1. Notation and statemant of results

Let G be a simply connected semisimple split algebraic group over $\mathbb{Q}$. Let A be a split maximal torus of G.

Let $G^\vee$ be the Langlands dual of G, considered as a split Chevalley group over $\mathbb{Z}$. Let $T^\vee$ be a split maximal torus of $G^\vee$.

For a compact open subgroup $K \subset G(\mathbb{A}_f)$, let

$$Y(K) := G(\mathbb{Q})\backslash G(\mathbb{A})/KK_\infty .$$

Let $\mathbb{T}_K$ be the image of the Hecke algebra in $\operatorname{End} R\Gamma(Y(K), \mathbb{Z}_p)$.

We will fix a level K_0 as our "base level" and write $Y(1)$ for $Y(K_0)$. The level K_0 need not be a maximal compact subgroup of $G(\mathbb{A}_f)$.

Venkatesh makes quite a few technical assumptions. I will not attempt to list all of them, but I will try to indicate when I make use of an assumption.

Assume that we are given a homomorphism $\mathbb{T}_{K_0} \to \mathbb{Z}_p$ and a corresponding Galois representation $\rho\colon \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to G^\vee(\mathbb{Z}_p)$. We will write $\bar\rho$ for the mod p reduction of ρ. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}_{K_0}$ given by

$$\mathfrak{m} := \ker(\mathbb{T}_{K_0} \to \mathbb{F}_p) .$$

Let

$$V := H^1_f\left(\mathbb{Z}\left[\frac{1}{S}\right], \operatorname{Ad}^* \rho(1)\right)^\vee .$$

We will define an action of V on $H^*(Y(1), \mathbb{Z}_p)_\mathfrak{m}$.

Recall that for $n \in \mathbb{N}$, a Taylor-Wiles prime of level n is a prime q such that:

1. K_0 is hyperspecial at q.
2. $q \equiv 1 \pmod{p^n}$,
3. $\bar\rho(\operatorname{Frob}_q)$ is conjugate to a strongly regular element of $T^\vee(k)$ (i.e. an element whose centralizer inside $G^\vee$ is equal to $T^\vee$).

For q is a Taylor-Wiles prime of level n, we define

$$T_q := A(\mathbb{F}_q)/p^n .$$

We will construct a natural embedding

$$\iota_{q,n}\colon H^1(T_q, \mathbb{Z}/p^n) \hookrightarrow \left(\mathscr{H}^{(1)}_{q,\mathbb{Z}/p^n}\right)_\mathfrak{m}$$

as well as a map

$$f_{q,n}\colon H^1(T_q, \mathbb{Z}/p^n) \to V/p^n .$$

Theorem 1. *There exists a function $a\colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ and an action of V on $H^*(Y(1), \mathbb{Z}_p)_\mathfrak{m}$ by endomorphisms of degree 1 having the following property:*

- (*) *For any $n \geq 1$ and any prime $q \equiv 1 \pmod{p}^{a(n)}$, equipped with a strongly regular element of $T^\vee(k)$ conjugate to $\bar\rho(\operatorname{Frob}_q)$, the actions of $H^1(T_q, \mathbb{Z}/p^n)$ on $H^*(Y(K), \mathbb{Z}/p^n)_\mathfrak{m}$ via $f_{q,n}$ and $\iota_{q,n}$ coincide.*

The property (*) uniquely characterizes the V-action.

Moreover, V freely generates an exterior algebra inside the ring of endomorphisms of $H^*(Y(K), \mathbb{Z}_p)_\mathfrak{m}$, and the global derived Hecke algebra $\tilde{\mathbb{T}}$ coincides with this exterior algebra.

Remark 2. Venkatesh also shows that if $H^*(Y(1), \mathbb{Z}_p)_\mathfrak{m} \cong \mathbb{Z}_p$, then $\wedge^* V \to \tilde{\mathbb{T}}_{K_0}$ is an isomorphism. In general, we only expect this map to be an isomorphism after tensoring with $\mathbb{Q}_p$.

