Derived Deformation Theory and the Derived Hecke Algebra

These notes are in fairly rough shape, tread with caution

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1 Introduction

I am going to start by recapping some of the things that happened in the study group so far. Let $\mathbf{G}/\mathbb{Q}$ be a split reductive algebraic group that is semi-simple and simply-connected and let π be a cuspidal, cohomological and tempered automorphic representation of $\mathbf{G}(\mathbb{A})$. Then π contributes to the cohomology of the locally symmetric space $Y(K)$ for G, where $K \subset \mathbf{G}(\mathbb{A}^\infty)$ is a neat compact open subgroup such that $\pi^K \neq 0$. If we let $\mathbb{T}$ be the Hecke algebra of level K and $\chi : \mathbb{T} \to \mathbb{C}$ be the character associated to π (taking operators to their eigenvalues on π) then we have seen that

$$H^*(Y(K), \mathbb{C})_\chi = 0 \text{ if } i \notin [q_0, q_0 + l_0]$$

and

$$\dim_{\mathbb{C}} H^{q_0+i}(Y(K), \mathbb{C})_\chi = \binom{l_0}{i} H^{q_0}(Y(K), \mathbb{C})_\chi.$$

Venkatesh then conjectures that this should be explained by a 'natural' action of an exterior algebra on a vector space of dimension l_0 (and much more). In the Hecke track we defined a derived Hecke algebra $\mathcal{H}$ which acts by degree increasing endomorphisms on $H^*(Y(K)), \mathbb{Z}_p)$ for primes p. However, we cannot prove much about the action without using the existence of sufficiently nice Galois representations (c.f. the section of assumptions). Let $\mathfrak{m}$ be the maximal ideal of the Hecke algebra $\mathbb{T}$ associated to the mod p Galois representation ρ associated to π, then we can prove that the action of $\mathcal{H}_\mathfrak{m}$ makes

$$H^*(Y(K), \mathbb{Z}_p)_\mathfrak{m}$$

into a free module generated in degree q_0 (the fact that we can prove this integrally should be an artefact of the our simplifying assumptions). Under these same assumptions we proved that

$$\mathcal{H}_\mathfrak{m} \approx \bigwedge^* V$$

where

$$V = H^1_f(\mathbb{Z}[\tfrac{1}{S}], \operatorname{Ad} \rho_{\mathcal{O}})$$

is a Bloch-Kato Selmer group, which is conjecturally of dimension l_0 (this is not quite right, see the discussion after Proposition 8.6 in the newest version of [Ven16]). Meanwhile in the Galois track we introduced a derived deformation ring $\mathcal{R}_S$ and we showed last week that it (to be precise its associated graded ring) also acts on

$$H^*(Y(K), \mathbb{Z}_p)_{\mathfrak{m}},$$

by degree decreasing endomorphisms. We did this by identifying $\pi_*\mathcal{R}_S$ with

$$\mathrm{Tor}_*^{S_\infty}(R_\infty, \mathbb{Z}_p)$$

which acts on $H^*(Y(K), \mathbb{Z}_p)_{\mathfrak{m}}$. Here the rings R_∞, S_∞ are outputs of the Calegari-Geraghty method and depend on the choices made, which means that we cannot prove that the action of $\pi_*\mathcal{R}_S$ is canonical. Now we note that

$$V \cong \left(\mathrm{Tor}_*^{S_\infty}(R_\infty, \mathbb{Z}_p)\right)^*$$

and so we get an isomorphism

$$\pi_*\mathcal{R}_S \cong \bigwedge^* V^*. \tag{1}$$

The goal of today's talk is to show that this isomorphism is canonical, i.e., does not depend on the choice of Taylor-Wiles data. We will achieve this by defining a canonical map

$$\pi_1\mathcal{R}_S \to V^*$$

and showing that it agrees with (1) in degree one. This then shows that the action of $\pi_*\mathcal{R}_S$ on $H^*(Y(K), \mathbb{Z}_p)_{\mathfrak{m}}$ is canonical and "dual" to the action of $\mathcal{H}_{\mathfrak{m}}$.

