DERIVED STRUCTURES IN THE LANGLANDS PROGRAM - INTRODUCTION II

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A talk in the Derived Structures in the Langlands Program study group at UCL in Spring 2019. These are notes taken by Ashwin Iyengar (ashwin.iyengar@kcl.ac.uk). They were edited by the speaker. Carl Wang-Erickson (c.wang-erickson@imperial.ac.uk) takes responsibility for the content of these notes; corrections and suggestions are welcome.

We continue, starting where James's talk left off.

Again assume $\mathbf{T}_{\mathfrak{m}} = \mathbf{Z}_p$, and assume the conjecture on the existence of the Galois representation ρ_χ attached to the cuspidal and tempered system of Hecke eigenvalues χ. As previously stated, Venkatesh proves that the free graded action of the exterior algebra on H^* gives an isomorphism

$$\bigwedge^{*} H^1_f(G_F, \mathrm{Ad}^* \rho(1))^* \otimes_{\mathbf{Q}_p} H^{q_0}(Y(K), \mathbf{Q}_p)_\chi \xrightarrow{\sim} H^*(Y(K), \mathbf{Q}_p)_\chi. \tag{1}$$

As we will see, this is the "($\ell \neq p$) étale realization" of a more general conjecture for motivic cohomology, which has various realizations.

1. Motivic Hidden Action

1.1. **The coadjoint motive.** The (conjectural) coadjoint representation $\mathrm{Ad}^* \rho_\chi : G_F \to \mathrm{GL}(\mathfrak{g}^*_{/\mathbf{Q}_p})$ induced by ρ_χ is part of a compatible system of Galois representations (by varying p). Thus, it is standard to expect that there exists a "co-adjoint motive" M_{coad} which induces various realizations, one of which is $\mathrm{Ad}^* \rho_\chi$.

To a general motive M over $\mathbf{Q}$, with coefficients in $\mathbf{Q}$, there should be attached various realizations, including the following:

(i) **Betti**: this is a $\mathbf{Q}$-vector space M_B, which is the singular cohomology.

(ii) **p-adic Étale**: this is a Galois representation

$$\rho_{M,p} : G_F \to \mathrm{GL}(M_B \otimes_{\mathbf{Q}} \mathbf{Q}_p).$$

(iii) **Hodge**: this is a $\mathbf{Q}$-Hodge structure based on M_B

Part of the theory of motives is that there exist motivic cohomology groups of M, which are $\mathbf{Q}$-vector spaces with various regulator maps under realizations. In particular, given M_{coad}, we expect a motivic cohomology group $H^1_{\mathrm{mot}}((M_{\mathrm{coad}})_{O_F}, \mathbf{Q}(1))$ and we get an étale regulator map

$$r_{\text{ét}} : H^1_{\mathrm{mot}}((M_{\mathrm{coad}})_{O_F}, \mathbf{Q}(1)) \otimes_{\mathbf{Q}} \mathbf{Q}_p \to H^1_f(G_F, \mathrm{Ad}^* \rho_\chi(1)),$$

which conjecturally should give an isomorphism. Here $()_{O_F}$ refers to a category of motives with extensions of good reduction.

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1.2. **The motivic hidden action conjecture.** We are now ready to state the conjecture.

Conjecture 1.2.1 ("Hidden action")**.** *There exists a "natural" action of*

$$\bigwedge^{*} H^1_{\mathrm{mot}}((M_{\mathrm{coad}})_{O_F}, \mathbf{Q}(1))$$

on $H^*(Y(K), \mathbf{Q})_\chi$ *inducing an isomorphism*

$$\bigwedge^{*} H^1_{\mathrm{mot}}((M_{\mathrm{coad}})_{O_F}, \mathbf{Q}(1)) \otimes_{\mathbf{Q}} H^{q_0}(Y_K, \mathbf{Q})_\chi \to H^*(Y(K), \mathbf{Q})_\chi$$

The word "natural" is made more precise in each "realization" of this hidden action conjecture. For example, we call the isomorphism (1) the "($\ell \neq p$) étale realized action."