Now we state a conjecture about the rationality of this action. The Langlands program predicts that there is a motive M_{coad} associated with the automorphic representation Π and the coadjoint representation $G^\vee$. We assume that this motive has $\mathbb{Q}$-coefficients. The étale realization of M_{coad} should be $\mathrm{Ad}^* \rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. There should be a regulator map

$$H^1_{\mathrm{mot}}(\mathbb{Q}, M_{\mathrm{coad},\mathbb{Z}}(1)) \to H^1_f\left(\mathbb{Z}\left[\frac{1}{S}\right] \mathrm{Ad}^* \rho(1)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\,.$$

We assume that the above map becomes an isomorphism after tensoring the left-hand side with $\mathbb{Q}_p$. Let $V_{\mathbb{Q}_p} := V \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let $V_{\mathbb{Q}}$ consist of those classes in $V_{\mathbb{Q}_p}$ whose pairing with the image of the above map lies in $\mathbb{Q}$.

Conjecture 3. *The action of $\bigwedge^* V_{\mathbb{Q}_p}$ on $H^*(Y(K), \mathbb{Q}_p)_\Pi$ defined in Theorem 1 has the property that the action of $V_{\mathbb{Q}}$ preserves $H^*(Y(K), \mathbb{Q})_\Pi$.*

2. The map $\iota_{q,n}$

Now we explain the construction of the map $\iota_{q,n}$. Let $\mathrm{Frob}_q^T \in T^\vee(\mathbb{F}_p)$ be an element conjugate to $\rho(\mathrm{Frob}_q)$. A rough description of $\iota_{q,n}$ is "pull back to $Y_0(q)$, project to Frob_q^T-eigenspace, cup with α, push down to $Y(1)$."

Recall that we previously defined a Satake isomorphism

$$\mathscr{H}_q \xrightarrow{\sim} ((\mathbb{Z}/p^n)[X_*] \otimes H^*(\mathbf{A}(\mathbb{F}_q), \mathbb{Z}/p^n))^W$$

Note that $H^1(T_q, \mathbb{Z}/p^n) \to H^1(\mathbf{A}(\mathbb{F}_q), \mathbb{Z}/p^n)$ is an isomorphism, so in degree 1 the isomorphism can be written

$$\mathscr{H}_q^{(1)} \xrightarrow{\sim} \left((\mathbb{Z}/p^n)[X_*] \otimes H^1(T_q, \mathbb{Z}/p^n)\right)^W$$

Now let $\chi\colon X_*(\mathbf{A}) = X^*(T^\vee) \to \mathbb{F}_p^\times$ be the character determined by Frob_q^T. This character determines a map $(\mathbb{Z}/p^n)[X_*] \to \mathbb{F}_p$. Let $\mathfrak{m}$ denote the kernel of the restriction $(\mathbb{Z}/p^n)[X_*]^W \to \mathbb{F}_p$, and let $\tilde{\mathfrak{m}} = \mathfrak{m}(\mathbb{Z}/p^n)[X_*]$. Then

$$(\mathbb{Z}/p^n)[X_*]_{\tilde{\mathfrak{m}}} \cong \bigoplus_{w \in W} (\mathbb{Z}/p^n)[X_*]_{w\chi}$$

where $(\mathbb{Z}/p^n)[X_*]_{w\chi}$ is the competion of $(\mathbb{Z}/p^n)[X_*]$ at the kernel of $w\chi$. The composite

$$((\mathbb{Z}/p^n)[X_*] \otimes H^*(T_q, \mathbb{Z}/p^n))^W_{\mathfrak{m}} \to (\mathbb{Z}/p^n)[X_*]_{\tilde{\mathfrak{m}}} \otimes H^*(T_q, \mathbb{Z}/p^n) \to (\mathbb{Z}/p^n)[X_*]_\chi \otimes H^*(T_q, \mathbb{Z}/p^n)$$

is an isomorphism. So to define the action of $H^1(T_q, \mathbb{Z}/p^n)$, we just need to exhibit a map

$$H^1(T_q, \mathbb{Z}/p^n) \to \mathbb{Z}/p^n[X_*]_\chi \otimes H^*(T_q, \mathbb{Z}/p^n)\,.$$

