DERIVED DEFORMATION THEORY IN GENERAL

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A talk in the Derived Structures in the Langlands Program study group at UCL in Spring 2019. These are notes taken by Ashwin Iyengar (ashwin.iyengar@kcl.ac.uk).

1. Introduction

Last week, Raffael told us about simplicial Artinian rings (category sArt_k). He wrote down what it means for a functor

$$F : \mathrm{sArt}_k \to \mathrm{sSet}$$

to be pro-representable.

This week is more or less about working out whether a given functor is pro-representable. So we will talk about a *derived version of Schlessinger's criterion.*

So the plan is

(1) A bit of extra background (homotopy limits/colimits) which we will need.

(2) Derived Schlessinger's criterion, and how to formulate it.

(3) Some examples of functors F.

(4) Sketch of proof of the derived Schlessinger's criterion.

2. Homotopy Limits and Colimit

Let C be a nice model category and I be a small category. Then let C^I denote the category of functors $I \to C$, i.e. the category of diagrams of shape I. Then there is also a model structure on C^I (in fact there are several) such that the map

$$\mathrm{colim} : C^I \to C$$

is a left Quillen functor.

That means there is a left derived functor $\mathrm{hocolim} : C^I \to C$, which is "well-defined up to homotopy".

Dually, there's another model structure with the same weak equivalences (but different fibrations and cofibrations), such that $\lim : C^I \to C$ is right Quillen, with derived functor

$$\mathrm{holim} : C^I \to C.$$

We won't need this much generality though: instead, we focus on the following special cases:

(1) Homotopy pullback. Here we take I to be the category

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

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So given a diagram $Z \to Y \leftarrow X$, we write $X \times^h_Y Z$ for the homotopy limit. Then given a diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

we say that W is a homotopy pullback if $W \to X \times^h_Y Z$ is a weak equivalence. But actually the situation simplifies if X, Y, Z are all fibrant and $X \to Y$ is a fibration: in this case

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

is a homotopy pullback already.

(2) If R is a commutative ring, then there is a nice model structure on $\mathrm{Ch}(R)$ (unbounded chain complexes of R-modules) such that a weak equivalence is a quasi-isomorphism. Suppose $X \in \mathrm{sSet}$ and

$$D : \mathrm{Simp}(X) \to \mathrm{Ch}(R)$$

is a diagram (where $\mathrm{Simp}(X)$ is the category of simplices of X whose objects are maps $\Delta^n \to X$ and whose morphisms are maps $\Delta^n \to \Delta^m$ respecting the map to X), then the homotopy limit is $\mathrm{holim}(D) \cong C^*(X, D)$, where

$$C^n(X, D) = \prod_{p+q=n, \Delta^p \to X} D(\sigma)^q$$

and $(d\alpha)_{p,q,\sigma} = (-1)^p d(\alpha_{p,q-1,\sigma}) + \sum_{i=0}^p (-1)^i d^i \alpha_{p-1,q,d_i\sigma}$.

So for example if $D(\sigma) = R$, then $C^*(X, D) = C^*(X, R)$ (ordinary singular cochains).

2.1. **Facts.** If $F : C \to D$ is a right/left Quillen functor, then F preserves homotopy limits/homotopy colimits. For example,

$$\underline{\mathrm{Hom}}_{\mathrm{sSet}}(-, -) : \mathrm{sSet}^{\mathrm{op}} \times \mathrm{sSet} \to \mathrm{sSet}$$

sends hocolims in the first factor and holims in the second factor to holims in sSet.

3. Derived Schlessinger Criterion

Definition 3.1. Let $F : \mathrm{sArt}_k \to \mathrm{sSet}$ be a homotopy invariant functor. We say that F *preserves pullbacks* if for every homotopy pullback square

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & A \end{array}$$

such that $B \to A$ is surjective, then

$$\begin{array}{ccc} F(D) & \longrightarrow & F(B) \\ \downarrow & & \downarrow \\ F(C) & \longrightarrow & F(A) \end{array}$$

is a homotopy pullback square. We say that F is *formally cohesive* if F preserves pullbacks and $F(k)$ is contractible.

Formally cohesive functors have a good theory of tangent spaces (this is analogous to the underived setting: recall the conditions for $F(k[\epsilon]/\epsilon^2)$ to have a canonical k-vector space structure if $F : \mathrm{Art}_k \to \mathsf{Set}$ is a classical deformation functor).

