THE DERIVED DEFORMATION RING AND PATCHING

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The goal of this talk is to prove the main theorem (Theorem 14.1) of [GV18], which says that a certain Tor algebra appearing in the Calegari-Geraghty method, which is constructed non-canonically via patching (and so depends on a lot of arbitrary choices) is actually isomorphic to the graded homotopy ring of a derived version of the global deformation ring unramified outside a finite set of bad primes (and p) with a crystalline condition at p:

$$\mathrm{Tor}_*^{S_\infty^\circ}(R_\infty, W(k)) \cong \pi_* \mathcal{R}_S^{\mathrm{cris}}$$

In the rest of the talk we will describe where these things come from and then construct the isomorphism.

1. Background on the Calegari-Geraghty Method

In this section we recall the Calegari-Geraghty method (at least as presented in Section 13 in [GV18]) and the objects it gives us to work with, so that we may perform a derived version of its patching argument.

Let S' be a finite set of primes. $\mathbf{G}$ is a split semisimple algebraic group over $\mathbf{Q}$ with a smooth reductive model over $\mathbf{Z}[\frac{1}{S'}]$. Let Y_0 denote the usual locally symmetric space attached to $\mathbf{G}$ with hyperspecial level outside of S' and Iwahori level at primes in S'. We define the Hecke algebra $\mathbf{T}_0$ to be the $\mathbf{Z}$-algebra generated by the Hecke operators away from S'. First fix a cuspidal and tempered system of Hecke eigenvalues

$$\mathbf{T}_0 \twoheadrightarrow \mathscr{O}$$

landing in the ring of integers $\mathscr{O}$ of some number field. We then fix a prime $\mathfrak{p}$ lying over a prime $p \notin S'$ satisfying the following (one expects all but finitely many $\mathfrak{p}$ to satisfy these conditions)

(1) We assume $H^*(Y_0, \mathbf{Z})$ is p-torsion free.

(2) p is larger than the order of Weyl group of $\mathbf{G}$.

(3) The induced map $(\mathbf{T}_0)_\mathfrak{m} \to \mathscr{O}_\mathfrak{p}$ is an isomorphism, where $\mathfrak{m} = \ker(\mathbf{T}_0 \to \mathscr{O}/\mathfrak{p} =: k)$.

(4) $\mathscr{O}$ is unramified at $\mathfrak{p}$.

(5) The localization $H_j(Y_0, \mathbf{Z}_p)_\mathfrak{m}$ vanishes outside of $[q_0, q_0 + \ell_0]$.

Now let $S = S' \cup \{p\}$, and let G denote the group dual to $\mathbf{G}$, defined over $W := W(k)$, where $k = \mathscr{O}/\mathfrak{p}$: we furthermore assume that G is adjoint in order to state the conjecture on the existence of Galois representations, and in order to ensure the derived deformation functors are actually pro-representable without some modification taking into account the center $Z(G)$ (which is now 0 by assumption). Let T denote a maximal k-split torus of G. Conjecturally, we expect there to exist a Galois representation $\rho : G_{\mathbf{Q},S} \to G(\mathscr{O})$ associated to $\mathfrak{m}$, such that $\mathrm{ad}\,\overline{\rho}$ is torsion crystalline, and such that there exists a subfunctor $D_{\overline{\rho}_p}^{\mathrm{cris}}$ of the usual unframed deformation functor for $\overline{\rho}_p : G_{\mathbf{Q}_p} \to G(k)$ which is formally smooth with tangent space given by

the Selmer group $H^1_f(G_{\mathbf{Q}_p}, \operatorname{ad}\overline{\rho}_p)$. We make the following further assumptions on $\overline{\rho}$, mentioned by Rebecca last week.

(1) $H^0(G_{\mathbf{Q}_p}, \operatorname{ad}\overline{\rho}_p) = H^2(G_{\mathbf{Q}_p}, \operatorname{ad}\overline{\rho}_p) = 0$, i.e. the local deformation ring at p is representable and formally smooth.

(2) For $q \in S'$, $H^{\{0,1,2\}}(G_{\mathbf{Q}_q}, \operatorname{ad}\overline{\rho}_q) = 0$, i.e. the local deformation ring at S' is W.

(3) $\overline{\rho}$ has big image, which ensures that $\overline{\rho}$ is Schur (i.e. the centralizer of $\overline{\rho}$ in $G(k)$ is $T(k)$).

Finally, we define a Taylor-Wiles datum.

Definition 1.1. A *Taylor-Wiles datum of level n* is a set Q_n of primes disjoint from S and a choice of elements $t_q \in T(k)$ for each $q \in Q_n$ such that

(1) $p^n | (q-1)$ for all $q \in Q_n$

(2) $\overline{\rho}(\operatorname{Frob}_q)$ (recall $\overline{\rho}$ is unramified at q) is conjugate to t_q, a strongly regular element of $T(k)$, i.e. the centralizer of t_q in $G(k)$ is $T(k)$.

