Notes on simplicial objects and things

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Introduction

These notes were produced while preparing a talk for the LNTSG on the derived structures in the Langlands programme. They are not meant to be in anyway comprehensive. They should serve as a quick survey of basic definitions and results. For a more extensive discussion we refer the reader to [GJ09].

1 Basic definitions

Definition 1.1 (Simplex category)**.** *The* simplex category, *denoted by* $\underline{\Delta}$, *is the category with*

- *the totally ordered sets (tosets)*

$$[n] = \{0 < 1 < 2 < \cdots < n\}, \quad \forall n \geqslant 0,$$

as objects

- *and the order preserving maps, that is, maps of posets, as morphisms.*

Remark 1.2. Notice that the tosets $[n]$, as all posets, can be considered categories themselves. In this sense, $\underline{\Delta}$ can be thought of as a subcategory of $\underline{\text{Cat}}$, the category of all small categories. This point of view can sometimes be helpful, see Example 2.5 below.

Lemma 1.3 (Combinatorial lemma)**.** *Any morphism* $f\colon [n] \to [m]$ *in* $\underline{\Delta}$ *can be written as the composition of maps of two kinds:*

- *the* coface maps $\delta^i\colon [n-1] \to [n]$, *"missing* i*",* $0 \leqslant i \leqslant n$*,*

- *the* codegeneracy maps $\sigma^i\colon [n] \to [n-1]$ *sending* $i, i+1$ *to* i, $0 \leqslant i < n$.

Sketch of proof. We won't include a full proof, but we remark that it can be obtained by *enumerating* the image of the map f. ☐

Definition 1.4 (Simplicial object). *A* simplicial object *in a small category* $\mathcal{C}$ *is a functor* $\underline{\Delta}^{op} \to \mathcal{C}$.

Notation 1.5. We will usually write

$$\mathrm{s}\mathcal{C} = \mathrm{Fct}(\underline{\Delta}^{\mathrm{op}}, \mathcal{C})$$

for the category of simplicial objects in $\mathcal{C}$.

Notice that giving a simplicial object $C \in \mathrm{s}\mathcal{C}$ is the same as giving:

1. a sequence of objects $\{C_n\}_{n\geqslant 0}$ in $\mathcal{C}$,
2. *face maps* $d_i\colon C_n \to C_{n-1}$ and *degeneracy maps* $s_i\colon C_{n-1} \to C_n$.

2 Important examples

We are mainly interested in simplicial sets, that is, elements of $\mathrm{s}\underline{\mathrm{Set}}$.

Example 2.1 (Standard n-simplex). The *standard n-simplex* Δ^n is the simplicial set given by

$$\Delta^n = \mathrm{Hom}_{\underline{\Delta}}(\cdot, [n]) = h_{[n]}\colon \underline{\Delta}^{\mathrm{op}} \to \underline{\mathrm{Set}}.$$

We remark that, by Yoneda's Lemma, for any $S \in \mathrm{s}\underline{\mathrm{Set}}$ we have

$$S_n = \mathrm{Hom}_{\mathrm{s}\underline{\mathrm{Set}}}(\Delta^n, S).$$

Example 2.2 (Boundary of the n-simplex). The *boundary* $\partial\Delta^n$ of Δ^n is the subobject of Δ^n generated by the $(n-1)$-simplices $\delta^k\colon [n-1] \to [n]$, for all k. Notice that $\mathrm{id}_{[n]} \notin (\partial\Delta^n)_n$, so the boundary is *strictly* contained in Δ^n.