Conjecture 1.2.2 ("Hidden action via $\ell \neq p$ étale realization")**.** *The* $\mathbf{Q}$*-rational structure isomorphisms*

$$H^*(Y_K, \mathbf{Q})_\chi \to H^*(Y_K, \mathbf{Q}_p)_\chi, \quad \bigwedge^{*} H^1_{\mathrm{mot}}((M_{\mathrm{coad}})_{O_F}, \mathbf{Q}(1)) \to H^1_f(G_F, \mathrm{Ad}^* \rho_\chi(1))$$

are preserved by étale realized action (1)*. That is,* (1) *induces the top arrow of a commutative diagram*

$$\begin{array}{ccc} \bigwedge^* H^1_{\mathrm{mot}}((M_{\mathrm{coad}})_{O_F}, \mathbf{Q}(1)) \otimes_{\mathbf{Q}} H^{q_0}(Y_K, \mathbf{Q})_\chi & \longrightarrow & H^*(Y_K, \mathbf{Q})_\chi \\ \downarrow & & \downarrow \\ \bigwedge^* H^1_f(G_F, \mathrm{Ad}^* \rho(1))^* \otimes_{\mathbf{Q}_p} H^{q_0}(Y(K), \mathbf{Q}_p)_\chi & \xrightarrow{\sim} & H^*(Y(K), \mathbf{Q}_p)_\chi. \end{array}$$

As Venkatesh notes, the point of his paper [6] is to reach the point where this conjecture can be made (by establishing the isomorphism (1)).

2. Hodge Case

2.1. **The Hodge realization of the hidden action conjecture.** See [5]: this will be covered in talks H8 and H9.

Theorem 2.1.1 ([5])**.** *There is an* ℓ_0*-dimensional* $\mathbf{C}$*-vector space* $\mathfrak{a}_G^*$ *and a graded action of* $\bigwedge^* \mathfrak{a}_G^*$ *on* $H^*(Y(K), \mathbf{C})_\chi$ *inducing an isomorphism*

$$\bigwedge^{*} \mathfrak{a}_G^* \otimes_{\mathbf{C}} H^{q_0}(Y(K), \mathbf{C})_\chi \xrightarrow{\sim} H^*(Y(K), \mathbf{C})_\chi.$$

where $\mathfrak{a}_G^*$ *is the target of an* $\mathbf{R}$*-regulator map*

$$r_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} : H^1_{\mathrm{mot}}((M_{\mathrm{coad}})_{O_F}, \mathbf{Q}(1)) \otimes \mathbf{QC} \xrightarrow{\sim} \mathfrak{a}_G$$

In analogy to Conjecture 1.2.2, we have

Conjecture 2.1.1 ("Hidden action via Hodge realization")**.** *The isomorphism in Theorem 2.1.1 preserves* $\mathbf{Q}$*-rational structures.*

Remark 2.1.1. Prasanna-Venkatesh explain that this implies that certain period integrals are equal to certain L-values up to $\mathbf{Q}^*$. They give some computations that reflects this. This seems to be the only evidence for the hidden action conjecture in print at this time.

2.2. **A bit more about $\mathfrak{a}_G^*$.** What is this **C**-vector space $\mathfrak{a}_G^*$? There is a group-theoretic way to define it, but we will take the following perspective. Let

$$\rho_{\chi,\mathbf{R}} : W_{\mathbf{R}} \to \mathrm{PGL}_n(\mathbf{C}) \rtimes \mathrm{Gal}(\mathbf{C}/\mathbf{R})$$

be the Langlands parameter corresponding to the real part of π_∞ of the automorphic representation π (with Hecke eigensystem χ). Here $W_{\mathbf{R}}$ is the Weil group of $\mathbf{R}$:

$$\mathbf{C}^* \rtimes \langle j \rangle$$

where $j^2 = -1$. There is then a coadjoint action

$$(\mathrm{Ad}\,\rho_\chi)^\vee : W_{\mathbf{R}} \to \mathrm{GL}(\mathrm{Lie}(\mathrm{PGL}_n(\mathbf{C}))),$$

and $\mathfrak{a}_G$ is defined to be the vector space of fixed points of this action of $W_{\mathbf{R}}$ on $\mathrm{Lie}(\mathrm{PGL}_n(\mathbf{C}))$.