2 Notation and patching

3 A canonical map

In this section we will describe a canonical map

$$\pi^1\mathcal{R}_S \to V^*$$

which we will later compare to the non-canonical isomorphism coming from patching. We start by discussing a slightly generalised theory of tangent complexes. Just as with complete local Noetherian rings R over $W(k)$ it can be useful to consider tangent spaces at points $R \to A$ for A some Artinian quotient of $W(k)$, it will be useful for us to discuss A-valued tangent complex of deformation functor. Now let $\mathcal{R}$ be a pro object of $\mathbf{sArt_k}$ and fix a point

$$\phi : \pi_0\mathcal{R} \to A,$$

then there is a good theory of tangent complexes $\mathfrak{t}^\phi$ relative to ϕ. In particular we should have that

$$\pi_{-i}\mathfrak{t}^A$$

is given by homotopy classes of maps

$$\mathcal{R} \to A \oplus A[i]$$

inducing ϕ on π_0 (here we consider A as a simplicial artin ring). We will be most interesting in this theory when $A = W_n$ the length n Witt vectors of k in which case we will write

$$\mathfrak{t}_n \mathcal{R}.$$

I completely made up all this notation by the way, but it will save us a lot of space later (I will refrain from writing $\pi_{-i}\mathfrak{t}_n$ as $\mathfrak{t}_n^i$).

Lemma 1 (Lemma 15.1 in [GV18])**.** *Let $\mathcal{R}_S$ be the crystalline derived deformation ring. Fix a lift $\rho_n : G_{\mathbb{Q},S} \to G(W_n)$, classified by a map $\phi : \pi_0 \mathcal{R}_S to W_n$. Then the set of homotopy classes of maps*

$$\mathcal{R}_S \to W_n \oplus W_n[1]$$

which lift ϕ is in bijection with

$$H^2_f(\mathrm{Ad}\,\rho_n).$$

Proof. When $A = k$ and $M = k$ then the set of homotopy classes of maps is just $\pi_{-1}(\mathfrak{t}\mathcal{R}_S)$ which we have identified with $H^2_f(\mathrm{Ad}\,\rho)$. The proof in our case is exactly the same, given a good theory of tangent complexes $\mathfrak{t}^n$ as above. □

The second bit of homotopy theory that we will need is that there is a natural map

$$\pi_{-i}\mathfrak{t}_n\mathcal{R} \to \hom(\pi_i\mathcal{R}, W_n)$$

which is given by taking the map

$$\mathcal{R} \to W_n \oplus W_n[1]$$

and evaluating it on loops in $\mathcal{R}$. To be precise the map

$$\pi_0\mathcal{R} \to W_n$$

is fixed and then a homotopy class of maps

$$\mathcal{R} \to W_n \oplus W_n[1]$$

gives a map from the 1-simplices of $\mathcal{R}$ to W_n, which induces a map $\pi_1\mathcal{R} \to W_n$. Combining this with Lemma **??** we get a natural map

$$H^2_f(\mathrm{Ad}\,\rho_n) \to \hom(\pi_1\mathcal{R}, W_n)$$

which induces (in the limit over n) a map

$$V \to \hom(\pi_1\mathcal{R}, W).$$

4 The reciprocity law revisited

In this section we will recall some things from section 8 of [Ven16]. In particular, we will review the construction of the isomorphism

$$\mathfrak{t}_{R_n}/\mathfrak{t}_{S_n} \cong V/p^n$$

because we need to compare it with the isomorphism $\pi_*\mathcal{R}_S = \bigwedge^* V^*$ later. Let Q_n by a collection of Taylor-Wiles primes of level n and let $v \in Q_n$, then we define

$$T_v := \mathbb{A}(\mathbb{F}_q)/p^n, T_n := \prod_{q\in Q_n} T_q$$

where $\mathbb{A}$ is a maximal torus of $\mathbb{G}$. Then there are isomorphisms (the first comes from section 6.4 and the second holds by definition)

$$\mathfrak{t}_{S_n} \cong \operatorname{Ext}^1_{S_n}(\mathbb{Z}/p^n, \mathbb{Z}/p^n) \cong H^1(T_n, \mathbb{Z}/p^n).$$