Example 2.3 (i^{th} Horn). The i^{th} *horn* Λ^n_i of the standard simplex Δ^n is the smallest subobject, that is, subfunctor, of Δ^n containing the $(n-1)$-simplices

$$\delta^k \in \mathrm{Hom}_{\underline{\Delta}}([n-1], [n]), \quad k \neq i.$$

Notice that $\delta^i \notin (\Lambda^n_i)_{n-1}$, so, in particular, Λ^n_i is strictly contained in $\partial\Delta^n$, so we have the strict inclusions of simplicial sets

$$\Lambda^n_i \subset \partial\Delta^n \subset \Delta^n.$$

Example 2.4 (The singular simplicial set). Let $X \in \underline{\mathrm{Top}}$ be any topological space. The *singular simplicial set* $S(X)$, sometimes $\mathrm{Sing}(X)$, is defined by setting

$$S(X)_n := \mathrm{Hom}_{\underline{\mathrm{Top}}}(|\Delta^n|, X),$$

where $|\Delta^n| = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geqslant 0, \sum_{i=0}^n x_i = 1\}$ is the *geometric n-simplex*. Any morphism $f\colon [n] \to [m]$ defines a natural map between the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, which can be extended by linearity and then restricted to $|\Delta^n| \to |\Delta^m|$. These give the face and degeneracy maps on $S(X)$ by precomposition.

Example 2.5 (Nerve of a category). The *nerve* $\mathrm{N}\mathcal{C}$ of a small category $\mathcal{C}$ is the simplicial set defined by

$$\begin{aligned}(\mathrm{N}\mathcal{C})_n &= \mathrm{Fct}([n], \mathcal{C}) = \mathrm{Hom}_{\underline{\mathrm{Cat}}}([n], \mathcal{C}) = h_{\mathcal{C}}([n]) \\ &= \{c_0 \to c_1 \to \cdots \to c_n,\ c_i \in \mathcal{C}\}.\end{aligned}$$

Face and degeneracy maps are again defined by precomposition and, if need be, can be described explicitly.

3 Geometric realisation

We start this section with two lemmata.

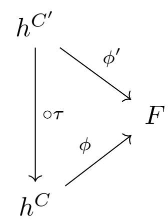
Lemma 3.1. *Let $\mathcal{C}, \mathcal{D}$ be two categories. Suppose that $\mathcal{C}$ is small. If $\mathcal{D}$ admits small (co)limits, then so does* $\mathrm{Fct}(\mathcal{C}, \mathcal{D})$.

Proof. We prove the statement for colimits only, but the proof in the case of limits is analogous. Let $(F_i)_{i\in I}$ be functors indexed by $I \to \mathrm{Fct}(\mathcal{C}, \mathcal{D})$, with I small. Then $\left(\varinjlim_I F_i\right)(C) = \varinjlim_I (F_i(C)) \in \mathcal{D}$, for all $C \in \mathcal{D}$. ☐

Lemma 3.2. *Let $F\colon \mathcal{C} \to \underline{\mathrm{Set}}$ be a functor from a small category $\mathcal{C}$. Consider the small category*

$$\mathcal{D}_F = \{(C, \phi) \mid C \in \mathcal{C}, \phi\colon h^C = \mathrm{Hom}_{\mathcal{C}}(C, \cdot) \longrightarrow F\},$$

with morphisms given by those $C \xrightarrow{\tau} C'$ of $\mathcal{C}$ such that



commutes. Then the natural map

$$\varinjlim h^C \longrightarrow F$$

is an isomorphism, the colimit on the left being indexed by $\mathcal{D}_F$.

Sketch of proof. Everything boils down to the fact that, by Yoneda, we have the bijection

$$\mathrm{Hom}(h^C, F) \xrightarrow{\sim} F(C).$$

Therefore, whenever we are given another functor $G\colon \mathcal{C} \longrightarrow \underline{\mathrm{Set}}$ that can take the place of F in the diagram above, we immediately get maps $F(C) \to G(C)$ for any $C \in \mathcal{C}$, naturally in C, that is, inducing $F \longrightarrow G$ uniquely. ☐

Remark 3.3. Lemma 3.2 can be described by the following slogan: gluing the C-points of F, for all $C \in \mathcal{C}$, one gets F back.

Notice that by taking $\mathcal{C} = \underline{\Delta}^{\mathrm{op}}$ in Lemma 3.2 we get that

$$\varinjlim_{\Delta^n \to S} \Delta^n \xrightarrow{\sim} S,$$

for any $S \in \mathrm{s}\underline{\mathrm{Set}}$.