(3) We have the following cohomological vanishing condition. In the exact sequence

$$H^1_f(G_{\mathbf{Q},SQ_n}, \operatorname{ad}\overline{\rho}) \to H^1(G_{\mathbf{Q},SQ_n}, \operatorname{ad}\overline{\rho}) \xrightarrow{A} \frac{H^1(G_{\mathbf{Q}_p}, \operatorname{ad}\overline{\rho}_p)}{H^1_f(G_{\mathbf{Q}_p}, \operatorname{ad}\overline{\rho}_p)} \to H^2_{f,Q_n} \to H^2(G_{\mathbf{Q},SQ_n}, \operatorname{ad}\overline{\rho}) \xrightarrow{B} \bigoplus_{q \in Q_n} H^2(G_{\mathbf{Q}_q}, \operatorname{ad}\overline{\rho}_q),$$

the term H^2_{f,Q_n} vanishes. This sequence is defined in Appendix B of [GV18], and should be thought of part of the long exact sequence coming from a "derived local condition". More precisely, they have some specific recipe to lift the underived local conditions $H^1_f(G_{\mathbf{Q}_p}, \operatorname{ad}\overline{\rho}_p) \subseteq H^1(G_{\mathbf{Q}_p}, \operatorname{ad}\overline{\rho}_p)$ and $0 \subseteq H^1(G_{\mathbf{Q}_q}, \operatorname{ad}\overline{\rho}_q)$ to subcomplexes $C_{f,q} \hookrightarrow C_q$ for $q \in Q_n$ and $C_{f,p} \hookrightarrow C_p$, and then take the cohomology of the mapping cone of $C \to \oplus_{q \in Q_n \cup \{p\}} C_q / C_{f,q}$ to get the above sequence. In fact, the big image assumption on $\overline{\rho}$ ensures that B is an isomorphism.

Note then that if q is a Taylor-Wiles prime of level n, then we have determined representations (via t_q)

$$\xymatrix{ & \overline{\rho}^T_q & \\ G_q \ar[r] & G^{\mathrm{ur}}_q \ar[r]^{\overline{\rho}^{T,\mathrm{ur}}_q} & T(k) }$$

which are isomorphic to $\overline{\rho}_q$ and $\overline{\rho}^{\mathrm{ur}}_q$ after composing with $T(k) \to G(k)$.

Now write $G^{\mathrm{ab,tame}}_{\mathbf{Q}_q} = I_q \times \widehat{\mathbf{Z}}$ where the tame inertia subgroup I_q is non-canonically isomorphic to $(\mathbf{Z}/q)^\times$.

Theorem 1.2. *For $s >> 0$ large enough, set*

$$S^\circ_\infty = W[[x_1, \ldots, x_s]], \quad R_\infty = W[[x_1, \ldots, x_{s-\ell_0}]].$$

Also let $\mathfrak{a}_n = (p^n, (1+x_i)^{p^n} - 1) \subset S^\circ_\infty$. If we assume a natural local-global compatibility criterion is true (a precise statement is in Section 13.5 in [GV18]), then we get Taylor-Wiles data Q_n for all n and a diagram

$$\xymatrix{ W \ar[d] & S^\circ_\infty \ar[l] \ar@{->>}[r] \ar[d]^{\iota} & S^\circ_\infty/\mathfrak{a}_n \ar[r]^{\sim} \ar[d] & \overline{S^\circ_n} \ar[d] \\ W = R^{\mathrm{cris}}_{\overline{\rho},S} & R_\infty \ar[l] \ar@{->>}[r] & R_\infty/\mathfrak{a}_n \ar[r]^{\sim} & \overline{R_n} }$$

where the leftmost square is a pushout diagram of commutative rings, $\overline{S^\circ_n} = S^\circ_n/p^n$ is the mod p^n reduction of the framed deformation ring S°_n of $\prod_{q \in Q_n} I_q/p^n \xrightarrow{\mathrm{triv}} T(k)$ and $\overline{R_n}$ is the quotient of $R^{\mathrm{cris}}_n/(p^n, \mathfrak{m}^{\alpha(n)})$ parametrizing deformations of inertial level $\leq n$ (defined in Andy's talk: this will become clear as the talk

goes on), where R_n^{cris} is the global crystalline deformation ring for $\bar{\rho} : G_{\mathbf{Q},SQ_n} \to G(k)$, and $\alpha(n)$ is some strictly increasing sequence.