3. "$\ell = p$ étale" or "crystalline" case

This is the work of Hansen-Thorne in [2], and will be covered in talk H10. The point here is that ℓ_0 appears in the geometry of an eigenvariety.

3.1. **Finding an étale action in the GL_n-eigenvariety.** There is an eigenvariety X and an underlying weight space W and a weight map $X \xrightarrow{w} W$. The points of X are pairs $x = (\chi, \lambda)$, where χ is the system of eigenvalues as before, and λ is a refinement, i.e. a choice of Hecke eigenvalues at p. The weight map w is locally finite, and it is expected (a "non-abelian version of the Leopoldt conjecture") of codimension ℓ_0.

Let $\mathbf{T}_x$ denote the local ring $\mathscr{O}^\wedge_{X,x}$, and Λ be the local ring $\mathscr{O}^\wedge_{W,w(x)}$, so that w induces a homomorphism $\Lambda \to \mathbf{T}_x$. Hansen–Thorne prove the following theorem:

Theorem 3.1.1 ([2])**.** *Under mild assumptions, and if* $\dim \mathbf{T}_x = \dim \Lambda - \ell_0$*, then*

(1) $\Lambda \to \mathbf{T}_x$ is surjective. Let I denote the kernel.

(2) If we let $V_x = I/\mathfrak{m}_\Lambda I$ ("conormal module"), there exists an action

$$\bigwedge^* V_x \otimes_{\mathbf{Q}_p} H^{q_0}(Y(K,p), \mathbf{Q}_p)_{\chi,\lambda} \xrightarrow{\sim} H^*(Y(K,p), \mathbf{Q}_p)_{\chi,\lambda},$$

where now $Y(K,p)$ is a locally symmetric space with some specified level at p.

Under an additional "trianguline $R = \mathbf{T}$" assumption, we get a natural isomorphism $V_x \cong H^1_f(G_F, \mathrm{Ad}^*\rho_\chi(1))$.

3.2. **Comparison of the two étale cases: $\ell \neq p$ vs. $\ell = p$.** We can compare the $\ell = p$ and $\ell \neq p$ étale realizations:

- They have the same Bloch–Kato Selmer group $H^1_f(G_F, \mathrm{Ad}^*\rho_\chi(1))$.
- They have different "forms": the $\ell = p$ case fixes a U_p-eigenvalue, restricting the forms that appear.
- Most importantly, the variation (i.e. deformation) of the level differs – this variation is where the labels "$\ell \neq p$" and "$\ell = p$" come from.
  - The $\ell \neq p$-étale action isomorphism (1) is proved using the Taylor–Wiles method: we add level at primes ℓ such that $p^n \nmid \ell - 1$, and use rings like $\mathbf{Z}/p^n[\mathbf{Z}/(\ell-1)]$ to form the rings S_∞. These are coordinate rings for modules of forms. In particular, this is deforming a mod p Galois representation in the p-adic direction.
  - On the other hand, the $\ell = p$-étale version uses variation in the eigenvariety – varying the level (and weight) at p, and only at p" – and stays in characteristic 0.

4. Derived Deformation Theory

There is a difference between derived deformation theory in characteristic zero vs. positive/mixed (i.e. arbitrary) characteristic. While much of the Galois track will be dedicated to understanding the arbitrary characteristic case, it is possible to get some flavor of what we are aiming for with a brief explanation. The main idea is that under the Bloch–Kato conjecture, $\bigwedge^* H^1_f(G_F, \mathrm{Ad}\,\rho_\chi)$ *is* the derived deformation ring of ρ_χ.