Definition 3.4 (Geometric realisation)**.** *The* geometric realisation *of a simplicial set S is defined as the colimit*

$$|S| := \varinjlim_{\Delta^n \to S} |\Delta^n| \in \underline{\mathrm{Top}}.$$

This defines a functor $|\cdot|\colon \mathrm{s}\underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Top}}$.

Example 3.5 (or rather an observation)**.** Since, by Yoneda again, the object $([n], \mathrm{id}_{[n]})$ is final in the index category $\mathcal{D}_{\Delta^n}$, we have

$$\varinjlim_{\Delta^m \to \Delta^n} |\Delta^m| = |\Delta^n|,$$

which also shows that the notation $|S|$ for the geometric realisation is compatible with the definition of geometric n-simplex given above.

Proposition 3.6 (Adjunction)**.** *We have*

$$\mathrm{Hom}_{\underline{\mathrm{Top}}}(|T|, Y) \cong \mathrm{Hom}_{\mathrm{s}\underline{\mathrm{Set}}}(T, S(Y)),$$

naturally in $T \in \mathrm{s}\underline{\mathrm{Set}}, Y \in \underline{\mathrm{Top}}$.

Proof. It is just a formal consequence of the definitions and the properties of limits and colimits.

$$\begin{aligned}
\mathrm{Hom}_{\underline{\mathrm{Top}}}(|S|, Y) &= \mathrm{Hom}_{\underline{\mathrm{Top}}}(\varinjlim_{\Delta^n \to T} |\Delta^n|, Y) \\
&\cong \varprojlim_{\Delta^n \to T} \mathrm{Hom}_{\underline{\mathrm{Top}}}(|\Delta^n|, Y) \\
&= \varprojlim_{\Delta^n \to T} \mathrm{Hom}_{\mathrm{s}\underline{\mathrm{Set}}}(\Delta^n, S(Y)) \\
&\cong \mathrm{Hom}_{\mathrm{s}\underline{\mathrm{Set}}}(\varinjlim_{\Delta^n \to T} \Delta^n, S(Y)) \\
&= \mathrm{Hom}_{\mathrm{s}\underline{\mathrm{Set}}}(T, S(Y)).
\end{aligned}$$

□

Remark 3.7. One can make the description of $|S|$ more explicit, presenting it as some quotient

$$|S| = \bigsqcup_{n \geqslant 0} S_n \times |\Delta^n| \big/_{\sim},$$

where the equivalence relation $\sim$ is "generated" by the identifications induced by face and degeneracy maps. This perhaps resembles more closely the intuitive idea of the geometric realisation as a "gluing-up of simplices".

4 Kan fibrations

Definition 4.1 (Kan fibration)**.** *A map* $f\colon S \to T$ *in* $\mathrm{s}\underline{\mathrm{Set}}$ *is called a* (Kan) fibration *if it satisfies the right lifting property (RLP) described by the following commutative diagram:*

$$\begin{array}{ccc}
\Lambda^n_i & \longrightarrow & S \\
\big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow f \\
\Delta^n & \longrightarrow & T.
\end{array}$$

Remark 4.2. Recall that, according to one out of many equivalent definitions, a *Serre fibration* is a continuous map $f\colon X \to Y$ of topological spaces with the RLP given by:

$$\begin{array}{ccc}
|\Lambda^n_i| & \longrightarrow & X \\
\big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow f \\
|\Delta^n| & \longrightarrow & Y.
\end{array}$$

So, by the adjunction 3.6, $f\colon X \to Y$ is a Serre fibration if and only if $S(f)\colon S(X) \to S(Y)$ is a Kan fibration.

Notation 4.3. We will use $\bullet$ to denote both the simplicial set Δ^0 and the "one point" topological space.