Moreover, there is a chain complex $C_\infty \in D(\mathsf{Mod}_{S_\infty^\circ})$ with homology M concentrated in degree q with an R_∞-action compatible with ι, and

$$\mathrm{Tor}_*^{S_\infty^\circ}(M, W) := H_*(C_\infty \otimes_{S_\infty^\circ}^{\mathbf{L}} W) = H_*(Y_0, W)_{\mathfrak{m}},$$

and we get a free action of $\mathrm{Tor}_*^{S_\infty^\circ}(R_\infty, W)$ on $H_*(Y_0, W)_{\mathfrak{m}}$, generating it in degree q.

In particular, we should be able to "realize $R_\infty \otimes_{S_\infty^\circ}^{\mathbf{L}} W$" in some natural way: in other words, there should be some derived interpretation of what's going on *before* we take homology. Today we will construct a map $\mathcal{R}_S^{\mathrm{cris}} \to R_\infty \otimes_{S_\infty^\circ}^{\mathbf{L}} W$ (which exists at least up to replacing the rings with weakly equivalent ones) which induces an isomorphism on the graded homotopy rings.

2. Allowing Ramification at Taylor-Wiles Primes

We define the following derived deformation functors:

(1) Let $\mathcal{F}_S^{(\mathrm{cris})} := \mathcal{F}_{G_{\mathbf{Q}},S,\bar{\rho}}^{(\mathrm{cris})}$ be the derived deformation functor of $\bar{\rho}$ (with the crystalline condition imposed), as defined by Rebecca last week. This is pro-representable because $\bar{\rho}$ is Schur and $Z(G) = 0$.

(2) For $q \in Q_n$ let $\mathcal{F}_q := \mathcal{F}_{G_{\mathbf{Q}_q},\bar{\rho}_q}$ denote the derived deformation functor of the pullback of $\bar{\rho}$ to $G_{\mathbf{Q}_q}$.

(3) For $q \in Q_n$ let $\mathcal{F}_q^T := \mathcal{F}_{G_{\mathbf{Q}_q},\bar{\rho}_q^T}$ denote the derived deformation functor of $\bar{\rho}_q^T$: here we replace G with T in the definition of the derived deformation functor. In particular, this is never going to be pro-representable because T is abelian.

(4) For $q \in Q_n$, $\bar{\rho}$ is unramified at q, so let $\mathcal{F}_q^{\mathrm{ur}} := \mathcal{F}_{G_q^{\mathrm{ur}},\bar{\rho}_q^{\mathrm{ur}}}$ denote the derived deformation functor of $\bar{\rho}_q^{\mathrm{ur}}$.

(5) If q belongs to a Taylor-Wiles datum, then let $\mathcal{F}_q^{T,\mathrm{ur}} := \mathcal{F}_{G_q^{\mathrm{ur}},\bar{\rho}_q^{T,\mathrm{ur}}}$ denote the derived deformation of $\bar{\rho}_q^{T,\mathrm{ur}}$, again valued in T.

Furthermore, if we add a $\square$ in any of the superscripts, that means that we're doing a framed version: these will always be pro-representable.

Now fix a Taylor-Wiles prime q.

The point of this section is to show that there is a diagram of homotopy pullback squares

$$
\begin{array}{ccccccccc}
\mathcal{F}_S^{\mathrm{cris}} & \longrightarrow & \mathcal{F}_S & \longrightarrow & \mathcal{F}_q^{\mathrm{ur}} & \xleftarrow{\ \sim\ } & \mathcal{F}_q^{T,\mathrm{ur}} & \xleftarrow{} & \mathcal{F}_q^{T,\mathrm{ur},\square} \\
\downarrow & & \downarrow & {\scriptstyle(a)} \ \downarrow & & {\scriptstyle(b)} \ \downarrow & & {\scriptstyle(c)} \ \downarrow & \downarrow \\
\mathcal{F}_{Sq}^{\mathrm{cris}} & \longrightarrow & \mathcal{F}_{Sq} & \longrightarrow & \mathcal{F}_q & \xleftarrow{\ \sim\ } & \mathcal{F}_q^T & \xleftarrow{} & \mathcal{F}_q^{T,\square}
\end{array}
$$

where the curvy maps in (c) are splittings (i.e. $\mathcal{F}_q^{T,\mathrm{ur}} \to \mathcal{F}_q^{T,\mathrm{ur},\square} \to \mathcal{F}_q^{T,\mathrm{ur}} = \mathrm{id}$). We need to construct these maps and prove that these are actually homotopy pullbacks, which I won't do in full detail here, but I'll give some indication of how they're proven. First note that the crystalline condition is given by taking the usual underived local crystalline condition and lifting it to a derived local condition, as described by Rebecca last week, and the ith homotopy group of the tangent complex is then just $H_f^i(G_{\mathbf{Q}_q}, \mathrm{ad}\,\bar{\rho}_q)$. But this is defined using a homotopy pullback, so it's not hard to see that this is also a homotopy pullback.

First, we note two technical lemmas.

