation isomorphism $X_*(\mathbb{T}) \xrightarrow{\sim} X_*(T)$. But then θ defines an element of

$$\operatorname{Hom}(X_*(\mathbb{T}), \mu_{\infty}(\overline{\mathbb{Q}_{\ell}}^{\times})) = \operatorname{Hom}(X_*(\mathbb{T}), k^{\times}) \cong X^*(\mathbb{T}) \otimes k^{\times}$$

and construction this element is *F*-invariant. Note the Weyl group $W = N_G(\mathbb{T})/\mathbb{T}$ acts on $X^*(\mathbb{T})$ by precomposition with the action of W on \mathbb{T} . We let

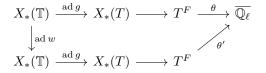
$$\mathcal{S} := [(X^*(\mathbb{T}) \otimes k^{\times})/W]^F$$

and denote the image of θ in S by $[\theta]$.

Remark 2.5. $X^*(\mathbb{T}) = X_*(\mathbb{T}^{\vee})$ where \mathbb{T}^{\vee} is the corresponding torus in the dual group G^{\vee} . Thus $X^*(\mathbb{T}) \otimes k^{\times} \cong (\mathbb{T}^{\vee})$.

Proposition 2.6. The association $(T, \theta) \mapsto [\theta]$ induces a bijection from the set of geometric conjugacy classes of pairs (T, θ) to S. Furthermore, the number of geometric conjugacy classes is exactly $|(Z^0)^F|q^r$ where r is the semisimple rank of G and Z^0 is the connected component of the center of G.

Proof. The map is well-defined by Proposition 2.3(1). It is injective because if $(T, \theta), (T', \theta')$ are two pairs with $|\theta| = |\theta'|$ in $(X^*(\mathbb{T}) \otimes k^{\times})/W$ then the following diagram commutes



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and thus the pairs are geometrically conjugate. For surjectivity, note that if the W-orbit of a map θ : $X_*(\mathbb{T}) \to \overline{\mathbb{Q}_\ell}^{\times}$ is invariant under F, there must exist $w \in W$ such that $F \cdot w \cdot \theta = \theta$. Therefore, θ descends to a character $\mathbb{T}(w)^F \to \overline{\mathbb{Q}_\ell}$. But $\mathbb{T}(w)^F$ is equal to T^F for some F-stable maximal torus of G.

Next, we show that $|(\mathbb{T}^{\vee}/W)^F| = |(Z^0)^F|q^r$. By the Lefschetz fixed point theorem in étale cohomology, we have

$$|(\mathbb{T}^{\vee}/W)^{F}| = \sum_{i=1}^{N} (-1)^{i} \operatorname{tr}(F, H_{c}^{i}(\mathbb{T}^{*}/W)) = \sum_{i=1}^{N} (-1)^{i} \operatorname{tr}(F, H_{c}^{i}(\mathbb{T}^{*})^{W})$$

But there is an isogeny $T^{\vee} \cong (T')^{\vee} \times (Z^0)^{\vee}$ where T' is the torus in the derived sugroup of G. But W acts trivially on $(Z^0)^{\vee} = Z^0$, so

$$|(\mathbb{T}^{\vee}/W)^{F}| = |(Z^{0})^{F}| = \sum_{i=1}^{N} (-1)^{i} \operatorname{tr}(F, H^{i}_{c}((\mathbb{T}')^{*})^{W})$$

Then using Poincaré duality, cohomology of a torus, and the fact that $X_*(T')^W = 0$, one can compute that the alternating sum is just q^r (for this see [DL76, Proposition 5.7]).

Example 2.7. Let's do the example of GL_2 over \mathbb{F}_p . Pick the maximal torus T to be just the upper triangular matrices: this is F-stable. Then $X_*(T) = \mathbb{Z}^2$, where (a, b) corresponds to

$$x \mapsto \begin{pmatrix} x^a & \\ & x^b \end{pmatrix}$$

Note F(x, y) = (px, py) and the cokernel of $F - id : \mathbb{Z}^2 \to \mathbb{Z}^2$ is clearly $\mathbb{Z}/p - 1 \oplus \mathbb{Z}/p - 1 \cong (\mathbb{F}_p)^{\times} = T^F$ (non-canonically).

In this case

$$\mathcal{S} \xrightarrow{\sim} (k^{\times}/S^2)^F$$

which consists of (unordered) pairs (x, x) for $x \in \mathbb{F}_p^{\times}$, (x, y) for distinct $x, y \in \mathbb{F}_p^{\times}$, and distinct (x, x^p) for $x \in \mathbb{F}_{p^2}^{\times} \setminus \mathbb{F}_p^{\times}$. Counting these up we get

$$(p-1) + {p-1 \choose 2} + \frac{(p^2-1) - (p-1)}{2} = p(p-1)$$

But note $Z^0 = \mathbb{G}_m$, which has p-1 *F*-rational points, and the semisimple rank is the rank of a maximal torus in $\operatorname{GL}_2/Z^0 = \operatorname{PGL}_2$, which is 1, so $|(Z^0)^F|q^r = p(p-1)$.

References

[DL76] P. Deligne and G. Lusztig. Representations of reductive groups over finite fields. Ann. of Math. (2), 103(1):103–161, 1976.