The Conjecture and its Complex Realization: Motivic Cohomology and the Beilinson Conjectures

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1 Recap

Recall the conjecture of Venkatesh.

Conjecture 1. *Let G be a semi-simple simply connected split algebraic group over $\mathbb{Q}$ and let π be a tempered cuspidal cohomological automorphic representations with associated p-adic Galois representation $\rho_{\pi,l}$. Then there is an action of*

$$\wedge^* H^1_f(\mathbb{Z}[1/S], \mathrm{Ad}^* \rho(1))^\vee$$

on

$$H^*(Y(1), \mathbb{Q}_p)_\Pi.$$

Assume that there is a motive M_{coad} associated to π. Then the image of the motivic cohomology of M_{coad} should define a $\mathbb{Q}$-structure on

$$\wedge^* H^1_f(\mathbb{Z}[1/S], \mathrm{Ad}^* \rho(1))^\vee$$

which we conjecture to preserve the $\mathbb{Q}$-structure on

$$H^*(Y(1), \mathbb{Q}_p)_\Pi$$

coming from $H^*(Y(1), \mathbb{Q})$.

A consequence of this action at the motivic level is an action in the complex case. This leads explicit verifiable statements that can give partial evidence for the conjecture. References is [PV16] sections 1,2,4

2 Classical Motives

Idea: Cohomology is functorial in correspondences and all the usual cohomology theories admit cycle class maps.

Definition 1. *A Chow motive (resp. homological motive) over K is a triple*

$$(X, e, n)$$

where X is a smooth projective variety over K of dimension d,the element e is an element of the rational Chow group $\mathrm{CH}^d(X \times X)$ of codimension d cycles, up to rational equivalence (respectively up to homological equivalence), that is an idempotent, and $n \in \mathbb{Z}$. A morphism of Chow motives (resp homological motives) $(X, e, n) \to (Y, f, m)$ is an element of

$$f\, \mathrm{CH}^{d_X+m-n}(X \times Y)e,$$

composition is given by composition of correspondences. This gives us categories $\mathcal{CHM}$ and $\mathrm{Hom}\,\mathcal{M}$.

Remark 1. Since homological equivalence is coarser than rational equivalence there is a forgetful functor

$$\mathcal{CHM} \to \mathrm{Hom}\,\mathcal{M}.$$

Moreover there is a functor $h : \mathrm{SmProjVar}^{op} \to \mathcal{CHM}$ given by

$$X \mapsto (X, \Delta_X, 0),$$

with Δ_X the class of the diagonal, and which sends a morphism $f : X \to Y$ to the graph

$$\Gamma_f \in \mathrm{CH}^{d_x}(X \times Y)$$

Example 1. Let C be a smooth projective curve, write $h(C)(1)$ for $(C, \Delta_C, 1)$ and let 1 denote $h(\operatorname{Spec} K)$. Then

$$\hom_{\mathcal{CHM}}(1, h(C)(1)) = \mathrm{Pic}(C)$$

and

$$\hom_{\mathrm{Hom}\,\mathcal{M}}(1, h(C)(1)) = \frac{\mathrm{Pic(C)}}{\mathrm{Pic^0(C)}} \cong \mathbb{Q}.$$

Remark 2. By definition we obtain a factorisation

$$\begin{array}{ccc} \mathrm{SmProjVar}^{op} & \xrightarrow{h} & \mathrm{Hom}\,\mathcal{M} \\ & \searrow^{\bigoplus H_B^i} & \downarrow{r_{\mathcal{H}}} \\ & & \mathrm{PHS}_{\mathbb{Q}}, \end{array}$$

where H_B^i denotes the i-th singular cohomology and $\mathrm{PHS}_{\mathbb{Q}}$ denotes the category of $\mathbb{Q}$-linear pure Hodge structures. It is conjectured that $r_{\mathcal{H}}$ is fully faithful (the Hodge conjecture).