We just take

$$x \mapsto 1 \otimes x\,.$$

3. The map $f_{q,n}$

Now we define the map $f_{q,n}$. Venkatesh shows that for all $q \in Q_n$, any deformation of $\rho|_{G_{\mathbb{Q}_q}}$ is conjugate to one with image in $T^\vee$, so in particular the deformation must be abelian. (The proof is by an explicit computation. The representation must factor through the tame quotient of G_q, with has the presentation $FtF^{-1} = t^q$. Since F is strongly regular, it is conjugate to an element in $T^\vee$. After conjugating,

t must be in the normalizer of $T^\vee$. But any element of the normalizer that reduces to the identity must be in $T^\vee$.)

Let ρ_n be the mod-p^n reduction of ρ. We can compute the cohomology of $(\operatorname{Ad}\rho_n)|_{G_{\mathbb{Q}_q}}$ using $\operatorname{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)^{\mathrm{ab}} \cong \widehat{\mathbb{Q}_q^\times} \cong \hat{\mathbb{Z}} \times \mathbb{F}_q^\times \times \mathbb{Z}_q$. In particular, we get an isomorphism

$$\frac{H^1(\mathbb{Q}_q, \operatorname{Ad}\rho_n)}{H^1_{\mathrm{ur}}(\mathbb{Q}_q, \operatorname{Ad}\rho_n)} \cong H^1(\mathbb{F}_q^\times, \operatorname{Ad}\rho_n) \cong \operatorname{Hom}(\mathbb{F}_q^\times, \operatorname{Lie}(T^\vee)\otimes\mathbb{Z}/p^n) \cong \operatorname{Hom}(T_q, \mathbb{Z}/p^n) \cong H^1(T_q, \mathbb{Z}/p^n)\,.$$

There is a pairing

$$H^1_f(\mathbb{Z}[1/S], \operatorname{Ad}^*\rho(1)) \times \frac{H^1(\mathbb{Q}_q, \operatorname{Ad}\rho_n)}{H^1_{\mathrm{ur}}(\mathbb{Q}_q, \operatorname{Ad}\rho_n)} \to \mathbb{Z}/p^n$$
$$\alpha, \beta_q \mapsto (\alpha_q \bmod p^n, \beta_q)_q$$

which induces a map

$$H^1(T_q, \mathbb{Z}/p^n) \to \operatorname{Hom}\left(H^1_f(\mathbb{Z}[1/S], \operatorname{Ad}^*\rho(1)), \mathbb{Z}/p^n\right) \cong V/p^n\,;$$

here we used the fact that $H^1_f(\mathbb{Z}[1/S], \operatorname{Ad}^*\rho(1))$ has no torsion. (This uses the assumption that $H^0(\mathbb{Z}[1/S], \operatorname{Ad}\bar\rho) = H^2(\mathbb{Z}[1/S], \operatorname{Ad}\bar\rho) = 0$.)

4. Convergent sequences of Taylor-Wiles data

We still need to show that $\iota_{q,n}$ factors through $f_{q,n}$. We will need to make use of convergent sequences of Taylor-Wiles data.