Lemma 2.1 (Lemma 4.30 and Lemma 3.10 in [GV18]). *A homotopy commutative square of formally cohesive functors*

$$
\begin{array}{ccc}
\mathcal{G} & \longrightarrow & \mathcal{F}_0 \\
\downarrow & & \downarrow \\
\mathcal{F}_1 & \longrightarrow & \mathcal{F}_{01}
\end{array}
$$

is a homotopy pullback if and only if the natural map

$$
\mathcal{G} \to \mathcal{F}_0 \times_{\mathcal{F}_{01}}^h \mathcal{F}_1
$$

induces a quasi-isomorphism on tangent complexes.

This lets us reduce the proof of homotopy pullback to something about the tangent complexes, which we understand, because they should compute Galois cohomology, as discussed by Rebecca last week.

Now we say a word about the homotopy pullback squares.

(a) This says that a deformation of $G_{\mathbf{Q},Sq}$ unramified at q is actually a deformation of $G_{\mathbf{Q},S}$. This is proven in Section 8.2 of [GV18]: essentially one reduces this to the statement that

$$
C^*(\operatorname{Spec}\mathbf{Z}[1/S], \operatorname{ad}\overline{\rho}) \to C^*(\operatorname{Spec}\mathbf{Z}[1/Sq], \operatorname{ad}\overline{\rho}) \oplus C^*(V, \operatorname{ad}\overline{\rho}) \to C^*(\operatorname{Spec}\mathbf{Z}[1/Sq] \times_{\operatorname{Spec}\mathbf{Z}[1/S]} V, \operatorname{ad}\overline{\rho})
$$

is an exact triangle, for $V \to \operatorname{Spec}\mathbf{Z}[1/S]$ a finite étale cover, and then uses Lemma 2.1. These complexes are some cochain complexes computing étale cohomology.

(b) After a slight rephrasing, this follows from a square

$$
\begin{array}{ccc}
H^*(G_q^{\mathrm{ur}}, \operatorname{ad}\overline{\rho}_q^T) & \longrightarrow & H^*(G_q^{\mathrm{ur}}, \operatorname{ad}\overline{\rho}_q) \\
\downarrow & & \downarrow \\
H^*(G_q, \operatorname{ad}\overline{\rho}_q^T) & \longrightarrow & H^*(G_q, \operatorname{ad}\overline{\rho}_q)
\end{array}
$$

It turns out that the horizontal maps are isomorphisms (this follows from the fact that t_q is strongly regular: see Section 8 of [GV18] for an outline of the proof), and this is enough to show that the square is a homotopy pullback.

(c) The fact that this is a homotopy pullback essentially follows from the fact that

$$
\mathcal{F}_q^{T,\square}(A) = \mathcal{F}_q^T(A) \times_{BG(A)}^h *
$$

is defined by a homotopy fiber product. I won't explicitly construct the splitting: it's some technical homotopy theoretic thing I don't think lends enough intuition for the purposes of this talk (the idea is to choose $1 \in T(A)$ as the framing, but since we're really working with $BT(A)$ which isn't a simplicial group, we replace $BT(A)$ functorially with a weakly equivalent simplicial group). The important thing is that it still gives a homotopy pullback square, simply because it's a splitting.

Now, up to replacing diagram (b) with a weakly homotopy equivalent diagram to "invert the arrows", we get a homotopy pullback square

$$
\begin{array}{ccc}
\mathcal{F}_S^{\mathrm{cris}} & \longrightarrow & \mathcal{F}_q^{T,\mathrm{ur},\square} \\
\downarrow & & \downarrow \\
\mathcal{F}_{Sq}^{\mathrm{cris}} & \longrightarrow & \mathcal{F}_q^{T,\square}
\end{array}
$$

Roughly why we go through this process is that we eventually want to compare the ring representing $\mathcal{F}_S^{\mathrm{cris}}$ to the rings $\overline{S_n^\circ}$ appearing in the Calegari-Geraghty method. But note that S_n° deforms I_q/p^n, and $\mathcal{F}_q^{T,\square}$

deforms $I_q \times \widehat{\mathbf{Z}}$, so there are a few more intermediate steps we need before we can write down the map $\mathcal{R}_S^{\mathrm{cris}} \to \overline{R_n} \otimes_{\overline{S_n^\circ}}^{\mathbf{L}} W_n$ that we want to define (and then take the limit over n in a "pro"-sense to get the map that we ultimately want).

3. Descending from Taylor-Wiles Level

So why did we bother with Section 2? The point is that if we replace the prime q in the previous analysis with a Taylor-Wiles datum Q_n, then everything still works: we get an analogous diagram of homotopy pullbacks, where we now take the product over $q \in Q_n$ of all of the local deformation functors. So now fix the following notation:

(1) $\mathcal{F}_n^{\mathrm{cris}}$ is the crystalline deformation functor unramified outside $S \cup Q_n$ with representing pro-ring $\mathcal{R}_n^{\mathrm{cris}}$. Similarly, say $\mathcal{R}_S^{\mathrm{cris}}$ pro-represents $\mathcal{F}_S^{\mathrm{cris}}$.

