DERIVED HECKE ALGEBRAS FOR WEIGHT 1 MODULAR FORMS

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A talk in the Derived Structures in the Langlands Program study group at UCL in Spring 2019. These are notes taken by Ashwin Iyengar (ashwin.iyengar@kcl.ac.uk).

The plan for today is the following:

(1) Formulation of the conjecture

(2) Numerical evidence for the conjecture

Venkatesh predicts an action of certain motivic cohomology groups on $H^*(Y(K), \mathbf{Q})$. I'll talk about an analogue for coherent cohomology, motivated by the appearance of the same systems of Hecke eigenvalues, but now coming from weight 1 modular forms.

Here is a rough table of analogies.

	Singular Cohomology	Coherent Cohomology
Derived Hecke Operators	Usual definition via correspondences, but we add in a cup product with a congruence class	Same thing, but now a cup product with a "Shimura class"
Action of a rational group	Conjecturally, a motivic cohomology group	Stark unit
Evidence for the conjecture	Tori, complex realization case, not much else	Numerical evidence from **?**, Proofs for forms of dihedral projective image in soon-to-be-published work of Darmon, Harris, Rotger, and Venkatesh.

1. Formulation

1.1. **Setup.** Let g be a weight 1 newform of level N and Nebentypus character χ. Write

$$g = \sum_{n \geq 0} a_n q^n$$

where $a_n \in E/\mathbf{Q}$ land in some number field E with ring of integers $\mathscr{O}$. We can define an odd Galois representation

$$\rho_g : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathscr{O})$$

attached to g (by work of Deligne-Serre in weight 1). Actually ρ_g factors through a finite extension L (we call this the splitting field of ρ_g):

$$\rho_g : G_{\mathbf{Q}} \to G_{L/\mathbf{Q}} \to \mathrm{GL}_2(\mathscr{O}).$$

Let $\mathrm{Ad}^0 \rho = \{\varphi \in \mathrm{End}(\mathscr{O}^2) : \mathrm{Tr}\, \varphi = 0\}$ denote the representation of $G_{L/\mathbf{Q}}$ acting by conjugation, and let $\mathrm{Ad}^* \rho = \mathrm{Hom}_{\mathscr{O}}(\mathrm{Ad}^0 \rho, \mathscr{O})$.

Concrete example: L is a Galois closure of a cubic field, and we have $\mathrm{Gal}(L/\mathbf{Q}) \cong S_3 \hookrightarrow \mathrm{GL}_2(\mathbf{Z})$.

Date: March 20, 2019.

1.2. **The Stark Unit.** Let

$$U_g = (U_L \otimes \mathrm{Ad}^* \rho)^{G_{L/\mathbf{Q}}} = \mathrm{Hom}_{\mathscr{O}[G_{L/\mathbf{Q}}]}(\mathrm{Ad}^0 \rho, U_L \otimes \mathscr{O})$$

where $U_L = \mathscr{O}_L^\times$. This is the "$\mathrm{Ad}^0 \rho$-isotypic part of the unit group."

Claim 1.2.1.

(1) $\dim_E(U_g \otimes \mathbf{Q}) = 1$.

(2) *Furthermore, if we assume that* $\mathbf{Q}(\mu_p) \not\subset L$*, then* $U_g \otimes \mathbf{Z}_p$ *is a free* $\mathscr{O} \otimes \mathbf{Z}_p$*-module.*

Proof.

$$\dim_E(U_L \otimes \mathrm{Ad}^* \rho) = \dim(\mathrm{Ad}^* \rho)^{c=1} - \dim(\mathrm{Ad}^* \rho)^{G_{L/\mathbf{Q}}} = 1 - 0 = 1$$

The second part follows from the fact that U_g has no p-torsion (?) □

In **?**, there was an action of a Bloch-Kato Selmer group on cohomology. We can try to do the same thing here. Take $\mathfrak{p}$ a prime dividing $p \nmid N$, and unramified in E. Then consider the representation

$$\rho_\mathfrak{p} : G_{L/\mathbf{Q}} \to \mathrm{GL}_2(\mathscr{O}_\mathfrak{p}).$$