Lemma 4.4. *For any topological space* X*, the unique map* $S(X) \to \bullet = \Delta^0 = S(\bullet)$ *is a fibration.*

Proof. If, using adjunction, we work in $\underline{\mathrm{Top}}$, then we only need to prove that we can find the diagonal map in the diagram

$$\begin{array}{ccc} |\Lambda^n_i| & \longrightarrow & X \\ \Big\downarrow & \overset{\exists ?}{\nearrow} & \Big\downarrow f \\ |\Delta^n| & \longrightarrow & \bullet, \end{array}$$

but the existence of such a map follows at once from the fact that $|\Lambda^n_i|$ is a strong deformation retract of $|\Delta^n|$. □

This leads to a definition whose importance will become clear in the discussion on simplicial homotopy.

Definition 4.5 (Kan complex)**.** *A simplicial set is called* fibrant*, or a* Kan complex*, if $S \to \bullet$ is a fibration.*

Remark 4.6 (Fibrations & pull-backs)**.** Notice that, whenever we have a Cartesian diagram

$$\begin{array}{ccc} F & \longrightarrow & S \\ \Big\downarrow & & \Big\downarrow f \\ \bullet & \longrightarrow & T, \end{array}$$

if f is a fibration, then F (the fibre) is fibrant, as one can verify immediately using the universal property of the pull-back and and the RLP of fibrations. More generally, the family of Kan fibrations is closed under pull-backs.

5 Self-enrichment

Remark 5.1. One of the implicit consequences of Lemma 3.1, which is worth spelling out here, is that one can define (fibred) products of simplicial sets in a rather natural way, that is, for $S, T \in \mathrm{s}\underline{\mathrm{Set}}$ we have

$$(S \times T)_n = S_n \times T_n,$$

with face and degeneracy maps defined componentwise.

Let S, T be two simplicial sets. We can define a simplicial object $\underline{\mathrm{Hom}}(S,T)$, called the *function complex*, by setting

$$\underline{\mathrm{Hom}}(S,T)_n \coloneqq \mathrm{Hom}_{\mathrm{s}\underline{\mathrm{Set}}}(S \times \Delta^n, T).$$

Given any $f\colon \Delta^m \to \Delta^n$, we can define the corresponding morphism $\underline{\mathrm{Hom}}(S,T)_n \to \underline{\mathrm{Hom}}(S,T)_m$ by pre-composition with $\mathrm{id} \times f\colon S \times \Delta^m \to S \times \Delta^n$.

One interesting thing about this simplicial set is that it comes with a natural *evaluation map* defined by

$$\begin{aligned} \mathrm{ev}\colon S \times \underline{\mathrm{Hom}}(S,T) &\longrightarrow T, \\ (s_n, f_n)_{n \geqslant 0} &\longmapsto ((f_n)_n(s_n, \mathrm{id}_{[n]}))_{n \geqslant 0}. \end{aligned}$$

One can check that this is actually a morphism of simplicial sets and that the following holds.

Proposition 5.2 (Exponential law)**.** *Let U be another simplicial set. The function*

$$\begin{aligned} \mathrm{ev}_*\colon \mathrm{Hom}_{\mathrm{s}\underline{\mathrm{Set}}}(U, \underline{\mathrm{Hom}}(S,T)) &\longrightarrow \mathrm{Hom}_{\mathrm{s}\underline{\mathrm{Set}}}(S \times U, T), \\ g &\longmapsto (S \times U \xrightarrow{\mathrm{id} \times g} S \times \underline{\mathrm{Hom}}(S,T) \xrightarrow{\mathrm{ev}} T), \end{aligned}$$

is a bijection which is natural in all of S, T and U.

Proof. One can verify that inverse of ev_* is defined by sending a map $f\colon S \times U \to T$ to $f_*\colon U \to \underline{\mathrm{Hom}}(S,T)$ which, for any $x\colon \Delta^n \to U$, acts as

$$f_*(x) \mapsto (S \times \Delta^n \xrightarrow{\mathrm{id}\times x} S \times U \xrightarrow{f} T).$$

$\square$

We state without proof an interesting consequence of the Exponential law, which we will use later.