Recall that a Taylor-Wiles datum Q is a collection of Taylor-Wiles primes. We want to consider a sequence $\{Q_n\}$ of Taylor-Wiles data. Define

$$T_n := \prod_{q\in Q} T_q$$
$$S_n := (\mathbb{Z}/p^n)[T_n]$$

R_n, $\bar{R}_n$, certain quotients of the Galois deformation ring at level Q

C_n, a complex computing the cohomology of $Y_0(Q_n)$:

$$R\Gamma(\operatorname{Hom}_{S_n}(C_n, \mathbb{Z}/p^n)) \xrightarrow{\sim} R\Gamma(Y(1), \mathbb{Z}/p^n)$$

It is possible to find a sequence of Taylor-Wiles data that are convergent, so that we can find limits S, R, C along with maps $S \to S_n$, $R \to R_n$, $C \to C_n$ such that

(1) R and S are power series rings over $\mathbb{Z}_p$.
(2) C is quasi-isomorphic to a shift of R.
(3) $H^*(\operatorname{Hom}_S(C, \mathbb{Z}/p^n)) \cong H^*(\operatorname{Hom}_{S_n}(C_n, \mathbb{Z}/p^n)) \cong H^*(Y(1), \mathbb{Z}/p^n)_{\mathfrak{m}}$.

5. Relation between ι_{Q_n} and f_{Q_n}

Given a Taylor-Wiles datum Q_n, we can assemble the maps $\iota_{q,n}$, $f_{q,n}$ for $q \in Q_n$ into maps

$$\iota_{Q_n}\colon H^1(T_n, \mathbb{Z}/p^n) \to \bigotimes_{q\in Q_n} (\mathscr{H}_q)_{\mathfrak{m}}$$
$$f_{Q_n}\colon \operatorname{Hom}(T_n, \mathbb{Z}/p^n) \to V/p^n\,.$$

We want to show that if Q_n is part of a convergent sequence, then ι_{Q_n} factors through f_{Q_n}. We claim that:

(1) $H^1(T_n, \mathbb{Z}/p^n) \cong \mathfrak{t}_{S_n}$, the tangent space to S_n

(2) Under this identification, $\mathfrak{t}_{R_n}$, the tangent space to R_n, acts trivially.
(3) Let $W_n := \operatorname{coker}(\mathfrak{t}_{R_n} \to \mathfrak{t}_{S_n})$; then $W_n \cong V/p^n$.

The first claim is straightforward to prove. Recall that

$$\mathfrak{t}_{S_n} \cong \operatorname{Hom}(I_n/I_n^2, \mathbb{Z}/p^n) \cong \operatorname{Hom}_{S_n}(I_n, \mathbb{Z}/p^n) \cong \operatorname{Ext}^1_{S_n}(\mathbb{Z}/p^n, \mathbb{Z}/p^n) \cong H^1(T_n, \mathbb{Z}/p^n)$$

where I_n is the augmentation ideal of S_n.

6. Vanishing of $\mathfrak{t}_{R_n}$-action

Now we prove the second claim. In the following diagram (which also appeared in section 7 of the paper), the first row is the cup product that appears in the definition of the Hecke operator. The second row shows that the action factors through $\operatorname{Ext}^j_S(\mathbb{Z}/p^n, \mathbb{Z}/p^n)$. Because we assumed that $H^i(\operatorname{Hom}_S(C, \mathbb{Z}/p^n))$ is torsionfree, we can lift to $H^i(\operatorname{Hom}_S(C, \mathbb{Z}_p))$, thus allowing us to compute the action using the third row. Using the torsionfree assumption again allows us to move to the fourth row.

$$\begin{array}{ccccc}
H^i\left(\operatorname{Hom}_{S_n}(C \otimes_S S_n, \mathbb{Z}/p^n)\right) & \times & \operatorname{Ext}^j_{S_n}(\mathbb{Z}/p^n, \mathbb{Z}/p^n) & \longrightarrow & H^{i+j}\left(\operatorname{Hom}_{S_n}(C \otimes_S S_n, \mathbb{Z}/p^n)\right) \\
\downarrow \wr & & \downarrow & & \downarrow \wr \\
H^i\left(\operatorname{Hom}_S(C, \mathbb{Z}/p^n)\right) & \times & \operatorname{Ext}^j_S(\mathbb{Z}/p^n, \mathbb{Z}/p^n) & \longrightarrow & H^{i+j}\left(\operatorname{Hom}_S(C, \mathbb{Z}/p^n)\right) \\
\twoheaduparrow & & \downarrow & & \| \\
H^i\left(\operatorname{Hom}_S(C, \mathbb{Z}_p)\right) & \times & \operatorname{Ext}^j_S(\mathbb{Z}_p, \mathbb{Z}/p^n) & \longrightarrow & H^{i+j}\left(\operatorname{Hom}_S(C, \mathbb{Z}/p^n)\right) \\
\| & & \twoheaduparrow & & \twoheaduparrow \\
H^i\left(\operatorname{Hom}_S(C, \mathbb{Z}_p)\right) & \times & \operatorname{Ext}^j_S(\mathbb{Z}_p, \mathbb{Z}_p) & \longrightarrow & H^{i+j}\left(\operatorname{Hom}_S(C, \mathbb{Z}_p)\right)
\end{array}$$