4.1. **Classical deformation theory.** Suppose we are given a Galois representation $\rho_\chi : G_F \to \mathrm{PGL}_n(\mathbf{Q}_p)$. Then it is standard that first order deformations are given by

$$\{\rho : G_F \to \mathrm{PGL}_n(\mathbf{Q}_p[\epsilon]/(\epsilon^2) : \rho \equiv \rho_\chi \mod \epsilon \text{ (+ local conditions)}\}/\sim \longleftrightarrow H^1_{(f)}(G_F, \mathrm{Ad}\,\rho_\chi)$$

where $\sim$ denotes strict equivalence, and the "(local conditions)" on the Galois representations correspond with the condition (f). We are using this notation to allow for either imposing the unramified/crystalline local conditions, which corresponds to the Bloch–Kato Selmer group; or for imposing the empty local condition, which corresponds to the unmodified global Galois cohomology $H^1(G_F, \mathrm{Ad}\,\rho_\chi)$.

We will write $\widehat{\mathrm{Sym}^*}_{\mathbf{Q}_p} V$ for the completed symmetric algebra of a $\mathbf{Q}_p$-vector space V. When a basis for V with n elements is chosen, this induces an isomorphism $\widehat{\mathrm{Sym}^*}_{\mathbf{Q}_p} V \cong \mathbf{Q}_p[\![x_1, \dots, x_n]\!]$. We also let $\mathfrak{m}_S$ denote the maximal ideal of a local ring S. Assume that the deformation problem with (local conditions) is represented by $R^{\mathrm{cris}}_{\rho_\chi}$. Then we may express our statement on first-order deformations as follows.

Fact 4.1.1 (Tangent space)**.** *There is a canonical isomorphism*

$$R^{(cris)}_{\rho_\chi}/\mathfrak{m}_R^2 \cong \mathbf{Q}_p \oplus H^1_{(f)}(G_F, \mathrm{Ad}\,\rho_\chi)^* \cong \widehat{\mathrm{Sym}^*}_{\mathbf{Q}_p} H^1_{(f)}(G_F, \mathrm{Ad}\,\rho_\chi)^*/\mathfrak{m}^2_{\widehat{\mathrm{Sym}^*}}.$$

Fact 4.1.2 (Obstruction theory)**.** *There exists a (non-canonical) presentation of $R^{(cris)}_{\rho_\chi}$ as follows: there exists a surjection*

$$\widehat{\mathrm{Sym}^*}_{\mathbf{Q}_p} H^1_{(f)}(G_F, \mathrm{Ad}\,\rho_\chi)^* \twoheadrightarrow R^{(cris)}_{\rho_\chi}$$

lifting the tangent space isomorphism. Letting J denote its kernel, there is a canonical surjection

$$H^2_{(f)}(G_F, \mathrm{Ad}\,\rho_\chi)^* \twoheadrightarrow J/\mathfrak{m}_{\widehat{\mathrm{Sym}^*}} J.$$

Thus, by Nakayama's lemma, the minimal number of generators of J is at most $\dim H^2_{(f)}(G_F, \mathrm{Ad}\,\rho_\chi)$. It is sometimes said that the "expected" Krull dimension of $R^{(\mathrm{cris})}_\rho$ is

$$\dim H^1_{(f)}(G_F, \mathrm{Ad}\,\rho_\chi) - \dim H^2_{(f)}(G_F, \mathrm{Ad}\,\rho_\chi).$$

This dimension is actually conjectured for two-dimensional irreducible representations in place of ρ_χ, when $F = \mathbf{Q}$ and with the trivial local conditions, which amounts to conjecturing that these deformation rings are complete intersection rings. In general, with the unramified/crystalline condition, this difference is $-\ell_0$. So when $\ell_0 > 0$, we can only expect this "expected" dimension to make sense in some derived way.

For example, if we do impose the unramified/crystalline condition on our deformation ring, then the Bloch–Kato conjecture predicts that $H^1_f(G_F, \mathrm{Ad}\,\rho_\chi) = 0$; assuming this, we have vector spaces of dimension ℓ_0

$$H^2_f(G_F, \mathrm{Ad}\,\rho_\chi) \cong H^1_f(G_F, \mathrm{Ad}^*\,\rho_\chi(1))^*,$$

where the isomorphism uses a version of global Tate duality due to Nekovár [4]. Since $H^1_f(G_F, \mathrm{Ad}\,\rho_\chi) = 0$, the classical crystalline deformation ring is trivial: $R^{(\mathrm{cris})} = \mathbf{Q}_p$. Therefore it loses the information about the deformation problem, namely, at least that of $H^2_f(G_F, \mathrm{Ad}\,\rho_\chi)$.

