The Taylor-Wiles method (the unobstructed case)

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We want to prove an $R = \mathbb{T}$ theorem, where R is a Galois deformation ring and $\mathbb{T}$ is a Hecke-algebra. The ring R will be a deformation ring which parametrizes lifts of $\overline{\rho}$ that should should arise from some space of automorphic forms S (which essentially means the lifts are 'geometric'). The space of automorphic forms S is acted on by a Hecke algebra $\mathbb{T}$. If we can prove an $R = \mathbb{T}$ theorem then such a lift $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{Gl}_n(\mathcal{O})$ gives rise to $\mathbb{T} \to \mathcal{O}$ which should correspond to an eigenform $f \in S$.

1 R

Let F be a totally real field of degree $d = [F : \mathbb{Q}]$, let p be an odd prime unramified in F and let

$$G_{F,\Sigma_Q} = \mathrm{Gal}(F_{\Sigma_Q}/F)$$

where $F_{\Sigma_Q} \subset \overline{F}$ is the maximal unramified outside of Σ_Q extension of F inside $\overline{F}$. Let $K/\mathbb{Q}_p$ be a field of coefficients with ring of integers $\mathcal{O}$ and residue field k. Consider

$$\overline{\rho} : G_{F,\Sigma_Q} \to \mathrm{Gl}_2(k)$$

and assume that:

- The determinant of $\overline{\rho}$ is the inverse of the cyclotomic character $\overline{\epsilon}$.
- The representation $\overline{\rho}$ is unramified outside places $v \mid p$.
- The representation $\overline{\rho}\big|_{G_{F(\zeta_p)}}$ is absolutely irreducible.
- The representation $\overline{\rho}\big|_{G_{F_v}}$ is finite flat for all $v \mid p$ (here G_{F_v} is the absolute Galois group of the completion F_v of F at v). Finite flat means that there is a finite flat group scheme $\mathcal{G}/\mathcal{O}_{F_v}$ such that

$$\overline{\epsilon} \otimes \overline{\rho}\big|_{G_{F_v}} \sim \mathcal{G}(\overline{F_v}).$$

  This condition is satisfied if for example $\overline{\rho}\big|_{G_{F_v}}$ is of the form

$$\begin{pmatrix} 1 & \star \\ 0 & \overline{\epsilon}^{-1} \end{pmatrix}$$

  split by $F_v(\mu_p, \sqrt[p]{u_1}, \cdots \sqrt[p]{u_k})$ for some $u_i \in \mathcal{O}_{F,v}^\times$.

Define $R_{\bar{\rho},Q}$ as in Misja's talk parametrizing deformations $\rho : G_{F,\Sigma_Q} \to \mathrm{Gl}_2(A)$ satisfying:

- The determinant $\det \rho = \epsilon^{-1}$.
- The restriction $\rho\big|_{G_{F_v}}$ is finite flat for all $v \mid p$, which means that $\rho \bmod \mathfrak{m}_A^n$ is finite flat in the previous sense, for all n.
- The representation ρ is unramified outside Σ_Q.

2 $\mathbb{T}$

Let $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{Gl}_{2/F}$ (or more generally consider multiplicative group associated to a quaternion algebra D over F ramified precisely at $2m$ Archimedean primes and no finite primes, in which case replace d by $d' := d - 2m$ below). Consider the associated symmetric space

$$Y(K_Q) = G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_f)/K_Q = \coprod_i \Gamma_i \setminus X$$

where $X = (\mathbb{H}^{\pm})^d$, $\mathbb{H}^{\pm}$ is the disjoint union of the upper and lower half plane and where K_Q is given by

$$\prod_{v \notin Q} \mathrm{Gl}_2(\mathcal{O}_{F_v}) \times \prod_{v \in Q} K_v$$

where K_v is to be described later. (Actually we need to fix some auxiliary level away from Q to make sure K_Q is neat or equivalently that Y_{K_Q} is smooth and that certain coverings will be étale.) Then $H^i(Y(K_Q), \mathcal{O})$ has a natural action of Hecke operators T_v for all places $v \notin Q$ and operators U_v for $v \in Q$. Let $\mathbb{T}_Q$ be the $\mathcal{O}$-subalgebra of