Assuming $p \nmid \mathrm{Cl}(L)$ and $p \nmid [L : \mathbf{Q}]$,

$$H^1_f(\mathbf{Q}, \mathrm{Ad}\,\rho_\mathfrak{p}(1)) \cong U_{g,\mathfrak{p}}.$$

Note this can also be compared with motivic cohomology. Denote by M_g the Chow motive attached to $\mathrm{Ad}^* \rho$ (which exists in this case, because ρ is an Artin representation), then we have a map

$$H^1_{\mathrm{mot}}(M_g, \mathbf{Q}(1)) \to U_g \otimes \mathbf{Q}.$$

Harris and Venkatesh believe this is an isomorphism.

1.3. **Taylor-Wiles Primes.** Choose a prime $\mathfrak{p}$ in E, $\mathfrak{p} \nmid p \geq 5$, assume $\mathbf{Q}(\mu_p) \not\subset L$, $p \nmid \# G_{L/\mathbf{Q}}$. Then one can look at

$$\overline{\rho} : G_{L/\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_\mathfrak{p}).$$

Also assume that all weight 1 forms of level $\Gamma_1(N)$ over $\mathbf{F}_\mathfrak{p}$ lift to characteristic 0. (will check)

Definition 1.3.1. As usual, a **Taylor-Wiles prime** q **of level** n **for** $(g, \mathfrak{p})$ is a prime q such that

- $(q, N) = 1$ and $p^n \mid q - 1$, and
- we fix an ordering $\alpha \neq \beta \in \mathbf{F}_\mathfrak{p}$ such that

$$\overline{\rho}(\mathrm{Frob}_q) \sim \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

Now let $k = \mathscr{O}/\mathfrak{p}^n$, and $(\mathbf{Z}/q\mathbf{Z})_p^\times$ the p-part of the units, $k\langle 1 \rangle = (\mathbf{Z}/q\mathbf{Z})_p^\times \otimes k$, and $k\langle -1 \rangle = \mathrm{Hom}((\mathbf{Z}/q\mathbf{Z})_p^\times, k)$. Note $k\langle 1 \rangle, k\langle -1 \rangle$ are non-canonically isomorphic to k. Finally, for M an abelian group, denote

$$M\langle n \rangle = M \otimes k\langle n \rangle,$$

where $k\langle n \rangle = k\langle 1 \rangle^{\otimes n}$.

1.4. **Reduction of U_g at Taylor-Wiles primes.** Now fix a prime $\mathfrak{q}$ of L lying over a Taylor-Wiles prime q, and pick Φ_q a Frobenius element for q. Fix lifts

$$\rho(\Phi_q) = \begin{pmatrix} \widetilde{\alpha} & 0 \\ 0 & \widetilde{\beta} \end{pmatrix}$$

Then

$$\operatorname{Ad}^* \rho(\Phi_q) = \begin{pmatrix} \widetilde{\alpha}/\widetilde{\beta} & 0 & 0 \\ 0 & \widetilde{\beta}/\widetilde{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and let

$$e_q = 2\rho(\Phi_q) - \operatorname{Tr} \rho(\Phi_q) \in (\operatorname{Ad}^0 \rho)^{\Phi_q = 1}.$$

We have a map

$$\theta_q : U_g = (U_L \otimes \operatorname{Ad}^* \rho)^{G_{L/\mathbf{Q}}} \to (\mathbf{F}_q^\times \otimes \operatorname{Ad}^* \rho)^{\Phi_q=1} \xrightarrow{e_q} (\mathbf{F}_q^\times \otimes k)^{\Phi_q=1} = \mathbf{F}_q \otimes k = k\langle 1 \rangle,$$

which should be thought of as reduction at the Taylor-Wiles prime q.

