The GLn eigenvariety and the p-adic realisation of Venkatesh's conjecture

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1 Setup and Motivation

The main reference for this talk is [**HansenThorne**]. Let $\mathcal{G} = \mathrm{GL}_n / \mathbb{Z}$ for $n \geq 2$ and in practice we will work with $n > 3$. Let π be a regular and (C)-algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. This means that π contributes to the cohomology of the locally symmetric space associated to G, with coefficients in a local system with weight the highest weight of. Let $E_\pi/\mathbb{Q}$ be the number field over which π is defined. For all places l of E_π we have Galois representations (constructed by [**HLTT**])

$$\rho_{\pi,l} : G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(E_{\pi,l}).$$

These are not known to be geometric, but expected to come from a motive $M(\pi)$ over $\mathbb{Q}$, giving rise to the compatible system $(\rho_{\pi,l})_l$. Let $Y_{\mathcal{K}}$ be the locally symmetric space for $\mathcal{G}$ defined by

$$G(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A})/\mathcal{K}\mathcal{K}_\infty \mathbb{R}_{>0}.$$

If $\mathcal{K}$ is neat, then $Y(K)$ has the structure of a Riemannian manifold. The representation π contributes to the Betti/singular cohomology of $Y_{\mathcal{K}}$:

$$H^*(\mathcal{K}, \mathcal{L}_{\lambda,L}) := H^*(Y_{\mathcal{K}'}, \Lambda_{\lambda,L})^{\mathcal{K}}$$

where $\mathcal{K}' \subset \mathcal{K}$ is sufficiently small. Fix a torus and Borel $\mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$ and a dominant weight $\lambda \in X^*(T)^+$. Then we can define a local system of L-vector spaces $\mathcal{L}_{\lambda,L}$ on $Y_{\mathcal{K}}$ where L is a number field or a p-adic field. The Hecke algebra $\mathbb{T}_K$ of level K acts on $H^*(\mathcal{K}, \mathcal{L}_{\lambda,L})$

More precisely, since π_∞ is tempered we know that it contributes to

$$H^i(\mathcal{K}, \mathcal{L}_{\lambda,L}) \text{ for } i \in [q_0, q_0 + l_0]$$

where $q_0 = \frac{1}{2}(\dim_{\mathbb{R}} Y_{\mathcal{K}} - l_0)$ and l_0 is the defect of $\mathcal{G}$ which is equal to $\lfloor \frac{n-1}{2} \rfloor$. Moreover the contribution in degree $q_0 + i$ has dimension

$$\binom{l_0}{i} \cdot \dim(\pi_f)^{\mathcal{K}}$$

Conjecture 1. *There is a natural action of*

$$\bigwedge^* \mathrm{Ext}_{MM_{\mathbb{Z}}}(M(\pi), M(\pi)(1)$$

on $H^*(K, \mathcal{L}_{\lambda,l}[\mathfrak{m}]$. Here the ext-group is taken in the category of mixed motives over $\mathbb{Z}$, i.e., it is a motivic cohomology group. It has expected dimension l_0 (by the Block-Kato conjecture) and should make

$$\bigwedge^{*} \mathrm{Ext}_{\mathrm{MM}_{\mathbb{Z}}}(M(\pi), M(\pi)(1)$$

into a free module of rank 1 (or $\dim(\pi_f)^K$ over

$$\bigwedge^{*} \mathrm{Ext}_{\mathrm{MM}_{\mathbb{Z}}}(M(\pi), M(\pi)(1).$$

The goal of this talk is to study the p-adic realisation of this conjecture, via properties of eigenvarieties.

