SECTION 5

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1. Review

Let G be a connected reductive group defined over $\mathbb{F}_q$ and let $k = \overline{\mathbb{F}_q}$; as always, we implicitly identify G with its k-points. Last time we studied virtual representations constructed as follows.

Pick a maximal F-stable torus T in G. Temporarily fix a Borel B containing T. Last time, we constructed a G^F-equivariant T^F-torsor

$$Y_{T \subset B} \to X_{T \subset B}$$

Then, after fixing a character

$$\theta : T^F \to \overline{\mathbb{Q}_\ell}$$

we defined a virtual representation

$$R_T^\theta = R_{T \subset B}^\theta$$

which lives in $R(G^F)$, the Grothendieck group of representations of G^F. In particular, it was mentioned in Andy's talk that this representation does not depend on the choice of B containing T.

Today we will define an equivalence relation on the pairs (T, θ) called *geometric conjugacy*. The goal of the next talk is to show the following (somewhat vaguely stated) result.

Theorem 1.1. *If (T, θ) and (T', θ') are not geometrically conjugate, then $\left\langle R_T^\theta, R_{T'}^{\theta'} \right\rangle = 0$. On the other hand, if (T, θ) and (T', θ') are geometrically conjugate, then $\left\langle R_T^\theta, R_{T'}^{\theta'} \right\rangle$ can be computed explicitly in a simple way.*

2. Geometric Conjugacy

Consider an F-stable maximal torus T in G. Let $F : T \to T$ denote the q-Frobenius. This map induces a map

$$F : X_*(T) \to X_*(T)$$

where $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ denotes the cocharacter lattice of T. We have an F-equivariant isomorphism

$$X_*(T) \otimes_{\mathbb{Z}} k^\times \xrightarrow{h \otimes \alpha \mapsto h(\alpha)} T$$

where F acts on $X_*(T)$ via the map just defined, and *trivially* on $k^\times$. Thus we get an exact sequence

$$0 \to T^F \to X_*(T) \otimes k^\times \xrightarrow{F - \mathrm{id}} X_*(T) \otimes k^\times \to 0$$

But using the fact that $k^\times \cong (\mathbb{Q}/\mathbb{Z})_{p'}$ and a clever application of the snake lemma, one can show that we get an exact sequence

$$0 \to X_*(T) \xrightarrow{F - \mathrm{id}} X_*(T) \to T^F \to 0$$

Before stating the definition, we give another definition.

Definition 2.1. For $n > 0$, the norm map is the map

$$N = \frac{F^n - 1}{F - 1} = \sum_{i=0}^{n-1} F^i : T^{F^n} \to T^F$$

Lemma 2.2. *The map* $X_*(T) \to T^{F^n} \xrightarrow{N} T^F$ *is the same as* $X_*(T) \to T^F$.

Proposition 2.3. *Let* (T, θ) *and* (T', θ') *be two pairs such that* T, T' *are* F*-stable maximal tori and* θ *and* θ' *are characters of* T^F *and* $(T')^F$. *The following are equivalent:*

(1) *There exists* $g \in G$ *such that* $gTg^{-1} = T'$ *and the following diagram commutes:*

$$\begin{array}{ccccc} X_*(T) & \longrightarrow & T^F & \xrightarrow{\theta} & \overline{\mathbb{Q}_\ell} \\ \big\downarrow{\scriptstyle \mathrm{ad}(g)} & & & \nearrow{\scriptstyle \theta'} & \\ X_*(T') & \longrightarrow & (T')^F & & \end{array}$$

(2) *For large enough* n, *there exists* $g \in G^{F^n}$ *such that* $gTg^{-1} = T'$ *and the following diagram commutes*

$$\begin{array}{ccc} T^{F^n} & \xrightarrow{\theta} & \overline{\mathbb{Q}_\ell} \\ \big\downarrow{\scriptstyle \mathrm{ad}(g)} & \nearrow{\scriptstyle \theta'} & \\ (T')^{F^n} & & \end{array}$$

Definition 2.4. A pair (T, θ) and (T', θ') are *geometrically conjugate* if the two equivalent conditions in Proposition 2.3 hold.

Now fix an absolute torus $\mathbb{T}$ ("the torus"). Now given a pair (T, θ) as above, we get a character of $X_*(\mathbb{T})$ by taking

$$X_*(\mathbb{T}) \xrightarrow{\sim} X_*(T) \to T^F \to \overline{\mathbb{Q}_\ell}$$

and one can check that this doesn't depend on the choice of conjugation isomorphism $X_*(\mathbb{T}) \xrightarrow{\sim} X_*(T)$. But then θ defines an element of

$$\mathrm{Hom}(X_*(\mathbb{T}), \mu_\infty(\overline{\mathbb{Q}_\ell}^\times)) = \mathrm{Hom}(X_*(\mathbb{T}), k^\times) \cong X^*(\mathbb{T}) \otimes k^\times$$

and construction this element is F-invariant. Note the Weyl group $W = N_G(\mathbb{T})/\mathbb{T}$ acts on $X^*(\mathbb{T})$ by precomposition with the action of W on $\mathbb{T}$. We let

$$\mathcal{S} := [(X^*(\mathbb{T}) \otimes k^\times)/W]^F$$

and denote the image of θ in $\mathcal{S}$ by $[\theta]$.