3 Motivic Cohomology

The categories of Chow motives and homological motives don't have the correct Ext groups. We can try to correct this by comparing with topology. Grothendieck showed that

$$K_0(X)_{\mathbb{Q}} \cong \bigoplus_i \mathrm{CH}^i(X)$$

and also gave an intrinsic definition of the piece $K_0^{(i)}(X)_{\mathbb{Q}}$ mapping to $\mathrm{CH}^i(X)$. This definition extends to higher K-groups (always with rational coefficients). The Atiyah–Hirzebruch spectral sequence for a CW-complex X is

$$E_2^{p,q} := H_B^{p-q}(X, \mathbb{Q}(-q)) \Rightarrow K_{-p-q}(X)_{\mathbb{Q}}.$$

It degenerates to give isomorphisms

$$H_B^i(X, \mathbb{Q}(n)) \cong K_{2n-i}^{(n)}(X)_{\mathbb{Q}}$$

where K denotes topological K-theory. If X is a scheme then we set

$$H_{\mathcal{M}}^i(X, \mathbb{Q}(n)) = K_{2n-i}^{(n)}(X)_{\mathbb{Q}}$$

to be the motivic cohomology of X, this extends to any Chow motive. There are issues with defining motivic cohomology for homological motives. Conjecturally, this can be resolved by taking graded pieces with respect to a filtration on motivic cohomology (see below), but we will ignore this issue.

Example 2. If F is a number field. Then

$$K_0(F)_{\mathbb{Q}} = \mathbb{Q} \subset K_0^{(0)}(F)_{\mathbb{Q}}.$$

More generally

$$K_n^M(F)_{\mathbb{Q}} = K_n^{(n)}(F)_{\mathbb{Q}},$$

which in particular gives

$$K_1(F)_{\mathbb{Q}} = K_1^M(F)_{\mathbb{Q}} = F^\times \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Borel calculated for $j > 1$ that

$$K_j(F)_{\mathbb{Q}} = \begin{cases} 0 & \text{if } j \text{ even} \\ \mathbb{Q}^{r_1+r_2} & \text{if } j = 1 \mod 4 \\ \mathbb{Q}^{r_2} & \text{if } j = 3 \mod 4 \end{cases},$$

and these classes lie in $K_{2t-1}^{(t)}(F)_{\mathbb{Q}}$.

As a table $H_{\mathcal{M}}^i(F, \mathbb{Q}(n))$ is given by

$i \setminus n$	0	1	2	3	4	5
0	$\mathbb{Q}$	0	0	0	0	$\cdots$
1	0	$F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$	0	$\mathbb{Q}^{r_2}$	0	$\mathbb{Q}^{r_1+r_2}$
2	0	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 1: *Motivic cohomology of a number field F.*

Note that the motivic cohomology vanishes in negative degrees. This is not clear from the definition. It is not supported in degrees $[0, 2d]$, in this sense it is an "arithmetic cohomology theory" as a opposed to a "geometric cohomology theory" (for example a Weil cohomology theory). For motives of weight zero the group $H_{\mathcal{M}}^1(M, \mathbb{Q}(1))$ is of particular interest. Having said this, "$F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ needs to be modified later to be considered "correct".

4 Deligne cohomology

Cycle class maps

$$\mathrm{cl} : H^{2n}_{\mathcal{M}}(X, \mathbb{Q}(n)) \to H^{2n}_{B}(X, \mathbb{Q}(n))$$

exists and assign cohomology classes to motivic classes. The kernel of this map consists of homologically trivial cycles. There is another method for creating cohomology classes, called Abel–Jacobi maps.

Definition 2. *Assume that $X/\mathbb{C}$ is a smooth projective variety. Then the p-th intermediate Jacobian is defined to be*

$$J^p(X) := \frac{H_B^{2p-1}(X, \mathbb{C})}{F^p H_B^{2p-1}(X, \mathbb{C}) + H_B^{2p-1}(X, \mathbb{Z})}.$$

This is a complex analytic torus of complex dimension $1/2 \dim H_B^{2p-1}(X, \mathbb{C})$.