4.2. **Motivation.** This motivates the need for a "derived" deformation theory that takes into account all of the higher cohomology groups of the adjoint representation. To do derived deformation theory, we need to

(1) Define a broader category of rings to work with, such that they carry extra derived information (examples include simplicial rings, differential graded rings).

(2) Define the derived deformation problem, which properly extends the classical deformation problem to this larger category of rings.

For the moment, we will focus on issue (1) and ignore issue (2). For a discussion of issue (2) in the characteristic zero case, see e.g. the paper of Kapranov [3].

4.3. **Derived deformation theory in characteristic zero.** In characteristic zero, it suffices to use commutative differential graded algebras as coefficient rings. Given the proper definition (2), the derived crystalline deformation ring is (up to quasi-isomorphism)

$$\mathcal{R}_{\rho_\chi}^{\text{cris, char } 0} \cong \widehat{\mathrm{Sym}^*}_{\mathbf{Q}_p} \Sigma H_f^*(G_F, \mathrm{Ad}\,\rho_\chi)^*$$

where Σ denotes suspension. Here we are using $\mathcal{R}$ to denote a derived deformation ring, in contrast to the classical deformation ring denoted by R.

In degree -1, this gives $(H_f^2)^*$, in degree 0 it gives $(H_f^1)^*$ and in degree 1 it gives $(H_f^0)^*$, so we recover all of the relevant cohomological data from this representing ring. In fact, in our case $H_f^i(G_F, \mathrm{Ad}\,\rho_\chi) = 0$ for $i \geq 3$, so actually

$$\mathcal{R}_{\rho_\chi}^{\text{cris, char } 0} = \widehat{\mathrm{Sym}^*}_{\mathbf{Q}_p} \Sigma H_f^2(\mathrm{Ad}\,\rho_\chi)^* = \bigwedge^* H_f^1(G_F, \mathrm{Ad}\,\rho_\chi^*(1)),$$

because a free commutative differential graded algebra generated in an odd degree is an exterior algebra. That is, upon the Bloch–Kato conjecture's prediction that $H_f^1(G_F, \mathrm{Ad}\,\rho_\chi) = 0$, the exterior algebra that appeared at the outset of this talk *is* the derived deformation ring. Likewise, the duality between this exterior algebra and the derived Hecke algebra in the first talk is a derived generalization of "$R = \mathbf{T}$."

Remark 4.3.1. This is compatible with a a philosophy attributed to Deligne: "every characteristic zero (classical) deformation problem comes from a differential graded Lie algebra." In fact there is a dg-Lie algebra structure on $H_{(f)}^*(G_F, \mathrm{Ad}\,\rho)$. This is "Koszul dual" (in the sense of the bar equivalence) to the commutative differential graded object $\mathcal{R}_{\rho_\chi}^{\text{cris, char } 0}$.

4.4. **Motivation for the homotopy background in the Galois track.** We will spend a good deal of the Galois track developing background to do derived deformation theory in mixed characteristic. Why do we need to do this when the content of étale hidden action conjecture (in particular, $\bigwedge^* H_f^1(G_F, \mathrm{Ad}\,\rho_\chi^*(1))$) seems to be entirely in characteristic zero? The reason is that, as discussed above in §3.2, we use the Taylor–Wiles method to prove that (1) is an isomorphism (the main result of [6]). This involves mixed characteristic rings.

It is well-understood that commutative differential graded algebras do not suffice as coefficient rings for derived deformation problems, once we no longer want to work over $\mathbf{Q}$. The basic problem is that $d(x^p) = 0$ in characteristic p. An alternative approach is needed.

Homotopical algebra supplies various frameworks for derived deformation theory in mixed characteristic. Following [1], we will use simplicial commutative rings. Homotopy algebra background will be given In talks G5, G6, and G7. This will then be applied toward derived deformation problems in G8, and derived deformation problems for Galois representations in talk G9.

References

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