(2)
$$\mathcal{F}_n^{\mathrm{loc}} = \prod_{q \in Q_n} \mathcal{F}_q^{T,\square}$$
with representing pro-ring $\mathcal{S}_n$, and

(3)
$$\mathcal{F}_n^{\mathrm{ur,loc}} = \prod_{q \in Q_n} \mathcal{F}_q^{T,\mathrm{ur},\square}$$
with representing pro-ring $\mathcal{S}_n^{\mathrm{ur}}$.

Therefore, following what we did earlier, we get an equality (ok really this is only a strict equality up to replacing the functors with weakly equivalent ones, but for all practical purposes we can ignore this subtlety: these are homotopy theoretic issues that only crop up because of the way we set up the homotopy theoretic framework, and are not really an essential feature of the argument)
$$\mathcal{F}_n^{\mathrm{cris}} \times_{\mathcal{F}_n^{\mathrm{loc}}}^h \mathcal{F}_n^{\mathrm{ur,loc}} = \mathcal{F}_S^{\mathrm{cris}}.$$

Let $S_n^{\mathrm{ur}} := \pi_0 \mathcal{S}_n^{\mathrm{ur}}$, which is the ring parametrizing framed deformations of $(\overline{\rho}_q^{T,\mathrm{ur}})_{q \in Q_n}$. Let $S_n = \pi_0 \mathcal{S}_n$. Let S_n' denote the ring parametrizing framed deformations of the representation
$$\prod_{q \in Q_n} (I_q/p^n \times \widehat{\mathbf{Z}}) \twoheadrightarrow \prod_{q \in Q_n} \widehat{\mathbf{Z}} \xrightarrow{\prod \overline{\rho}_q^{T,\mathrm{ur}}} T(k)$$

Lemma 3.1. *There is a diagram*

$$\begin{array}{ccc} S_n^\circ & \longrightarrow & S_n' \\ \downarrow & & \downarrow \\ W & \longrightarrow & S_n^{\mathrm{ur}} \end{array}$$

inducing an isomorphism $S_n^{\mathrm{ur}} = S_n' \widehat{\otimes}_{S_n^\circ} W$.

Proof. This is clear from the identities $S_n^\circ = W[[\prod_{q \in Q_n} I_q/p^n]]$ and $S_n' = S_n^{\mathrm{ur}}[[\prod_{q \in Q_n} I_q/p^n]]$. $\qquad \square$

Furthermore, we also have
$$\overline{S_n^{\mathrm{ur}}} = \overline{S_n'} \widehat{\otimes}_{\overline{S_n^\circ}} W_n,$$
where W_n denotes truncated Witt vectors, and $\overline{(\cdot)}$ denotes reduction mod p^n.

Note the map $I_q \times \widehat{\mathbf{Z}} \twoheadrightarrow I_q/p^n \times \widehat{\mathbf{Z}}$ induces a map $S_n \to S_n'$.

Now we can finally construct the map we want:

$$\phi_n : \mathcal{R}_S^{\mathrm{cris}} = \mathcal{R}_n^{\mathrm{cris}} \otimes_{\mathcal{S}_n}^{\mathbf{L}} \mathcal{S}_n^{\mathrm{ur}} \to R_n^{\mathrm{cris}} \otimes_{S_n}^{\mathbf{L}} S_n^{\mathrm{ur}} \to \overline{R_n} \otimes_{\overline{S_n'}}^{\mathbf{L}} \overline{S_n^{\mathrm{ur}}} = \overline{R_n} \otimes_{\overline{S_n^\circ}}^{\mathbf{L}} W_n \cong R_\infty/\mathfrak{a}_n \otimes_{S_\infty^\circ/\mathfrak{a}_n}^{\mathbf{L}} W_n$$

The equality (again, ignoring homotopy theoretic subtleties) in the second to last map follows from the discussion above: a completed tensor product is presented by a level-wise tensor product of pro-objects, and after passing to a derived setting this should still behave well.

Remark 3.2. What do we mean by derived tensor product of pro-objects in sArt_k? Really it should be some thing, well-defined up to homotopy, which represents the homotopy fiber product of the representing functors. In practice, it is possible to give an explicit construction of this thing, but it's not really useful to think about the exact details, since it's only well-defined up to weak equivalence. Therefore, all we will need is that it represents the fiber product of the representing functors, and that if the indexing category of the pro objects is the same, then the construction is functorial in all three variables. Alternatively, one could just avoid working with rings almost entirely, and instead only consider maps between their representing functors, but it's somehow more instructive/familiar to think about the rings themselves.