We also get an isomorphism, which depends on the choice of strongly regular element, (this is basically an identification of S_n with the framed deformation ring of the $\operatorname{Ad}\rho$ into the torus)

$$\mathfrak{t}_{S_n} \cong \bigoplus_{q\in Q_n} \frac{H^1(\mathbb{Q}_q, \operatorname{Ad}\rho_n)}{H^1_{ur}(\mathbb{Q}_q, \operatorname{Ad}\rho_n)}$$

which we will use to describe a canonical surjection

$$\psi : \mathfrak{t}_{S_n} \twoheadrightarrow V/p^n$$

with kernel $\mathfrak{t}_{R_n}$. Consider the following diagram

$$\begin{array}{ccccc} \mathfrak{t}_{R_n} & \longrightarrow & \mathfrak{t}_{S_n} & \longrightarrow & \mathfrak{t}_{S_n}/\mathfrak{t}_{R_n} \\ \downarrow\cong & & \downarrow\cong & & \| \\ H^1_f(\mathbb{Z}[1/SQ_n], \operatorname{Ad}\rho_n) & \xrightarrow{\phi} & \bigoplus_{q\in Q_n} \frac{H^1(\mathbb{Q}_q, \operatorname{Ad}\rho_n)}{H^1_{ur}(\mathbb{Q}_q, \operatorname{Ad}\rho_n)} & \longrightarrow & \mathfrak{t}_{S_n}/\mathfrak{t}_{R_n} \end{array}$$

where ϕ is the restriction map in Galois cohomology, the first vertical isomorphism is just the computation of the tangent space to a deformation ring and the bottom-right horizontal map is the induced one. This gives us a pairing

$$H^1_f(\mathbb{Z}[1/S], \operatorname{Ad}\rho_n^*(1)) \times \mathfrak{t}_{R_n}/\mathfrak{t}_{S_n} \to \mathbb{Z}/p^n \tag{2}$$

by

$$(\alpha, (\beta_v)_{v\in Q_n}) \mapsto \sum_v (\alpha_v, \beta_v)_v.$$

The local pairing $(\alpha_v, \beta_v)_v$ is just the cup product pairing

$$H^1(\mathbb{Q}_v, \operatorname{Ad}\rho_n^*(1)) \times H^1(\mathbb{Q}_v, \operatorname{Ad}\rho) \to H^2(\mu_{p^n}) = \mathbb{Z}/p^n.$$

Since the classes α_v are unramified, it means they pair trivially with

$$H^1_{ur}(\mathbb{Q}_v, \operatorname{Ad}\rho_n),$$

so the pairing (2) is well defined. Using the condition that the $v \in Q_n$ are Taylor-Wiles primes Venkatesh then proves that the pairing (2) is perfect which gives us an isomorphism

$$\mathfrak{t}_{R_n}/\mathfrak{t}_{S_n} \cong V/p^n,$$

using the fact that V is torsion-free. The goal of the next section is to show that the isomorphism

$$\pi_*\mathcal{R}_S \cong \operatorname{Tor}_*^{S_\infty}(R_\infty, W) \cong (\mathfrak{t}_{S_\infty}/\mathfrak{t}_{R_\infty})^* \cong \bigwedge^* V^*$$

is induced by the canonical map $\pi_1\mathcal{R}_S \to V^*$. Here the last identification comes from the maps

$$\mathfrak{t}_{R_n}/\mathfrak{t}_{S_n} \to V/p^n$$

described above.

5 Comparison and conclusion

We will work to identify the dual of the map constructed in Section 3. As always, we will work mod p^m and at Taylor-Wiles level n, with $n \gg m$. There is a natural map

$$\pi_0(\text{ lifts to } \mathcal{R}_S \to W_m \oplus W_m[1]) \to \hom(\pi_1\mathcal{R}_S, W_m)$$

and Lemma 1 shows that the left hand side is naturally identified with

$$H^2_f(\mathbb{Z}[1/S], \operatorname{Ad}\rho_m),$$

which is just a mod p^n version of the standard tangent complex computation. Now consider the maps

$$\mathcal{R}_S \to \overline{R}_n \underline{\otimes}_{\overline{S_n^\circ}} W_n \leftarrow R_\infty \underline{\otimes}_{S_\infty^\circ} W$$

from which we construct the following diagram

$$\begin{array}{ccc}
\pi_0\left(\text{ lifts to } R_\infty \underline{\otimes}_{S_\infty^\circ} W \to W_m \oplus W_m[1]\right) & \longrightarrow & \hom(R_\infty \underline{\otimes}_{S_\infty^\circ} W, W_m) \\
\uparrow & & \uparrow \\
\pi_0\left(\text{ lifts to } \overline{R}_n \underline{\otimes}_{\overline{S_n^\circ}} W_n \to W_m \oplus W_m[1]\right) & \longrightarrow & \hom(\pi_1\mathcal{R}_S \overline{R}_n \underline{\otimes}_{\overline{S_n^\circ}} W_n), W_m) \\
\downarrow & & \downarrow \\
\pi_0(\text{ lifts to } \mathcal{R}_S \to W_m \oplus W_m[1]) & \longrightarrow & \hom(\pi_1\mathcal{R}_S, W_m).
\end{array}$$