$$\mathrm{End}_{\mathcal{O}}(H^d(Y(K_Q), \mathcal{O})$$

generated by the operators T_v, U_v. There is a Hecke-equivariant map

$$S_{(2,\cdots,2)}(K_Q) \to H^d_{\mathrm{cusp}}(Y_{K_Q}, \mathbb{C}) \subset H^d(Y_{K_Q}, \mathbb{C})$$

where $S_{(2,\cdots,2)}(K_Q)$ is the space of parallel weight 2 Hilbert modular cuspforms of level K_Q. So if $f \in S_Q$ is an eigenform for the Hecke operators with eigenvalues in $\mathcal{O}_f \subset \mathbb{C}$ (and assume $\mathcal{O}_f \subset \mathcal{O}$ which can be achieved by enlarging $\mathcal{O}$), then we get a map

$$\begin{aligned} \mathbb{T}_Q &\to \mathcal{O} \\ T_v &\mapsto a_v \end{aligned}$$

where a_v is the T_v eigenvalue of f.

3 $R_Q \to \mathbb{T}_Q$

We start with a result due to Carayol and Taylor:

Theorem 1. *For f as above there is an associated Galois representation*

$$\rho_f : G_{F,\Sigma_Q} \to \mathrm{Gl}_2(\mathcal{O})$$

such that that for all $v \notin \Sigma_Q$ that characteristic polynomial of $\rho_f(\mathrm{Frob}_v)$ is

$$X^2 - a_v X + \mathrm{Nm}_{F/\mathbb{Q}}\, v.$$

Moreover the representation $\rho_f\big|_{G_{F_v}}$ is finite flat for all $v \mid p$ and the $\mathrm{Gl}_2(K)$ valued representation is irreducible.

Assume that $\overline{\rho} = \overline{\rho_f}$ for some f as above (in fact, then there is such an f with $Q = \emptyset$ by level lowering results of Jarvis, Rajaei, Fujiwara, Gee). Then for any such f we get a diagram

$$\begin{array}{ccccc} R_{\overline{\rho},Q} & \longrightarrow & \mathcal{O} & \longrightarrow & k \\ & \nearrow & & & \\ \mathbb{T}_Q & & & & \end{array}$$

such that $T_v \mapsto a_v \mapsto \mathrm{Tr}(\overline{\rho}(\mathrm{Frob}_v)$. This map does not depend on f and gives us a kernel $\mathfrak{m}_Q \subset \mathbb{T}_Q$. Using the fact that $\overline{\rho}$ is irreducible be know by Dimitrov (but it is easy if $d - 2m = 0, 1$) that

$$H^i(Y(K_Q), \mathcal{O})_{\mathfrak{m}_Q} = \begin{cases} 0 & \text{if } i \neq q \\ H^d_{\mathrm{cusp}}(Y(K_Q), \mathcal{O})_{\mathfrak{m}_Q} & \text{if } i = d \end{cases}$$

and moreover that this is torsionfree. But the Hecke algebra $\mathbb{T}_{Q,\mathfrak{m}_Q}$ acts faithfully on $H^d(Y(K_Q), \mathcal{O})_{\mathfrak{m}_Q}$, so in fact every map

$$\mathbb{T}_{Q,\mathfrak{m}_Q} \to \mathcal{O}$$

comes from f as above so we get a commutative diagram

$$\begin{array}{ccc} R_{\overline{\rho},Q} & \longrightarrow & \prod_f \mathcal{O}, \\ \big\downarrow\!\!\!\!\downarrow & \nearrow & \\ \mathbb{T}_{Q,\mathfrak{m}_Q} & & \end{array}$$

(using that $\mathbb{T}_{Q,\mathfrak{m}_Q} \to \prod_f \mathcal{O}$ is injective and $R_{\overline{\rho},Q}$ is generated by traces of images under the universal deformation).