Lemma 5.3 ([GJ09, Cor. I.5.3])**.** *If $U \subseteq S$ and T is fibrant, then the induced map $\underline{\mathrm{Hom}}(S,T) \to \underline{\mathrm{Hom}}(U,T)$ is a fibration.*

6 Simplicial homotopy

Definition 6.1 (Homotopy in $\mathrm{s}\underline{\mathrm{Set}}$)**.** *Let $f, g\colon S \to T$ be two morphisms in $\mathrm{s}\underline{\mathrm{Set}}$. We say that f and g are* homotopic *if we can find a morphism $h\colon S \times \Delta^1 \to T$ that fits in the following diagram*

$$\begin{array}{ccc} S \times \Delta^0 \cong S & & \\ \downarrow{\scriptstyle \mathrm{id}\times\delta^1} & \searrow^{f} & \\ S \times \Delta^1 & \overset{\exists h}{\dashrightarrow} & T \\ \uparrow{\scriptstyle \mathrm{id}\times\delta^0} & \nearrow_{g} & \\ S \times \Delta^0 \cong S. & & \end{array}$$

One denotes this relation between f and g as $f \sim g$. Moreover, if $U \subseteq S$ is a sub-simplicial set and $f|_U = g|_U$, we say that f and g are homotopic relative to U, *denoted $f \sim g\,(\mathrm{rel}\,U)$, if the map h described before also makes the following diagram commute:*

$$\begin{array}{ccc} S \times \Delta^1 & \xrightarrow{h} & S \\ {\scriptstyle \iota\times\mathrm{id}_{\Delta^1}}\uparrow & & \uparrow{\scriptstyle \alpha} \\ U \times \Delta^1 & \xrightarrow{\pi_U} & U, \end{array}$$

where $\alpha = f|_U = g|_U$.

Notice that, in general, being homotopic is *not* an equivalence relation.

Lemma 6.2. *Suppose that $S \in \mathrm{s}\underline{\mathrm{Set}}$ is fibrant. Then the simplicial homotopy of vertices $\Delta^0 \to S$ is an equivalence relation.*

Sketch of proof. Let $x, y\colon \Delta^0 \to S$ be two vertices. First notice that, rewriting the definition, $x \sim y$ if and only if there is some $\gamma\colon \Delta^1 \to S$ such that $d_1\gamma = x$, $d_0\gamma = y$.

We prove that $\sim$ is a symmetric relation. Suppose that $x \sim y$ and let γ be the path we just described. Then maps

$$\Lambda^2_0 \longrightarrow S$$

are in bijection with couples of 1-simplices (γ_0, γ_1) of S sharing a common vertex, that is, $d_1\gamma_0 = d_0\gamma_1$. The couple (s_0x, γ) is one of these and since S is fibrant we can lift the corresponding morphism to a 2-simplex $f\colon \Delta^2 \to S$ such that $d_2 f = s_0x$, $d_1 f = \gamma$. One can check that $d_0 f\colon \Delta^1 \to S$ gives the homotopy relation $y \sim x$. $\square$

Corollary 6.3. *Let $T \in \mathrm{s}\underline{\mathrm{Set}}$ be fibrant and $U \subseteq S$ be an inclusion in $\mathrm{s}\underline{\mathrm{Set}}$. Then:*

1. *homotopy of maps $S \to T$ is an equivalence relation,*

2. *homotopy of maps* $S \to T\,(\operatorname{rel} U\,)$ *is an equivalence relation.*

Sketch of proof. It is enough to prove 2. To do that we point out that an homotopy of maps $S \to T\,(\operatorname{rel} U\,)$ corresponds, via the Exponential law 5.2, to an homotopy of vertices in the fibres of the Kan fibration (Lemma 5.3)

$$\underline{\operatorname{Hom}}(S,T) \longrightarrow \underline{\operatorname{Hom}}(U,T).$$

So, to conclude, it is enough to use Remark 4.6 and Lemma 6.2. □

Fix now a simplicial set S and a vertex $v\colon \Delta^0 \to S$.