Example 3. When $p = 1$ this is the Picard variety and when $p = \dim X$ this is the Albanese variety (which are canonically isomorphic when X is a curve).

Fact 1. There is a canonical map

$$AJ^p : CH^p(X)_{\mathrm{hom}\sim 0} \to J^p(X).$$

Example 4. If X is a curve, then the Abel–Jacobi map is constructed as follows. If we have a divisor

$$[P] - [Q] \sim 0.$$

Then we can choose a path γ from P to Q which gives us a functional in the dual space

$$H^0(X, \Omega^1_X)^\vee = H^1(X, \mathcal{O}_X) = H^1(X, \mathbb{C})/H^0(X, \Omega^1_X),$$

where the first equality is given by Serre duality. Different choices of paths differ up to choices of loops, which lie in

$$H^1(X, \mathbb{Z}).$$

In general we use higher cycles.

Definition 3. *Let $\mathbb{Z}(p)_{\mathcal{D}}$ be the complex*

$$(2\pi i)^p \mathbb{Z} \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{p-1}_X.$$

We define the Deligne cohomology of X to be the hypercohomology of this complex, that is

$$H^i_D(X, \mathbb{Z}(p)) := \mathbb{H}^i(X, \mathbb{Z}(p)_{\mathcal{D}}).$$

Let $\mathbb{Z}(p)$ denote $(2\pi i)^p\mathbb{Z}$.

Lemma 1. *We have a short exact sequence*

$$0 \to J^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}_p) \to H^{2p}_B(X, \mathbb{Z}(p))^{(0,0)} \to 0.$$

Proof. There is a short exact sequence of complexes

$$0 \to \Omega_X^{\bullet, \leq p-1} \to \mathbb{Z}(p)_{\mathcal{D}} \to \mathbb{Z}(p)$$

which gives a long exact sequence in cohomology. Now the cohomology of the first complex is

$$H^{2p-1}_{dR}(X)/F^p H^{2p-1}_{dR}(X)$$

and its image in $H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$ is then

$$H^{2p-1}_{dR}(X)/(F^p H^{2p-1}_{dR}(X) + H^{2p-1}(X, \mathbb{Z})) = J^p(X).$$

Similarly, the kernel of

$$H^{2p}_B(X, \mathbb{Z}(p)) \to H^{2p}_{dR}(X)/F^p H^{2p}_{dR}(X)$$

is precisely the Hodge classes. □

Theorem 1 (Deligne)**.** *There exists a canonical construction filling in the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}^p(X)_{hom\sim 0} & \longrightarrow & \mathrm{CH}^p(X) & \longrightarrow & \frac{\mathrm{CH}^p(X)}{\mathrm{CH}^p(X)_{hom\sim 0}} & \longrightarrow & 0 \\ & & \downarrow AJ^p & & \downarrow \exists & & \downarrow \mathrm{cl} & & \\ 0 & \longrightarrow & J^p(X) & \longrightarrow & H^{2p}_{\mathcal{D}}(X, \mathbb{Z}_p) & \longrightarrow & H^{2p}_B(X, \mathbb{Z}(p))^{(0,0)} & \longrightarrow & 0 \end{array}$$

Write $H^i_{\mathcal{D}}(X, \mathbb{R}(n))$ for the analogous construction with $\mathbb{Z}$ replaces by $\mathbb{R}$. These are $\mathbb{R}$ vector spaces and this definition extends to motives.

Theorem 2 (Beilinson)**.** *This extends to a map*

$$r_{\mathcal{D}} : H^i_{\mathcal{M}}(M, \mathbb{Q}(n)) \to H^i_{\mathcal{D}}(M, \mathbb{R}(n))$$

called the Beilinson regulator.