Recall: There is an equivalence of categories

$$DK : \mathrm{Ch}_{\geq 0}(k) \to \mathrm{sVect}$$

given by the Dold-Kan correspondence. Furthermore this respects the homotopy theory, i.e.

$$\pi_n(DK(V)) = H_n(V).$$

Let $\mathrm{Ch}_{\geq 0}(k)^{fd} \subseteq \mathrm{Ch}_{\geq 0}(k)$ be the full subcategory of V such that $\dim H_*(V) \leq \infty$. If $V \in \mathrm{Ch}_{\geq 0}(k)^{fd}$ then define

$$k \oplus V := k \oplus DK(V)$$

with product $(a, u) \cdot (b, v) = (ab, av + bu)$. In fact, this thing is an object of sArt_k (note $k \oplus k = k[\epsilon]/\epsilon^2$, the ring of dual numbers in the classical, underived setting).

Proposition 3.2. *Let $F : \mathrm{sArt}_k \to \mathrm{sSet}$ be a formally cohesive functor. Then there exists some complex $\mathfrak{t}F \in \mathrm{Ch}(k)$, unique up to quasi-isomorphism, together with a natural weak equivalence*

$$F(k \oplus V) \xrightarrow{\sim} DK(\tau_{\geq 0}(\mathfrak{t}F \otimes V)).$$

of functors $\mathrm{Ch}_{\geq 0}(k)^{fd} \to \mathrm{sSet}$.

Definition 3.3. We call $\mathfrak{t}F$ the *tangent complex* of F.

Why does this make sense? Note there is a weak equivalence

$$F(k \oplus k[n]) \xrightarrow{\sim} DK(\tau_{\geq 0}(\mathfrak{t}F \otimes k[n])),$$

so $\pi_i F(k \oplus k[n]) = H_{i-n}(\mathfrak{t}F)$. In particular,

$$H_0(\mathfrak{t}F) = \pi_0(F \oplus k) = \pi_0(F(k[\epsilon]/\epsilon^2)).$$

Remark 3.4 (GV)**.** Galatius and Venkatesh don't state Proposition 3.2 (they use spectra instead, which are not allowed in this study group) but they more or less prove it by constructing $\mathfrak{t}F$ directly.

Now we may state the main theorem.

Theorem 3.5 (Basically due to Lurie, Derived Schlessinger)**.** *Let $F : \mathrm{sArt}_k \to \mathrm{sSet}$ be homotopy invariant. Then F is pro-representable if and only if*

(1) *F is formally cohesive, and*
(2) *$H_n(\mathfrak{t}F) = 0$ for $n > 0$.*

Moreover, if $H_*(\mathfrak{t}F)$ has countable dimension, then you can take a countable indexing category for the representing pro-object.

Slogan:

$$\{\text{pro-representable functors}\} \subseteq \{\text{formally cohesive functors}\}$$

is like

$$\{\text{schemes}\} \subseteq \{\text{stacks}\}.$$

I.e. the homological vanishing condition in Theorem 3.5 should be thought of as having no infinitesimal automorphisms.

4. Examples

Here's an example. Let $R \in \mathrm{sCR}$ be a cofibrant simplicial ring with a map $R \to k$. Then we can define

$$F_R : \mathrm{sArt}_k \to \mathrm{sSet}, \quad F_R(A) = \underline{\mathrm{Hom}}_{\mathrm{sCR}\,/k}(R, A).$$

This is formally cohesive because $\underline{\mathrm{Hom}}$ preserves limits in the second variable (as stated previously), so in particular it preserves pullbacks. Furthermore, $F_R(k) = \bullet$ by definition, so it's contractible.

The tangent complex is is

$$\mathfrak{t}D = \mathrm{DK}(\Omega^1_{R/\mathbf{Z}} \otimes_R k)^\vee.$$

This has homology only in negative degrees because we took the dual of something which had nonzero homology in positive degrees, and the fact that homotopy groups are the homology groups under Dold-Kan. Therefore, F_R is pro-representable by the "completion of R".

If $\pi_n(R) = 0$ for $n > 0$, and $\pi_0(R)$ is formally smooth over $\mathbf{Z}$, then in fact

$$\mathfrak{t}F_R = (\Omega^1_{\pi_0(R)/\mathbf{Z}} \otimes_{\pi_0(R)} k)^\vee,$$

the usual tangent space of R. On the other hand, if $\pi_0(R)$ is not formally smooth, then you need to replace $\Omega^1_{\pi_0(R)/\mathbf{Z}}$ with the cotangent complex of R.