Definition 6.4 (n^{th} Homotopy group). *Let* $\pi_n(S,v)$ *denote, for* $n > 0$, *the set of the homotopy classes of maps* $\alpha\colon \Delta^n \to S\,(\operatorname{rel} \partial\Delta^n\,)$ *that fit into the commutative diagram*

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\alpha} & S \\ \uparrow & & \uparrow{\scriptstyle v} \\ \partial\Delta^n & \longrightarrow & \bullet. \end{array}$$

We also write $\pi_0(S,v) = \{\Delta^0 \to S\}/\sim$.

Assume for a moment that $n \geqslant 1$ and let $\alpha, \beta\colon \Delta^n \to S$ represent two classes in $\pi_n(S,v)$. A construction similar to the one used in the proof of Lemma 6.2, gives us another $\omega\colon \Delta^n \to S$, as a face of an $(n+1)$-simplex in S having α, β and some degenerations of v as other faces. One can prove that this construction gives a well defined element $[\omega]_\sim \in \pi_n(S,v)$ which does not depend on the representatives α and β in $\pi_n(S,v)$. In short, we have a binary operation

$$\pi_n(S,v) \times \pi_n(S,v) \xrightarrow{*} \pi_n(S,v).$$

We conclude this section with the following fundamental result.

Theorem 6.5 ([GJ09, Thm. I.7.2])**.** *The couple* $(\pi_n, *)$ *is a group for* $n \geqslant 1$, *which is moreover abelian for* $n \geqslant 2$.

7 Some remarks on simplicial groups

We begin by mentioning, without proof, the following.

Lemma 7.1 (Moore, [GJ09, Lem. I.])**.** *The underlying simplicial set of any simplicial group G is fibrant.*

This, in particular, implies that for any simplicial group G, with $1 \in G_0$ the identity, it makes sense to consider the homotopy groups $\pi_n(G,1)$.

If A is a simplicial *Abelian* group, we let

$$(\mathrm{N}A)_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subseteq A_n.$$

Because of the identity $d_{n-1}d_n = d_{n-1}d_{n-1}$, the maps

$$(\mathrm{N}A)_n \xrightarrow{(-1)^n d_n} (\mathrm{N}A)_{n-1}$$

make $(\mathrm{N}A)_\bullet$ into a complex of Abelian groups. This defines a functor

$$\mathrm{s}\underline{\mathrm{Ab}} \xrightarrow{\mathrm{N}} \mathrm{Ch}_+(\underline{\mathrm{Ab}}).$$

Proposition 7.2 (Dold-Kan correspondence, [GJ09, Cor. III.2.3]). *The functor N induces an equivalence of categories.*

Given $A \in \mathrm{s}\underline{\mathrm{Ab}}$ one can also define another complex by taking

$$A_n \xrightarrow{\partial} A_{n-1},$$

with $\partial = \sum_{i=0}^{n}(-1)^i d_i$. The two constructions are not unrelated and in fact we have the following.

Proposition 7.3 ([GJ09, Thm. III.2.4, 2.5]). *The natural inclusion of complexes $\mathrm{N}A_\bullet \to A_\bullet$ is a chain-homotopy equivalence, which is natural in A.*

Moreover, we have natural isomorphisms

$$\pi_n(A, 0) \cong H_n(\mathrm{N}A_\bullet) \cong H_n(A_\bullet)$$

for any $n \geqslant 0$.

Remark 7.4. Notice that the simplicial group structure of A induces an alternative group operation on $\pi_n(A, 0)$ with the same identity as $*$, the vertex 0. One can prove that these two operations are naturally compatible, hence equal (this is known as the "Eckmann-Hilton argument"). This group structure can also be defined on $\pi_0(A)$ and it fits naturally in the isomorphisms described by Proposition 7.3 for $n = 0$.

References

[GJ09] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Reprint of the 1999 edition [MR1711612]. Birkhäuser Verlag, Basel, 2009, pp. xvi+510.