Remark 3. It is expected that there is a decreasing filtration on motivic cohomology such that

$$F^0 H^i_M(X, \mathbb{Q}(n)) = H^i_M(X, \mathbb{Q}(n))$$
$$F^1 H^{2n}_M(X, \mathbb{Q}(n)) = F^1 H^n(X) = \mathrm{CH}^n(X)_{\mathrm{hom}\sim 0}$$

and a bunch of other conditions. One hopes that F^0/F^1 can be seen using cycle class maps (e.g. Hodge/Tate type conjectures) and that F^1/F^2 can be seen using Abel-Jacobi maps (e.g. Beilinson/Bloch–Kato type conjectures).

Conjecture 2. *If X is an algebraic variety over a number field K, then*

$$F^2 = 0$$

5 Beilinson Conjectures

Notation (due to Scholl). We define a subspace

$$H^i_{\mathcal{M}}(X_{\mathbb{Z}}, \mathbb{Q}(n)) \subset H^i_{\mathcal{M}}(X, \mathbb{Q}(n))$$

denote the "unramified" or "integral" elements. If X admits a proper regular model $\mathcal{X}$, then

$$H^i_{\mathcal{M}}(X_{\mathbb{Z}}, \mathbb{Q}(n)) = \operatorname{im}\left(K^{(n)}_{2n-i}(\mathcal{X})_{\mathbb{Q}} \to K^{(n)}_{2n-i}(X)_{\mathbb{Q}}\right).$$

Example 5. When F is a number field then

$$K^{(1)}_1(\mathcal{O}_F)_{\mathbb{Q}} = \mathcal{O}_F^{\times} \otimes \mathbb{Q} \subset F^{\times} \otimes \mathbb{Q}$$

is a finite dimensional vector space by Dirichlet's unit theorem.

Notation: Let

$$H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) := H^i_{\mathcal{D}}(X, \mathbb{R}(n))^{\Phi=1}$$

where Φ is the tensor product of complex conjugation on $X(\mathbb{C})$ and of complex conjugation on the coefficients.

Remark 4. Deligne defines L-factors at ∞ for $L(M, s)$ in terms of the Hodge structure on M_B. Then $L_\infty(M, s)$ has a pole of order

$$\dim H^{i+1}_B(M_{\mathbb{R}}, \mathbb{R}(n))$$

at $s = i - n$

Conjecture 3 (Beilinson)**.** *If M is a motive of weight i and $i - 2n < -2$, then the Beilinson regulator*

$$r_{\mathcal{D}} : H^{i+1}_{\mathcal{M}}(M_{\mathbb{Z}}, \mathbb{Q}(n)) \otimes \mathbb{R} \to H^{i+1}_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(n))$$

is an isomorphism. If M has no Tate motives as summands then this should also be an isomorphism for $i - 2n = 2$.

Corollary 1. *If we have expected properties of $L(M, s)$ then the dimensions of the motivic cohomology groups can be computed as the order of vanishing of $L(M, s)$ at certain points (and vice versa).*

Remark 5. Since Deligne cohomology only depends on $M_{\mathbb{R}}$, a consequence of the conjecture is that these dimensions don't depend on the motive M, but only on the base change $M_{\mathbb{R}}$ of M to $\mathbb{R}$.

Fact 2. The conjecture is known for $M = \operatorname{Spec} F$ with F a number field (as a consequence of Borel's calculations of higher K-groups of number fields). Note that $\mathcal{O}_F^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$ only depends on $F \otimes_{\mathbb{Z}} \mathbb{R}$.

Remark 6. The Deligne cohomology groups have a canonical $\mathbb{Q}$-structure DR coming from the long exact sequence from earlier.