We have already identified the bottom left space with $H^2_f(\mathbb{Z}[1/S], \operatorname{Ad}\rho_m)$ and we are going to identify the other two spaces in a similar way. For this we have to compute with the mod p^m tangent complex of the rather terrifying looking rings

$$R_\infty \underline{\otimes}_{S_\infty^\circ} W, \overline{R}_n \underline{\otimes}_{\overline{S_n^\circ}} W_n. \tag{3}$$

Fortunately there is a "Mayer-Vietoris sequence" in the homology groups of (mod p^m) tangent complexes associated to derived tensor products

$$D = A \underline{\otimes}_B C$$

which looks like

$$\cdots \to \pi_{-n} \mathfrak{t} D \to \pi_{-n} \mathfrak{t} A \oplus \mathfrak{t} C \to \pi_{-n} \mathfrak{t} B \to \cdots$$

We are interested in computing π_{-1} of the tangent complexes of the rings in (3) and the relevant parts of the long exact sequences look like (recall that W and W_n have trivial tangent spaces)

$$\cdots \mathfrak{t}_{R_n} \otimes W_m \to \mathfrak{t}_{S_n} \otimes W_m \to \pi_{-1}\left(\mathfrak{t}\left(\overline{R}_n \underline{\otimes}_{\overline{S_n^\circ}} W_n\right)\right) \to \cdots$$

and

$$\cdots \mathfrak{t}_{R_\infty} \otimes W_m \to \mathfrak{t}_{S_\infty} \to \pi_{-1}\left(\mathfrak{t}\left(R_\infty \underline{\otimes}_{S_\infty^\circ} W\right)\right) \to \cdots$$

Putting all of this together we get the following diagram

$$\begin{array}{ccc}
(\mathfrak{t}_{S_\infty}/\mathfrak{t}_{R_\infty}) \otimes W_m & \xrightarrow{\cong} & \hom(\pi_1(R_\infty \underline{\otimes}_{S_\infty^\circ} W), W_m) \\
\uparrow h & & \uparrow f \\
(\mathfrak{t}_{R_n}/\mathfrak{t}_{S_n}) \otimes W_m & \longrightarrow & \hom(\pi_1(\overline{R}_n \underline{\otimes}_{\overline{S_n^\circ}} W_n), W_m) \\
\downarrow \theta & & \downarrow g \\
H^2_f(\mathbb{Z}[1/S], \operatorname{Ad}\rho_m) & \longrightarrow & \hom(\pi_1 \mathcal{R}_S, W_m).
\end{array}$$

We note that the top vertical map is just the mod p^m reduction of the dual of the isomorphism

$$(\mathfrak{t}_{S_\infty}/\mathfrak{t}_{R_\infty})^* \cong \operatorname{Tor}_1^{S_\infty}(R_\infty, W).$$

After passing to the inverse limit we get isomorphisms

$$\begin{array}{ccc}
(\mathfrak{t}_{S_\infty}/\mathfrak{t}_{R_\infty}) \otimes W_m & \xrightarrow{\cong} & \hom(\pi_1(R_\infty \underline{\otimes}_{S_\infty^\circ} W), W_m) \\
\downarrow F & & \downarrow G \\
\hom(\pi_1(\mathcal{R}_S), W_m) & \longrightarrow & H^2_f(\mathbb{Z}[1/S], \operatorname{Ad}\rho_m)
\end{array}$$

and we would like to show that the induced isomorphism G agrees with ψ. This means we have to identify θ with the map ϕ from the previous section

5.1 Conclusion

The action of V on $H^*(Y(K), \mathbb{Z}_p)_\mathfrak{m}$, defined via the Taylor-Wiles method in section 8 of [Ven16], does not depend on the choices of Taylor-Wiles data. This means that the action of $\pi_* \mathcal{R}_S$ on the same space, defined in section 13 of [GV18], does not depend on any choices either.

References

[Ven16] Akshay Venkatesh. "Derived Hecke algebra and cohomology of arithmetic groups". In: *arXiv e-prints*, arXiv:1608.07234 (Aug. 2016), arXiv:1608.07234. arXiv: `1608.07234 [math.NT]`.

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