4 Taylor-Wiles primes

Suppose that $v \in Q$ satisfy $\mathrm{Nm}(v) = 1 \mod p$ and that $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues in k. Then

$$\rho_Q^{\mathrm{univ}} : G_{F,\Sigma_Q} \to \mathrm{Gl}_2(R_{\overline{\rho},Q})$$

has the property that

$$\rho_Q^{\mathrm{univ}}|_{G_{F_v}} \sim \begin{pmatrix} \chi & 0 \\ 0 & \epsilon^{-1}\chi^{-1} \end{pmatrix}$$

where (here $I_{F,v} \subset G_{F_v}$ is the inertia group)

$$\chi|_{I_{F_v}}$$

factors as a character (the first map is local class field theory).

$$I_{F_v} \to \mathcal{O}_{F,v}^\times \to k_v^\times \to \Delta_v.$$

Where Δ_v is the maximal pro-p quotient of k_v^*, which is nontrivial because $\# k_v^*$ is divisible by p. All in all χ will determine a character $\Delta_v \to R_{\overline{\rho},Q}^\times$ leading to a ring homomorphism

$$\mathcal{O}[\Delta_Q] \to R_{\overline{\rho},Q}$$

where

$$\Delta_Q = \prod_{v \in Q} \Delta_v.$$

Recall that we haven't defined the groups K_v yet (for $v \in Q$). We define

$$K_{v,0} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod v$$

containing

$$K_v = \left\{ \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mod v \,\middle|\, ad^{-1} \mapsto 1 \in \Delta_v \right\}.$$

Note that we trivially get $K_v / K_{v,0} = \Delta_v$ for all $v \in Q$. So now we get a Δ_v torsor

$$\begin{array}{c} Y(K_Q) \\ \big\downarrow \\ Y(K_{Q,0}) \end{array}$$

inducing a map on cohomology

$$M_Q := H^d(Y(K_Q), \mathcal{O})_{\mathfrak{m}_Q} \to H^d(Y(K_{0,Q}, \mathcal{O})_{\mathfrak{m}_Q}.$$

This gives two actions of Δ_Q on the space of modular forms M_Q. One defined geometrically as above and one via

$$\mathcal{O}[\Delta_Q] \to R_{\overline{\rho},Q} \to \mathbb{T}_{Q,\mathfrak{m}_Q}.$$

The fact that the two actions coincide is local-global compatibility of Langlands correspondences at $v \in Q$. Furthermore M_Q is *free* over $\mathcal{O}[\Delta_Q]$ (this is a general fact about Galois covers and proved using the Hochschild-Serre spectral sequence).

5 Galois cohomology

Recall that $R^{\text{univ}}_{\overline{\rho},Q}$ is generated by

$$\dim_k H^1(G_{F,\Sigma_Q}, \operatorname{Ad}\overline{\rho})$$

generators as an $\mathcal{O}$-algebra. Once we start adding conditions we will need less generators:

- Fixing determinant $\det \rho = \epsilon^{-1}$ means we replace $\operatorname{Ad}\overline{\rho}$ with $\operatorname{Ad}^0\overline{\rho}$ (space of trace zero endomorphisms).
- The finite flat (think crystalline) condition tells us we get classes with image in a certain subspace

  $$L_v \subset H^1(G_{F,v}, \operatorname{Ad}^0\overline{\rho})$$

  of the local Galois cohomology groups.

Let us denote this subspace by H^1_Q and its dimension by r_Q. Local Tate duality tells us that

$$H^i(G_{F_v}, \operatorname{Ad}^0\overline{\rho}) \cong H^{2-i}(G_{F_v}, \operatorname{Ad}^0\overline{\rho}(1))^\vee.$$

For $i = 1$ the former space contains L_v and we let $L_v^\perp$ be its orthogonal complement, so we get Selmer groups (global classes that map to L_v or $L_v^\perp$ for all v)

$$H^1_Q, H^1_{Q^\perp}.$$

For $v \in Q$ we have that $L_v = H^1$ and $L_v^\perp = 0$, and in this case $\dim H^0 = 1$ and $\dim H^1 = 2$ by the local Euler characteristic formula

$$\dim H^1 - (\dim H^0 + \dim H^2) = \begin{cases} 0 & \text{if } v \nmid p \\ j \cdot 3 & \text{if } v \mid p \end{cases}$$

where $j = [F_v : \mathbb{Q}_p]$ and $3 = \dim \operatorname{Ad}^0$. For $v|p$, we have $\dim L_v = [F_v : \mathbb{Q}_p] + \dim H^0$. (This is harder and uses that p is unramified in F.) Now global duality and Euler characteristic computations gives us the formula of Wiles

$$\#H^1_Q/\#H^1_{Q^\perp} = \frac{\#H^0(G_F, \operatorname{Ad}^0\overline{\rho})}{\#H^0(G_f, \operatorname{Ad}^0\overline{\rho}(1))} \cdot \prod_v \frac{\#L_v}{\#H^0(G_{F,v}, \operatorname{Ad}^0\overline{\rho})}.$$