Conjecture 4. *With the same assumptions as in the previous conjecture, we have*

$$r_{\mathcal{D}}\left(\det H^{i+1}_M(M_{\mathbb{Z}}, \mathbb{Q}(n))\right) = L(M, n) \cdot \det DR \subset \det H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n))$$

of $\mathbb{Q}$ vector spaces inside $\det H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n))$

6 Motives associated to automorphic forms

Let G be a reductive group over F and let π be a tempered cuspidal cohomological automorphic representation with associated p-adic Galois representation $\rho_{\pi,l}$. Assume that π_v is defined over $\mathbb{Q}$ for almost all places v (we say that π has coefficient field $\mathbb{Q}$). By Langlands, π should give rise to a representation of G_F taking values in ${}^LG$. Composing with the adjoint representation of ${}^LG$ we obtain Galois represenions:

$$\operatorname{Ad}\rho_{\pi,l}\colon G_F \to \operatorname{Aut}(\hat{\mathfrak{g}}_{\overline{\mathbb{Q}}_l})$$

where $\hat{\mathfrak{g}} := \operatorname{Lie}({}^LG)$. Similarly, the archimedean parameter $W_{F_v} \to {}^LG$ (here W_{F_v} is the Weil group at v) for each infinite place allows us to define:

$$\operatorname{Ad}\rho_{\pi,v}\colon W_{F_v} \to \operatorname{Aut}(\hat{\mathfrak{g}}_{\mathbb{C}})$$

We use the representations $\operatorname{Ad}\rho_{\pi,l}, \operatorname{Ad}\rho_{\pi,v}$ to constrain the adjoint motive M_{ad}. This should be a motive over $\mathbb{Q}$, and the distinction between F and $\mathbb{Q}$ adds some technical complications to the conjecture.

Conjecture 5. *There exists a homological motive M_{ad} of weight 0 associated to π with the following properties*

1. *(rationality): there exists some Lie algebra $\mathfrak{g}'/\mathbb{Q}$ for which $\mathfrak{g}'_{\mathbb{C}}$ is a twist of $\hat{\mathfrak{g}}$ and we have isomorphisms for all infinite places v:*
$$i_v\colon H^i_B(M_{ad,v,\mathbb{C}}, \mathbb{Q}) \xrightarrow{\sim} \mathfrak{g}'_{\mathbb{C}}.$$
2. *(Galois equivariance of étale realisation): The isomorphism*
$$H^i_{\acute{e}t}(M_{ad,\bar{F}}, \mathbb{Q}_l) \cong H^i_B(M_{ad,v,\mathbb{C}}, \mathbb{Q}) \otimes \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathfrak{g}'_{\bar{\mathbb{Q}}_l} \cong \hat{\mathfrak{g}}_{\bar{\mathbb{Q}}_l}$$
respects the action of G_F on the left hand side given by étale cohomology and on the right hand side given by $\operatorname{Ad}\rho_{\pi,l}$.
3. *(Weil group equivariance of de Rham realisation):The isomorphism*
$$H^i_{dR}(M_{ad}) \otimes \mathbb{C} \cong H^i_B(M_{ad,v,\mathbb{C}}, \mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\sim} \mathfrak{g}'_{\mathbb{C}} \cong \hat{\mathfrak{g}}_{\mathbb{C}}$$
respects the action of W_{F_v} on the left hand side given by de Rham cohomology and on the right hand side given by $\operatorname{Ad}\rho_{\pi,v}$.
4. *(bilinear forms are motivic) Every $\mathbb{Q}$-valued bilinear form on $math\hat{f}rakg_{\mathbb{C}}$ which is "stable under ${}^LG$" arises from a pairing*
$$M_{ad} \otimes M_{ad} \to \mathbb{Q}(-m).$$

We then set M_{coad} to be $M_{\mathrm{ad}}^\vee$, where $(-)^\vee$ denotes dual within the category of homological motives. Since M_{ad} is of weight zero (if it exists), this coincides with the notion of "dual" which assigns to a motive N a motive N^* whose i^{th} cohomology is $H^i(N)^\vee$.

References

[PV16] Kartik Prasanna and Akshay Venkatesh. "Automorphic cohomology, motivic cohomology, and the adjoint L-function". In: *arXiv e-prints*, arXiv:1609.06370 (Sept. 2016), arXiv:1609.06370. arXiv: `1609.06370 [math.NT]`.