By dualizing, we get a map

$$\theta_q^\vee : k\langle -1 \rangle \to U_g^\vee \otimes k.$$

Then choose $u \in U_g$ such that $\gcd([U_g : \mathscr{O}u], p) = 1$, and choose $u^* \in U_g^\vee$ such that $\gcd(\langle u, u^* \rangle, p) = 1$. Then the map is

$$z \mapsto u^* \otimes \frac{\langle \theta_q(u), z \rangle}{\langle u, u^* \rangle}$$

1.5. **Derived Hecke Operators.** We have the usual covering $X_1(q) \to X_0(q)$, defined over $\mathbf{Z}[1/N]$, and we let $X_1(q)^\Delta$ be the subcovering whose Galois group over $X_0(q)$ is $(\mathbf{Z}/q\mathbf{Z})_p^\times$, which defines an étale covering over $\mathbf{Z}[1/qN]$.

This covering corresponds to a class $\sigma \in H^1_{\text{et}}(X_0(q)_k, k\langle 1 \rangle)$. But we have a map $k \to \mathbf{G}_a$ of étale sheaves over $X_0(q)_k$, so we get a map

$$H^1_{\text{et}}(X_0(q)_k, k\langle 1 \rangle) \to H^1_{\text{et}}(X_0(q)_k, \mathbf{G}_a\langle 1 \rangle) = H^1_{\text{Zar}}(X_0(q)_k, \mathscr{O}\langle 1 \rangle)$$

Now we can define the Hecke operator. Let $X = X(\Gamma_1(N)) \xleftarrow{\pi_1} X(\Gamma_0(q) \cap \Gamma_0(N)) \xrightarrow{\pi_2} X$. Let σ_X denote the pullback of σ in $H^1(X(\Gamma_1(N) \cap \Gamma_0(q))_k, \mathscr{O}\langle 1 \rangle)$. Then we have (for $z \in k\langle 1 \rangle$)

$$\Gamma_{q,z} : H^0(X_k, \omega) \xrightarrow{\pi_1^*} H^0(X_0(qN), \omega) \xrightarrow{\cup \sigma_X z} H^1(X_{01}(qN), \omega) \xrightarrow{\pi_2} H^1(X_k, \omega)$$

Remark 1.5.1. This is really a construction that works over characteristic p fields.

Conjecture 1.5.1 (?). *Denote $H^*(X, \omega)[g]$ the eigenspace for the system of Hecke eigenvalues of g. Then there exists an action $\star$ of $U_g^\vee$ on $H^*(X, \omega)[g]$ such that (let $\overline{g}$ denote the reduction to k)*

$$T_{q,z}\overline{g} = \alpha\overline{(\widetilde{\theta_q^\vee(z)} \star g)}$$

where the tilde over $\theta_q^\vee(z)$ denotes an arbitrary lift, and $\alpha \in E$ is independent of $(\mathfrak{p}, n, q, z)$.

Remark 1.5.2. To be clear, if $x, y \in V$ a k-vector space, then $x = \alpha y$ for $\alpha \in E$ if there exist elements $A, B \in \mathscr{O}$ such that $\alpha = A/B$ with A, B not both divisible by p, with $Ay = Bx$.

2. Numerical Evidence for the Conjecture

Say $x \sim y$ if there exists $\alpha \in E$ such that $x = \alpha y$, as previously defined.

Then the conjecture says that $T_{q,z}\overline{g} \sim \overline{\widetilde{\theta_q^\vee} \star g}$.

Let $g' = \sum_n \overline{a_n} q^n$ denote the mod p reduction of g. We have

$$[\cdot,\cdot]_R : H^1(X_R, \omega) \times H^0(X_R, \omega(-1)) \to H^1(X_R, \Omega^1) \cong R$$

where $\Omega^1 = \omega^{\otimes 2}(-D)$ for D a cusp divisor, and R is some $\mathbf{Z}[1/N]$-algebra.