In the fourth row, since S and R are power series rings over $\mathbb{Z}_p$, it is straightforward to check that

$$\operatorname{Ext}^*_S(\mathbb{Z}_p, \mathbb{Z}_p) \cong \wedge^* \mathfrak{t}_S$$

$$H^*\left(\operatorname{Hom}_S(C, \mathbb{Z}_p)\right) \cong \operatorname{Ext}^*_S(R, \mathbb{Z}_p) \cong \wedge^*(\mathfrak{t}_S/\mathfrak{t}_R)$$

and $\mathfrak{t}_R \subset \mathfrak{t}_S$ acts trivially on this module.

7. Relationship between W_n and V/p^n

Recall from earlier in the lecture that

$$\mathfrak{t}_{S_n} \cong \bigoplus_{q \in Q_n} \frac{H^1(\mathbb{Q}_q, \operatorname{Ad}\rho_n)}{H^1_{\mathrm{ur}}(\mathbb{Q}_q, \operatorname{Ad}\rho_n)}.$$

Similarly, we can identify

$$\mathfrak{t}_{R_n} \cong H^1_f\left(\mathbb{Z}\left[\frac{1}{SQ_n}\right], \operatorname{Ad}\rho_n\right).$$

The pairing that we defined previously extends to a pairing

$$H^1_f\left(\mathbb{Z}\left[\frac{1}{S}\right], \operatorname{Ad}^*\rho_n(1)\right) \times W_n \to \mathbb{Z}/p^n$$

$$\alpha, (\beta_v)_{v \in Q_n} \mapsto \sum_{v \in Q_n} (\alpha_v, \beta_v)_v$$

This pairing is well-defined since each α_v is unramified, and if (β_v) comes from a global class, then the pairing vanishes by global reciprocity.

Now we want to show that the above pairing is perfect. By our assumptions, both W_n and $H^1_f\left(\mathbb{Z}\left[\frac{1}{S}\right], \mathrm{Ad}^*\rho_n(1)\right)$ are free $\mathbb{Z}/p^n$-modules. An Euler characteristic computation shows that they both have rank δ, the defect of G.

So it suffices to check that

$$\mathfrak{t}_{S_n} \to H^1_f\left(\mathbb{Z}[1/S], \mathrm{Ad}^*\bar{\rho}(1)\right)^\vee$$

is surjective. Recall that we chose the Taylor-Wiles set Q_n so that

$$H^1_f\left(\mathbb{Z}\left[1/S\right], \mathrm{Ad}^*\bar{\rho}(1)\right) \hookrightarrow \prod_{v\in Q_n} H^1\left(\mathbb{Q}_v, \mathrm{Ad}^*\bar{\rho}(1)\right) .$$

In fact, the image is contained in the subspace of unramified classes. Applying local Tate duality gives us exactly the surjection that we need.

In conclusion, we have shown

$$W_n \cong V/p^n := H^1_f\left(\mathbb{Z}\left[1/S\right], \mathrm{Ad}^*\rho_n(1)\right)^\vee .$$

Defining the action of V requires a bit of additional work to show that if Q_n belongs to a convergent sequence of Taylor-Wiles data, then the action of V_n does not depend on any choices.