Remark 2.5. $X^*(\mathbb{T}) = X_*(\mathbb{T}^\vee)$ where $\mathbb{T}^\vee$ is the corresponding torus in the dual group $G^\vee$. Thus $X^*(\mathbb{T}) \otimes k^\times \cong (\mathbb{T}^\vee)$.

Proposition 2.6. *The association* $(T, \theta) \mapsto [\theta]$ *induces a bijection from the set of geometric conjugacy classes of pairs* (T, θ) *to* $\mathcal{S}$. *Furthermore, the number of geometric conjugacy classes is exactly* $|(Z^0)^F|q^r$ *where* r *is the semisimple rank of* G *and* Z^0 *is the connected component of the center of* G.

Proof. The map is well-defined by Proposition 2.3(1). It is injective because if $(T, \theta), (T', \theta')$ are two pairs with $[\theta] = [\theta']$ in $(X^*(\mathbb{T}) \otimes k^\times)/W$ then the following diagram commutes

$$\begin{array}{ccccccc} X_*(\mathbb{T}) & \xrightarrow{\mathrm{ad}\, g} & X_*(T) & \longrightarrow & T^F & \xrightarrow{\theta} & \overline{\mathbb{Q}_\ell} \\ \big\downarrow{\scriptstyle \mathrm{ad}\, w} & & & & & \nearrow{\scriptstyle \theta'} & \\ X_*(\mathbb{T}) & \xrightarrow{\mathrm{ad}\, g} & X_*(T) & \longrightarrow & T^F & & \end{array}$$

and thus the pairs are geometrically conjugate. For surjectivity, note that if the W-orbit of a map $\theta : X_*(\mathbb{T}) \to \overline{\mathbb{Q}_\ell}^\times$ is invariant under F, there must exist $w \in W$ such that $F \cdot w \cdot \theta = \theta$. Therefore, θ descends to a character $\mathbb{T}(w)^F \to \overline{\mathbb{Q}_\ell}$. But $\mathbb{T}(w)^F$ is equal to T^F for some F-stable maximal torus of G.

Next, we show that $|(\mathbb{T}^\vee/W)^F| = |(Z^0)^F|q^r$. By the Lefschetz fixed point theorem in étale cohomology, we have

$$|(\mathbb{T}^\vee/W)^F| = \sum(-1)^i \operatorname{tr}(F, H^i_c(\mathbb{T}^*/W)) = \sum(-1)^i \operatorname{tr}(F, H^i_c(\mathbb{T}^*)^W)$$

But there is an isogeny $T^\vee \cong (T')^\vee \times (Z^0)^\vee$ where T' is the torus in the derived sugroup of G. But W acts trivially on $(Z^0)^\vee = Z^0$, so

$$|(\mathbb{T}^\vee/W)^F| = |(Z^0)^F| = \sum(-1)^i \operatorname{tr}(F, H^i_c((\mathbb{T}')^*)^W)$$

Then using Poincaré duality, cohomology of a torus, and the fact that $X_*(T')^W = 0$, one can compute that the alternating sum is just q^r (for this see [DL76, Proposition 5.7]). □

Example 2.7. Let's do the example of GL_2 over $\mathbb{F}_p$. Pick the maximal torus T to be just the upper triangular matrices: this is F-stable. Then $X_*(T) = \mathbb{Z}^2$, where (a, b) corresponds to

$$x \mapsto \begin{pmatrix} x^a & \\ & x^b \end{pmatrix}$$

Note $F(x, y) = (px, py)$ and the cokernel of $F - \mathrm{id} : \mathbb{Z}^2 \to \mathbb{Z}^2$ is clearly $\mathbb{Z}/p-1 \oplus \mathbb{Z}/p-1 \cong (\mathbb{F}_p)^\times = T^F$ (non-canonically).

In this case

$$\mathcal{S} \xrightarrow{\sim} (k^\times/S^2)^F$$

which consists of (unordered) pairs (x, x) for $x \in \mathbb{F}_p^\times$, (x, y) for distinct $x, y \in \mathbb{F}_p^\times$, and distinct (x, x^p) for $x \in \mathbb{F}_{p^2}^\times \setminus \mathbb{F}_p^\times$. Counting these up we get

$$(p-1) + \binom{p-1}{2} + \frac{(p^2-1)-(p-1)}{2} = p(p-1)$$

But note $Z^0 = \mathbb{G}_m$, which has $p-1$ F-rational points, and the semisimple rank is the rank of a maximal torus in $\mathrm{GL}_2/Z^0 = \mathrm{PGL}_2$, which is 1, so $|(Z^0)^F|q^r = p(p-1)$.

References

[DL76] P. Deligne and G. Lusztig. Representations of reductive groups over finite fields. *Ann. of Math. (2)*, 103(1):103–161, 1976.