Then

$$[\overline{\widetilde{\theta_q^\vee} \star g}, \overline{g}']_k = [\overline{\widetilde{\theta_q^\vee}} \star g, \overline{g}']_k = \langle \theta_q(u), z\rangle \cdot \overline{\frac{[u^* g, g']_{\mathscr{O}[\frac{1}{N}]}}{\langle u, u^*\rangle}}$$

Furthermore,

$$[T_{q,z}\overline{g}, g']_k \sim \langle \theta_q(u), z\rangle.$$

But this is

$$[\pi_{2,*}(\pi_1^*\overline{g} \cup z\sigma_X), g']_k = [\pi_1^*\overline{g} \cup z\sigma_X, \pi_2^* g']_k = \langle \pi_1^*\overline{g} \cdot \pi_2^* g', z\sigma_X\rangle_k = \langle G^{\text{proj}}, z\sigma\rangle$$

where $\langle\cdot,\cdot\rangle_k : H^0(X_k, \Omega^1) \times H^1(X_k, \mathscr{O}) \to k$ is the usual Serre duality pairing, and G^{proj} is the pushforward of $\pi_1^* g \cdot \pi_2^* g$ to level $\Gamma_0(q)$.

So now we have $\langle G^{\text{proj}}, z\sigma\rangle_k \sim \langle \theta_q(u), z\rangle_k$, and we may as well get rid of the z, so $\langle G^{\text{proj}}, \sigma\rangle_k \sim \theta_q(u)$ in $k\langle 1\rangle$.

Now we've reduced the situation to something a bit more easily computable.

2.1. Morel's Computation.

Denote by E_2 the Eisenstein cusp form of weight 2 over k. Then $E_2 \in H^0(X_0(q)_k, \Omega^1)$. We want to use the pairing

$$\langle \sigma, E_2\rangle_k = \text{computed explicitly over } \mathbf{F}_p$$

and compare this with the pairing $\langle \sigma, G^{\text{proj}}\rangle_k$ that we care about.

We denote $\varpi_{\text{Morel}} = \zeta^2 \prod_{i=1}^{(q-1)/2} i^{-8i} \in (\mathbf{Z}/q\mathbf{Z})^\times$ where $\zeta = 1$ if q = 2 mod 3 or $2^{q-1}/3$ otherwise.

Claim 2.1.1 (Morel)**.** *$\langle \sigma, E_2\rangle_k \equiv \varpi_{\text{Morel}}$ mod p.*

Claim 2.1.2 (Morel)**.** *$\varpi_{\text{Morel}} \neq 0$ if and only if $\operatorname{rank}_{\mathbf{Z}_p} \mathbf{T}_I = 1$ where $\mathbf{T}$ is the Hecke algebra acting over $\mathbf{Z}_p$ for weight 2 cusp forms, and I is the Eisenstein ideal, defined as*

$$0 \to I \to \mathbf{T} \to \mathbf{F}_p \to 0$$

where $\mathbf{T} \to \mathbf{F}_p$ sends $T_\ell \mapsto (\ell + 1)$.

Write

$$\mathbf{T} \to \oplus_{\mathfrak{m} \neq I} \mathbf{T}_{\mathfrak{m}} \oplus \mathbf{T}_I$$

and take $G^{\text{proj}} = G_I^{\text{proj}} + G_{I'}^{\text{proj}}$. Then, finally, the Shimura class is killed by the Eisenstein ideal I (this is due to Mazur), so

$$\langle \sigma, G^{\text{proj}}\rangle_k = \langle \sigma, G_I^{\text{proj}}\rangle_k$$

Now assuming $\varpi_{\text{Morel}} \neq 0$, so $a_1(G_I^{\text{proj}})E = G^{\text{proj}}$ and we finally conclude that

$$\langle \sigma, G^{\text{proj}}\rangle = \varpi_{\text{Morel}} \otimes a_1(G_I^{\text{proj}}) \mod p.$$