Theorem 3.3. *Assume $n > m > 1$. The map $\mathcal{R}_S^{\mathrm{cris}} \xrightarrow{\phi_n} \overline{R_n} \otimes_{\overline{S_n'}}^{\mathbf{L}} \overline{S_n^{\mathrm{ur}}} \xrightarrow{e_{n,m}} \overline{R_n}/\mathfrak{a}_m \otimes_{\overline{S_n'}/\mathfrak{a}_m}^{\mathbf{L}} \overline{S_n^{\mathrm{ur}}}/\mathfrak{a}_m$ induces an isomorphism on $\mathfrak{t}^0$ and a surjection on $\mathfrak{t}^1$.*

Proof. The maps

$$\pi_0 \mathcal{R}_n^{\mathrm{cris}} \twoheadrightarrow \overline{R_n} \twoheadrightarrow \overline{R_n}/\mathfrak{a}_m$$
$$\pi_0 \mathcal{S}_n \twoheadrightarrow \overline{S_n'} \twoheadrightarrow \overline{S_n'}/\mathfrak{a}_m$$
$$\pi_0 \mathcal{S}_n^{\mathrm{ur}} \twoheadrightarrow \overline{S_n^{\mathrm{ur}}} \twoheadrightarrow \overline{S_n^{\mathrm{ur}}}/\mathfrak{a}_m$$

all induce isomorphisms on $\mathfrak{t}^0$: they are injective on tangent spaces and the ideals defining these quotients all live in the square of the maximal ideals: this is straightforward to check after unwinding the definitions, because after choosing natural presentations for each of the rings, you only really quotient by p^m and terms looking like $(1+x)^{p^m} - 1$, which is in the square of the maximal ideal in the ring if $m > 1$.

The induced map between Mayer-Vietoris exact sequences for tangent complexes (see Section 4.30 (iv) in [GV18]) gives

$$
\begin{array}{ccccccccc}
\mathfrak{t}^0(\overline{R_n} \otimes_{\overline{S_n'}}^{\mathbf{L}} \overline{S_n^{\mathrm{ur}}}) & \hookrightarrow & \mathfrak{t}^0(\overline{R_n}) \oplus \mathfrak{t}^0(\overline{S_n^{\mathrm{ur}}}) & \to & \mathfrak{t}^0(\overline{S_n'}) & \to & \mathfrak{t}^1(\overline{R_n} \otimes_{\overline{S_n'}}^{\mathbf{L}} \overline{S_n^{\mathrm{ur}}}) & \to & \mathfrak{t}^1(\overline{R_n}) \oplus \mathfrak{t}^1(\overline{S_n^{\mathrm{ur}}}) \\
\downarrow{j_1} & & \| f & & \| g & & \downarrow{j_2} & & \downarrow \\
\mathfrak{t}^0(\mathcal{R}_n^{\mathrm{cris}} \otimes_{\mathcal{S}_n}^{\mathbf{L}} \mathcal{S}_n^{\mathrm{ur}}) & \to & \mathfrak{t}^0(\mathcal{R}_n^{\mathrm{cris}}) \oplus \mathfrak{t}^0(\mathcal{S}_n^{\mathrm{ur}}) & \to & \mathfrak{t}^0(\mathcal{S}_n) & \to & \mathfrak{t}^1(\mathcal{R}_n^{\mathrm{cris}} \otimes_{\mathcal{S}_n}^{\mathbf{L}} \mathcal{S}_n^{\mathrm{ur}}) & \to & \mathfrak{t}^1(\mathcal{R}_n^{\mathrm{cris}}) \oplus \mathfrak{t}^1(\mathcal{S}_n^{\mathrm{ur}}) \xrightarrow{h} \mathfrak{t}^1(\mathcal{S}_n)
\end{array}
$$

We want to show that j_1 is an isomorphism and that j_2 is surjective. Note f, g are isomorphisms by assumption. Then modulo a diagram chase, it suffices to show that h is injective.

Note $\mathcal{S}_n^{\mathrm{ur}}$ is formally smooth (we're deforming representations of $\mathbf{Z}$ into $T(k)$) so $\mathfrak{t}^1(\mathcal{S}_n^{\mathrm{ur}}) = 0$. It suffices to show that $\mathfrak{t}^1(\mathcal{R}_n^{\mathrm{cris}}) \to \mathfrak{t}^1(\mathcal{S}_n)$ is an isomorphism. But by definition

$$
\begin{array}{ccc}
\mathfrak{t}^1 \mathcal{R}_n^{\mathrm{cris}} & \longrightarrow & \mathfrak{t}^1 \mathcal{S}_n \\
\| & & \| \\
H_f^2(G_{\mathbf{Q}, SQ_n}, \mathrm{ad}\,\overline{\rho}) & \longrightarrow & \bigoplus_{q \in Q_n} H^2(G_{\mathbf{Q}_q}, \mathrm{ad}\,\overline{\rho}_q^T)
\end{array}
$$