Here's another example. If F, F', F'' are formally cohesive and

$$\begin{array}{ccc} G & \longrightarrow & F' \\ \downarrow & & \downarrow \\ F'' & \longrightarrow & F \end{array}$$

is a homotopy pullback, then G is formally cohesive. What about the tangent complex? For $V \in \mathrm{Ch}_{\geq 0}(k)^{fd}$,

$$\begin{aligned} G(k \oplus V) &= F'(k \oplus V) \times^h_{F(k \oplus V)} F''(k \oplus V) \\ &= DK(\tau_{\geq 0}(\mathfrak{t}F' \otimes V)) \times^h_{DK(\tau_{\geq 0}(\mathfrak{t}F \otimes V))} DK(\tau_{\geq 0}(\mathfrak{t}F'' \otimes V)) \\ &= DK(\tau_{\geq 0}((\mathfrak{t}F' \times^h_{TF} \mathfrak{t}F'') \otimes V)) \end{aligned}$$

So $\mathfrak{t}G = \mathfrak{t}F' \times^h_{\mathfrak{t}F} \mathfrak{t}F''$.

Now take resolutions so that $\mathfrak{t}F' \to \mathfrak{t}F$ is surjective (hence a fibration). Then we have a short exact sequence

$$0 \to \mathfrak{t}G \to \mathfrak{t}F' \oplus \mathfrak{t}F'' \to \mathfrak{t}F \to 0,$$

with associated long exact sequence

$$\cdots \to H_{n+1}(\mathfrak{t}F) \to H_n(\mathfrak{t}G) \to H_n(\mathfrak{t}F') \oplus H_n(\mathfrak{t}F'') \to \cdots.$$

So if F, F', F'' pro-representable, then so is G (by Theorem 3.5).

In particular this gives us a way to talk about the completed tensor product without computing it explicitly. I.e. if F, F', F'' are pro-represented by $R = (R_j)_{j \in J}$ $R' = (R'_j)_{j \in J'}$, $R'' = (R''_j)_{j \in J''}$, then G is pro-represented by $R \underline{\otimes}_R R''$ (if you read Galatius and Venkatesh, they construct an explicit pro-system realizing this).

Let $F : \mathrm{sArt}_k \to \mathrm{sSet}$ be a functor preserving pullbacks, but could have $F(k)$ not contractible. Now let X be a simplicial set, and let

$$\overline{\rho} : X \to F(k)$$

be some map. Now define a functor

$$F_{X,\overline{\rho}} : \mathrm{sArt}_k \to \mathrm{sSet}$$

by $F_{X,\overline{\rho}}(A) = \underline{\mathrm{Hom}}(X, F(A)) \times^h_{\underline{\mathrm{Hom}}(X, F(k))} \{\overline{\rho}\}$. Then $F_{X,\overline{\rho}}$ is formally cohesive. But now how do we compute $\mathfrak{t}F_{X,\overline{\rho}}$?

If $\sigma : \Delta^n \to F(k)$, then $F_\sigma(A) = F(A) \times^h_{F(k),\sigma} \Delta^n$ defines a formally cohesive functor. So we get a functor

$$\mathfrak{t}F : \mathrm{Simp}(F(k)) \to \mathrm{Ch}(k), \quad \sigma \mapsto \mathfrak{t}F_\sigma,$$

which sends all morphisms to quasi-isomorphisms. Think of this a "derived local system on $F(k)$".

(Exercise: convince yourself that a functor $\mathrm{Simp}(F(k)) \to \mathrm{Vect}_k$ sending all morphisms to isomorphisms is the same as a local system on $F(k)$.)

Thus, I get a functor $\overline{\rho}^*\mathfrak{t}F : \mathrm{Simp}(X) \xrightarrow{\overline{\rho}} \mathrm{Simp}(F(k)) \xrightarrow{\mathfrak{t}F} \mathrm{Ch}(k)$.

Claim: $\mathfrak{t}F_{X,\overline{\rho}} = C^*(X, \overline{\rho}^*\mathfrak{t}F)$.