2 Eigenvarieties

The idea of eigenvarieties goes back to Hida and Coleman and is (roughly speaking) that classical modular forms of weight $k \geq 2$ can be but into p-adic families that vary continuously with respect to the weight, in the case that they have finite slope (valuation of the U_p eigenvalue). We will use Hansen's construction of eigenvarieties in[**MR3692014**], which gives a pair of rigid spaces over $\mathbb{Q}_p$

$$w : \mathcal{X} \to \mathcal{W}$$

where $\mathcal{W}$ is the weight space, which represents the functor

$$X \mapsto \mathrm{Hom}_{\mathrm{cts}}(T(\mathbb{Z}_p), \mathcal{O}(X)^{\times}).$$

The eigenvariety $\mathcal{X}$ of tame level K^p is a rigid space whose closed points are in bijection with systems of Hecke eigenvalues occurring in

$$H^*(K^p I, \mathcal{D}_\lambda)$$

where I is the Iwahori subgroup of $\mathcal{G}(\mathbb{Q}_p)$ and $\mathcal{D}_\lambda$ is some big p-adic coefficient system. Moreover the map w is finite locally on $\mathcal{X}$ and has discrete fibers. The Hecke algebra that acts is

$$\mathbb{T}^(N, p) := \mathcal{H}^(Np) \otimes_{\mathbb{Z}} \mathbb{Z}[X_*(T)^-]$$

where $\mathcal{H}^(Np)$ is the spherical Hecke algebra away from Np.

An example of a point is a pair $(\pi, \mathcal{A})$ where π is as before and $\mathcal{A}$ is a "refinement" ordering of the eigenvalues of Satake parameters of π_p.

3 Main theorems

Assume that the pair $(\pi, \mathcal{A})$ satisfies

A 1 π is unramified at p and the Satake parameters of π_p are regular semisimple.

A 2 $\mathcal{A}$ is an ordering of the eigenvalues of π_p that has "small slope" (for n=2 this is just $\mathrm{val}(a_p) < k - 1$, which is related to Coleman classicality).

A 3 Parity condition on π_∞.

Assumption A1 is necessary to prove that the Hecke eigenvalues occur in l_0+1 degrees of cohomology, assumption A3 is not very serious.

If $x=(\pi,\mathcal{A})$ is above then $x\in\mathcal{X}(L)$ with $L/\mathbb{Q}_p$ finite and assumed sufficiently large. We choose tame level $K_1(N)$ where N is the conductor of π and $K_1(N)$ is some mirabolic subgroup. We have complete local Noetherian L-algebras

$$\mathcal{T}_x := \hat{\mathcal{O}}_{\mathcal{X},x}$$
$$\Lambda := \hat{\mathcal{O}}_{\mathcal{W},\lambda}$$

where $\lambda = w(x)$ and we note that $\mathcal{T}_x$ is a finite Λ-algebra.

Theorem 1 (Hansen-Thorne)**.** *Under assumptions A1,A2,A3 we have*

$$\dim \mathbb{T}_x \geq \dim \Lambda - l_0 \tag{1}$$

and if equality holds we moreover have:

- *The natural map $\Lambda \to \mathbb{T}_x$ is surjective and $\mathbb{T}_x$ is a complete intersection ring*
- *Let $\mathfrak{m}\subset \mathbb{T}^{(N),p}$ be the maximal ideal corresponding to $(\pi,\mathcal{A})$. Let $L_{\lambda,L}$ be the algebraic coefficient system of L-vector spaces of weight λ on $Y_{\mathcal{K}'}$ with $\mathcal{K}'\subset K_1(N)^pI$. Let*

  $$V_x := \ker(\Lambda \to \mathbb{T}_x)\otimes_\Lambda L$$

  which has dimension l_0 by (1). Then

  $$H^*(K_1(N)^pI, \mathcal{L}_{\lambda,L}[\mathfrak{m}]]$$

  has a canonical structure of a $\bigwedge^* V_x$-module that is free of rank 1.

Remark 1. The equality in (1) was conjecture by Hida/Urban and is a non-abelian analogue of the Leopoldt conjecture (also see Calegari-Emerton conjectures for completed cohomology).

Let $\rho_\pi : G_\mathbb{Q} \to \mathrm{GL}_n(L)$ be the Galois representation associated to π which satisfies:

- It is unramified outside of Np.
- For $l \nmid Np$ the characteristic polynomial of $\rho_\pi(\mathrm{Frob}_l)$ is determined by the Satake parameter of π_L.
- It is odd.