Here $H^2_f(G_{\mathbf{Q},S}, \operatorname{ad} \overline{\rho})$ fits into the exact sequence (again see appendix B for the construction of this, this again comes from a Selmer complex mapping cone type construction)

$$H^1_f(G_{\mathbf{Q},SQ_n}, \operatorname{ad} \overline{\rho}) \to H^1(G_{\mathbf{Q},SQ_n}, \operatorname{ad} \overline{\rho}) \to \frac{H^1(G_{\mathbf{Q}_p}, \operatorname{ad} \overline{\rho}_p)}{H^1_f(G_{\mathbf{Q}_p}, \operatorname{ad} \overline{\rho}_p)} \to H^2_f(G_{\mathbf{Q},SQ_n}, \operatorname{ad} \overline{\rho}) \to H^2(G_{\mathbf{Q},SQ_n}, \operatorname{ad} \overline{\rho}) \to H^2(G_{\mathbf{Q}_p}, \operatorname{ad} \overline{\rho}_p)$$

But $H^2(G_{\mathbf{Q}_p}, \operatorname{ad} \overline{\rho}_p) = 0$ by assumption and the second map is surjective by definition of a Taylor-Wiles datum, so $H^2_f(G_{\mathbf{Q},SQ_n}, \operatorname{ad} \overline{\rho}) = H^2(G_{\mathbf{Q},SQ_n}, \operatorname{ad} \overline{\rho})$. The proof that diagram (b) is a homotopy fiber product from earlier shows that $H^2(G_{\mathbf{Q}_q}, \operatorname{ad} \overline{\rho}_q^T) = H^2(G_{\mathbf{Q}_q}, \operatorname{ad} \overline{\rho}_q)$, so we are left with

$$H^2(G_{\mathbf{Q},SQ_n}, \operatorname{ad} \overline{\rho}) \xrightarrow{B} \bigoplus_{q \in Q_n} H^2(G_{\mathbf{Q}_q}, \operatorname{ad} \overline{\rho}_q),$$

which is an injective again by definition of a Taylor-Wiles datum. $\qquad\square$

4. Derived Patching

We come to the main result.

Theorem 4.1. *There is an isomorphism of graded rings*

$$\pi_* \mathcal{R}^{\mathrm{cris}}_S \cong \operatorname{Tor}^{S^\circ_\infty}_*(R^\circ_\infty, W)$$

Proof. Let $\mathcal{C}_n := R_\infty / \mathfrak{a}_n \otimes^{\mathbf{L}}_{S^\circ_\infty / \mathfrak{a}_n} W_n$. First, Galatius and Venkatesh show that the fact that $\pi_* \mathfrak{t}(\mathcal{R}^{\mathrm{cris}}_S)$ is finite dimensional implies that

$$[\mathcal{R}^{\mathrm{cris}}_S, \mathcal{C}_n] := \pi_0 \underline{\operatorname{Hom}}_{\mathrm{pro-sArt}_k}(\mathcal{R}^{\mathrm{cris}}_S, \mathcal{C}_n)$$

is finite. Thus by compactness (i.e. the axiom of choice), there is a compatible system of maps in the profinite set $\lim_n [\mathcal{R}^{\mathrm{cris}}_S, \mathcal{C}_n]_\mathfrak{t}$, where the subscript $\mathfrak{t}$ denotes the subset of maps inducing a an isomorphism on $\mathfrak{t}^0$ and a surjection on $\mathfrak{t}^1$.

So there are maps $g_n \in [\mathcal{R}^{\mathrm{cris}}_S, \mathcal{C}_n]_\mathfrak{t}$ compatible up to homotopy, which defines a map

$$\operatorname{hocolim}_n \underline{\operatorname{Hom}}(\mathcal{C}_n, -) \to \underline{\operatorname{Hom}}(\mathcal{R}^{\mathrm{cris}}_S, -).$$

which is still an iso on $\mathfrak{t}^0$ and surjective on $\mathfrak{t}^1$. But note that the homotopy colimit and colimit are weakly equivalent for filtered indexing categories, so we can just take colim. Note if we define $\mathcal{C} = (n \mapsto \mathcal{C}_n) \in \mathrm{pro-sArt}_k$, then we really have a map

$$\underline{\operatorname{Hom}}(\mathcal{C}, -) \to \underline{\operatorname{Hom}}(\mathcal{R}^{\mathrm{cris}}_S, -)$$

We claim that this actually induces a quasi-isomorphism of tangent complexes, for which is suffices to show that both sides vanish outside degrees $0, 1$ and that both sides have the same "Euler characteristic", i.e. $\dim \mathfrak{t}^1 - \dim \mathfrak{t}^0$ is the same on both sides. which induces a map (up to replacing things by weakly equivalent things etc etc) $\mathcal{R}^{\mathrm{cris}}_S \to \mathcal{C}$. Think of this as the map

$$\mathcal{R}^{\mathrm{cris}}_S \to R_\infty \otimes^{\mathbf{L}}_{S^\circ_\infty} W.$$