Proof. Write $X = \mathrm{colim}_{\Delta^n \to X} \Delta^n = \mathrm{hocolim}_{\Delta^n \to X} \Delta^n$ (basically the diagram you get is already cofibrant, with the projective model structure, so no replacement is needed). Then for $V \in \mathrm{Ch}_{\geq 0}(k)^{fd}$,

$$F_{X,\overline{\rho}}(k \oplus V) = \underline{\mathrm{Hom}}(X, F(k \oplus V)) \times^h_{\underline{\mathrm{Hom}}(X,F(k))} \{\overline{\rho}\}$$

But this is just

$$\mathrm{holim}_{\sigma:\Delta^n \to X} \underline{\mathrm{Hom}}(\Delta^n, F(k \oplus V)) \times^h_{\underline{\mathrm{Hom}}(\Delta^n,F(k))} \{\overline{\rho} \circ \sigma\} = \mathrm{holim}_{\sigma:\Delta^n \to X} F(k \oplus V) \times^h_{F(k),\overline{\rho}\circ\sigma} \Delta^n$$

But by definition this is

$$\mathrm{holim}_{\sigma:\Delta_n \to X} F_\sigma(k \oplus V)$$

which is

$$\mathrm{holim}_\sigma DK(\tau_{\geq 0}(\mathfrak{t}F\sigma \otimes V))$$

which is

$$DK(\tau_{\geq 0}((\mathrm{holim}_\sigma \mathfrak{t}F\sigma) \otimes V))$$

So $\mathfrak{t}F_{X,\overline{\rho}} = \mathrm{holim}_\sigma \mathfrak{t}F_\sigma = C^*(X, \overline{\rho}^*\mathfrak{t}F)$. □

So why did we care about this? Well, given a finite group H, we set $X = BH$, which is the classifying space of H. Then let

$$F(A) = B\,\mathrm{GL}_n(A).$$

Then F preserves pullbacks, but $F(k) = B\,\mathrm{GL}_n(k)$ is not contractible. But

$$\{\overline{\rho}^* BH \to F(k) = B\,\mathrm{GL}_n(k)\}/\text{homotopy} = \{n\text{-dimensional reps of } H\}/\sim.$$

For $A \in \mathrm{Art}_k$,

$$F_{X,\overline{\rho}}(A) = \{\text{lifts } \rho : H \to \mathrm{GL}_n(A) \text{ of } \overline{\rho}\}/\sim.$$

(but you need to use something with trivial center for this ever to be pro-representable).

5. Why does Derived Schlessinger Work

Key fact: If $B, A \in \mathrm{sArt}_k$ are cofibrant and a map $B \to A$ is surjective, and given some $n \geq 0$ such that

1. $\pi_i(B) = \pi_i(A) = 0$ for $i > n$,
2. $\pi_i(B) \xrightarrow{\sim} \pi_i(A)$ for $i < n$, and
3. $\pi_n(B) \to \pi_n(A)$ surjective with kernel I satsifying $m_{\pi_0(B)} I = 0$.

Then there exists a morphism $A \to k \oplus I[n+1]$ and a homotopy pullback

$$\begin{array}{ccc} B & \longrightarrow & k \oplus I\widetilde{[n+1]} \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & k \oplus I[n+1] \end{array}$$

Lemma 5.1. *Let F, G be formally cohesive and $\varphi : F \to G$ a natural transformation. Then φ is a natural weak equivalence if and only if $TF \to TG$ is a quasi-isomorphism.*

Sketch of proof of Derived Schlessinger. **Step 0:** Given $R \in \mathrm{sArt}_k$ and $\eta \in F(R)$, then we get $\eta : F_R \to F$. Set $C_\eta = \mathrm{cone}(\mathfrak{t}F_R \to R_F)$.

Step 1: Suppose we have R and $\eta \in F(R)$ as in Step 0 and some $n \geq 0$ such that $H_i(C_\eta) = 0$ for $i > -n$. Given $k \hookrightarrow H_{-n}(C_\eta) \hookrightarrow H_{-(n+1)}(\mathfrak{t}F_R)$, we get a homotopy pullback

$$\begin{array}{ccc} R' & \longrightarrow & k \oplus k\widetilde{[n+1]} \\ \big\downarrow & & \big\downarrow \\ R & \longrightarrow & k \oplus k[n+1] \end{array}$$

But F preserves homotopy pullbacks, so we get

$$\begin{array}{ccc} F(R') & \longrightarrow & F(k \oplus k\widetilde{[n+1]}) \\ \big\downarrow & & \big\downarrow \\ F(R) & \longrightarrow & F(k \oplus k[n+1]). \end{array}$$

By construction, the image of η in $F(k \oplus k[n+1])$ is homotopic to zero, so can be lifted to $F(k \oplus k\widetilde{[n+1]}) \simeq F(k)$ up to homotopy. So we get a lift η to $\eta' \in F(R')$ such that

$$k \hookrightarrow H_{-n}(C_\eta) \to H_{-n}(C_{\eta'})$$

is 0.

Step 2: Do this infinitely many times, starting with $R = k$ and $n = 0$. $\square$