Conjecture 2. *The representation ρ_π satisfies the further conditions:*

1. *It is absolutely irreducible*
2. *It satisfies local-global compatibility at p. More precisely it is crystalline with Hodge-Tate weights determined by λ and the Frobenius eigenvalues match the Satake parameters on the Weil Deligne representation associated to*

   $$\rho_\pi\big|_{G_{\mathbb{Q}_p}}$$

3. There is an isomorphism

$$\mathbb{T}_x \cong R_{\rho_\pi,\alpha}$$

where the latter is a trianguline deformation ring (this is an $R = \mathbb{T}$ type result).

Remark 2. We are far from knowing assumptions 1, but it is a reasonable thing to assume. It should be possible to prove assumption 2 with state of the art techniques. Kare and Thorne [**MR3702498**] sketch an idea for proving a result like 3 in the ordinary case by implementing the Calegari-Geragthy method.

Recall the Bloch-Kato Selmber groups for $\mathrm{Ad}\,\rho_\pi$ and $\mathrm{Ad}\,\rho_\pi(1)$ defined by

$$\ker\left[H^1(\mathbb{Z}[1/S], \mathrm{Ad}\,\rho_\pi) \to \frac{H^1(\mathbb{Q}_p, \mathrm{Ad}\,\rho_\pi)}{H^1_f(\mathbb{Q}_p, \mathrm{Ad}\,\rho_\pi)}\right] \oplus \bigoplus_{\substack{l\in S \\ l\neq p}} H^1(\mathbb{Q}_l^{\mathrm{ur}}, \mathrm{Ad}\,\rho_\pi)$$

Remark 3. This is the tangent space of the universal deformation ring of ρ_π which classifies deformations ρ_A of ρ_π such that

$$\rho_A\big|_{G_{\mathbb{Q}_p}} \text{ is crystalline}$$
$$\rho_A\big|_{G_{\mathbb{Q}_p}} = \rho_{\pi,l} \otimes_L A \text{ for } l \nmid N.$$

Theorem 2. *Let $x \in \mathcal{X}(L)$ correspond to (π, A) as in the previous theorem. Assume conjecture B and that $\dim \mathbb{T}_x = \dim \Lambda - l_0$. Then*

1. *The dimension of $H^1_f(\mathbb{Q}, \mathrm{Ad}\,\rho_\pi)$ is zero, the dimension of $H^1_f(\mathbb{Q}, \mathrm{Ad}\,\rho_\pi(1)) = l_0$ and we have a canonical exact sequence;*

$$H^1_f(\mathbb{Q}, \mathrm{Ad}\,\rho_\pi(1)) \to \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \to \mathfrak{m}_{\mathbb{T}_x}/\mathfrak{m}_{\mathbb{T}_x}^2 \to 0 \tag{2}$$

2. *Suppose that $\mathcal{X}$ is smooth at x. Then (2) is left exact inducing a canonical isomorphism*

$$H^1_f(\mathbb{Q}, \mathrm{Ad}\,\rho_\pi(1)) \cong V_x.$$

   Consequently, combining with Theorem 1

$$H^*(K_1(N;p), \mathcal{L}_{\lambda,L})[\mathfrak{m}]$$

   has a canonical structure of a free

$$\bigwedge H^1_f(\mathbb{Q}_p, \mathrm{Ad}\,\rho_\pi(1))$$

   module of rank 1.