Note $\mathfrak{t}^i \mathcal{R}^{\mathrm{cris}}_S$ vanishes for $i \neq 0, 1$ because the Galois cohomology is concentrated in degrees $0, 1, 2$, but actually vanishes at degree 0 because $\overline{\rho}$ is Schur and G is adjoint. The homotopy groups of the tangent complex of the other side is $\operatorname{colim}_n \mathfrak{t}^i \mathcal{C}_n$ by definition. But taking the Mayer-Vietoris sequence again gives

$$\cdots \to \mathfrak{t}^i \mathcal{C}_n \to \mathfrak{t}^i(R_\infty / \mathfrak{a}_n) \oplus \mathfrak{t}^i(W_n) \to \mathfrak{t}^i(S^\circ_\infty / \mathfrak{a}_n) \to \mathfrak{t}^{i+1} \mathcal{C}_n \to \ldots$$

Taking the colimit we get

$$\cdots \to \operatorname{colim}_n \mathfrak{t}^i \mathcal{C}_n \to \mathfrak{t}^i(R_\infty) \oplus \mathfrak{t}^i(W) \to \mathfrak{t}^i(S^\circ_\infty) \to \operatorname{colim}_n \mathfrak{t}^{i+1} \mathcal{C}_n \to \ldots$$

This uses the fact that the descending sequence of ideals $\mathfrak{a}_n$ defines the maximal ideal topology on both R_∞ and S_∞, which one can check directly. But $\mathfrak{t}^i(W) = 0$ (e.g. this parametrizes deformations of the trivial

group and thus has trivial cohomology), and $\mathfrak{t}^i(R_\infty) = \mathfrak{t}^i(S_\infty^\circ) = 0$ for $i \neq 0$ since these are power series rings, and thus formally smooth. Thus the exact sequence above tells us that $\mathrm{colim}_n \mathfrak{t}^i \mathcal{C}_n$ vanishes if $i \neq 0, 1$.

For the Euler characteristic part, note that

$$\dim \mathfrak{t}^1 \mathcal{R}_S^{\mathrm{cris}} - \dim \mathfrak{t}^0 \mathcal{R}_S^{\mathrm{cris}} = \dim H_f^2(G_{\mathbf{Q},S}, \mathrm{ad}\,\overline{\rho}) - \dim H_f^1(G_{\mathbf{Q},S}, \mathrm{ad}\,\overline{\rho}) = \ell_0$$

by the numerical criterion in the Calegari-Geraghty method (and the fact that H_f^2 is dual to the dual Selmer group $H_{f^\perp}^1$: see appendix B for a proof). But since the Mayer-Vietoris sequence for the other side is just (as above)

$$\cdots \to 0 \to \mathrm{colim}_n \mathfrak{t}^0 \mathcal{C}_n \to \mathfrak{t}^0(R_\infty) \to \mathfrak{t}^0(S_\infty^\circ) \to \mathrm{colim}_n \mathfrak{t}^1 \mathcal{C}_n \to 0 \to \cdots$$

So by exactness we have

$$\dim \mathfrak{t}^1 \mathcal{C} - \dim \mathfrak{t}^0 \mathcal{C} = \dim \mathrm{colim}_n \mathfrak{t}^1 \mathcal{C}_n - \dim \mathrm{colim}_n \mathfrak{t}^0 \mathcal{C}_n = \dim \mathfrak{t}^0 S_\infty^\circ - \dim \mathfrak{t}^0 R_\infty = \ell_0$$

because $\mathfrak{t}^0 = H_1$, and R_∞ and S_∞° are power series rings in $s - \ell_0$ and s variables, respectively.

Therefore the pro-objects $\mathcal{R}_S^{\mathrm{cris}}$ and $\mathcal{C}$ living in $\mathrm{pro}-\mathrm{sArt}_k$ represent equivalent functors and therefore the induced map

$$\pi_* \mathcal{R}_S \to \pi_i \mathcal{C} = \lim_n \pi_* \mathcal{C}_n$$

is an isomorphism. But

$$\pi_i \mathcal{C} = \lim_n \pi_i \mathcal{C}_n = \lim_n \mathrm{Tor}_i^{S_\infty^\circ/\mathfrak{a}_n}(R_\infty/\mathfrak{a}_n, W_n) \cong \mathrm{Tor}_i^{S_\infty^\circ}(R_\infty, W).$$

The first equality is Theorem 6 of [Qui67], and Lemma 7.6 in [GV18] gives the second equality. This concludes the proof. $\qquad\square$

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