The first fraction on the right hand side is zero by our irreducibility assumption on $\overline{\rho}$. Putting everything together we get

$$\dim H^1_Q - \dim H^1_{Q^\perp} = \sum_{v|\infty}(-1) + \sum_v [F_v : \mathbb{Q}_p] + \#Q.$$

The 'numerical coincidence' occurs since we are working with Gl_2 and since F is totally real we see that

$$\sum_{v|\infty}(-1) = \sum_{v|p}[F_v : \mathbb{Q}_p]$$

since they are both just equal to the the degree of F over $\mathbb{Q}$. Let $r = \dim H^1_\emptyset$ and use the Chebotarev density theorem to choose for each n a set of Taylor-Wiles primes $Q_n = \{v_1, \cdots, v_r\}$ such that $\mathrm{Nm}(v) \equiv 1 \mod p^n$ and

$$H^1_{\emptyset^\perp}(G_F, \mathrm{Ad}^0 \overline{\rho}) \to \bigoplus_{v \in Q} H^1(G_{F_v}, \mathrm{Ad}^0 \overline{\rho}(1))$$

is an isomorphism. The idea for the proof is to realize this in terms of conditions on $\mathrm{Frob}_v \in \mathrm{Gal}(L_n/F)$ where $L_n = L(\zeta_{p^n})$ and L is the splitting field of $\mathrm{Ad}^0 \overline{\rho}$. Then

$$H^1_{Q_n^\perp} = 0$$

and so

$$\# H^1_{Q_n} = \# H^1_\emptyset$$

and so

$$R_{\overline{\rho}, Q_n}$$

is generated by $r = \# Q_n$ elements.

6 Patching

Now for each $n \geq 1$ we have a map

$$\mathcal{O}[\Delta_{Q_n}] \to R_{\overline{\rho}, Q_n} \to \mathbb{T}_{Q_n} \to \mathrm{End}(M_{Q_n})$$

which fits into a diagram as follows (taking compatible presentations of both $R_{\overline{\rho}, Q_n}$ and $\mathcal{O}[\Delta_{Q_n}]$):

$$\begin{array}{ccccccc} \mathcal{O}[\Delta_{Q_n}] & \longrightarrow & R_{\overline{\rho}, Q_n} & \twoheadrightarrow & \mathbb{T}_{Q_n} & \longrightarrow & \mathrm{End}(M_{Q_n}) \\ \twoheaduparrow & & \twoheaduparrow & & & & \\ \mathcal{O}[\![S_1, \cdots, S_r]\!] & \xrightarrow{\exists} & \mathcal{O}[\![T_1, \cdots, T_r]\!] & & & & \end{array}$$

with $M_{Q_n}/(S_1, \ldots, S_r) = M_\emptyset$.

Now we essentially want to take the limit over n. Since there are only finitely many such data mod $\mathfrak{m}^n$ there is a compatible subsequence and we can take limits to obtain

$$\mathcal{O}[\![S_1, \cdots, S_r]\!] \to \mathcal{O}[\![T_1, \cdots, T_r]\!] \to R_\infty \twoheadrightarrow \mathbb{T}_\infty \twoheadrightarrow \mathrm{End}(M_\infty)$$

with $M_\infty/(S_1, \ldots, S_r) = M_\emptyset$. Moreover we know that M_∞ is free over the first ring (lets call it A and lets call the second ring B), we want to show that M_∞ is also free over B: The Auslander-Buchsbaum formula tells us that

$$\mathrm{depth}_B M_\infty + \mathrm{proj}\dim_B M_\infty = r + 1$$

and since $S_1, \cdots, S_r, \varpi_{\mathcal{O}}$ is a regular sequence we know that

$$\operatorname{depth}_B M_\infty = r + 1.$$

Hence the projective dimension is zero, i.e., M_∞ is free over B. This means that M_∞ is a faithful B-module and so the map from B to $\operatorname{End}_{M_\infty}$ is injective. This tells us immediately that

$$B \cong R_\infty \cong \mathbb{T}_\infty$$

over which M_∞ is free. Going back down this gives $R = \mathbb{T}$.

Remark 1. You don't actually need to patch. See the paper by Brochard (Compositio, 2017) which proves a commutative result that implies you only need Q_1.