4 Idea of proof of Theorem A

We sketch the construction of $\mathcal{X}, \mathcal{W}$ and the map w, following [**MR3692014**]. There is a homomorphism

$$\mathbb{T}^{(N),p} \to \mathcal{O}(\mathcal{X})$$

such that for all $\lambda \in \mathcal{W}(\overline{\mathbb{Q}_p})$ we have discrete set $w^{-1}(\lambda)$ which is in bijection with "finite slope eigenpackets" of weight λ, tame level $\mathcal{K}_1(N)$ such that for $x \in w^{-1}(\lambda)$ we have

$$\phi_{\mathcal{X},x} : \mathbb{T}^{(N),p} \to \mathcal{O}(\mathcal{X}) \to k(x).$$

If $\Omega \subset \mathcal{W}$ is an affinoid open subset then there is an

$$\mathcal{O}(\Omega)[\Delta_p]$$

module $\mathcal{D}_\Omega$ where Δ_p is the monoid

$$\Delta_P = \coprod_{\mu \in X_*(T)^-} I\mu I.$$

This module $\mathcal{D}_\Omega$ is a local system on $Y_{\mathcal{K}'}$ where $\mathcal{K}' \subset K_1(N)^p I$ as usual. This turns

$$H^*(K_1(N;p), \mathcal{D}_\Omega)$$

is a $\mathbb{T}^{(N),p}_{\mathcal{O}(\Omega)}$-module. Now choose (Ω, h) a slope datum where h is a rational number. There is a Hecke-equivariant decomposition

$$(H^*(K_1(N;p), \mathcal{D}_\Omega)_{\leq h} \oplus (H^*(K_1(N;p), \mathcal{D}_\Omega)_{>h}$$

and define

$$\mathbb{T}_{\Omega,h} := \operatorname{Im}\left(\mathbb{T}^{(N),p}_{\mathcal{O}(\Omega)} \to \operatorname{End}\left(H^*(K_1(N;p), \mathcal{D}_\Omega)_{\leq h}\right)\right).$$

Then locally on the eigenvariety, every point $x \in \mathcal{X}$ will have an affinoid neighborhood of the form

$$\operatorname{Sp} \mathbb{T}_{\Omega,h}.$$

Moreover, if $\lambda \in \Omega(\overline{\mathbb{Q}_p})$ then

$$\mathcal{D}_\lambda := \mathcal{D}_\Omega \otimes_{\mathcal{O}(\Omega)} k(\lambda)$$

and there is a natural surjective map

$$D_\lambda \to \mathcal{L}_{\lambda,L} \otimes_L L(\lambda^{-1}),$$

which induces

$$H^*(K_1(N;p), \mathcal{D}_\lambda) \to H^*(K_1(N;p), \mathcal{L}_{\lambda,L} \otimes_L L(\lambda^{-1})$$

that is $\mathbb{T}^{(N),p}$-equivariant.

Definition 1. *A finite slope eigenpacket*

$$\phi_{\mathcal{X},x} : \mathbb{T}^{(N),p} \to k(x) = L$$

is

1. *classical if it is in the support of*

$$H^*(K_1(N;p), \mathcal{L}_{\lambda,L} \otimes_L L(\lambda^{-1})$$

2. *non-critical if*

$$H^*(K_1(N;p), \mathcal{D}_\lambda)_\mathfrak{m} \to H^*(K_1(N;p), \mathcal{L}_{\lambda,L} \otimes_L L(\lambda^{-1})_\mathfrak{m}$$

   is an isomorphism, where $\mathfrak{m} = \ker \phi_{\mathcal{X},x}$. *The latter is supported in degrees* $q_0, \cdots, q_0 + l_0$

The key point is now that there is a spectral sequence

$$\mathrm{Tor}^{\Lambda}_{-i}\left(\left(H^j(K_1(N;p), \mathcal{D}_\Omega\right)_{\le h}, k(\lambda)\right) \Rightarrow H^{i+j}(K_1(N;p), \mathcal{D}_\lambda)_{\le h}$$

and because $\mathcal{O}(\Omega) \to \Lambda$ is flat we get

$$\mathrm{Tor}^{\Lambda}_{-i}\left(\left(H^j(K_1(N;p), \mathcal{D}_\Omega\right)_{\le h} \otimes_{\mathcal{O}(\Omega)} \Lambda, k(\lambda)\right) \Rightarrow H^{i+j}(K_1(N;p), \mathcal{D}_\lambda)_{\le h}.$$

Now we can apply the Calegari-Geragthy commutative algebra Lemma (slightly improved by Hansen) to get Theorem 1.