THE CLASSICAL AND ARITHMETIC STANDARD CONJECTURES

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1. Introduction

Intersection theory is an essential part of algebraic geometry, but becomes increasingly complicated as our objects of study become more general. For example, intersection theory on a nice smooth surface over an algebraically closed field is much simpler than intersection theory on a nice smooth variety of arbitrary dimension over an algebraically closed field, which is in turn simpler than a nice smooth variety of arbitrary dimension over $\operatorname{Spec} \mathbf{Z}$ (or any ring). And even then, we can allow singularities, and the theory becomes more complex.

The standard conjectures on algebraic cycles are conjectures that concern properties of the intersection pairings on the Chow groups of an algebraic variety, which are groups that house the basic objects of study in intersection theory: substructures of a geometric object that will be intersected with other ones. They were initially formulated by Alexander Grothendieck in the 1960s (see [Gro68] for an introduction) to prove that the category of pure motives is semisimple and abelian, but their full proofs have remained elusive since then.

The standard conjectures illustrate the aforementioned phenomenon nicely: in the case of surfaces over $k = \bar{k}$, the Hodge index theorem follows quite quickly after the development of basic intersection theory, using the Riemann-Roch theorem, and other auxiliary lemmas. But once we consider varieties of arbitrary dimension, we already cannot prove the more general Hodge standard conjecture for varieties in characteristic $p > 0$. Furthermore, all of the other conjectures remain unresolved in full generality, although some progress has been made towards their resolution.

The development of arithmetic intersection theory complicates the picture even more by considering schemes over $\operatorname{Spec} \mathbf{Z}$. Even in the case of arithmetic surfaces, there is already more information to keep track of, in the form of fibral and infinite divisors. Once we pass to higher dimensional arithmetic varieties, then even defining the basic objects that we work with (arithmetic Chow groups) takes significant work.

The aim of this thesis is to try to sew a thin thread through the evolution of intersection theory and the standard conjectures in these various cases. We start with classical intersection theory over $k = \bar{k}$ and prove the Hodge index theorem. We then define intersection theory on higher dimensional varieties, and state the standard conjectures. We then do the exact same thing for the arithmetic case, while trying to link the arithmetic and geometric stories back together at each step. This thesis is fairly self contained at the beginning, but appeals to other texts for certain technical proofs later on in the document.

In addition to the works cited in the paper, we also found [Tra07] and [Kle] useful in the preparation of this thesis.

2. Preliminaries

First, we will introduce some preliminary notions that are essential for a treatment of intersection theory.

2.1. **Cartier divisors.** We consider the notion of a Cartier divisor, which can be defined on an arbitrary scheme.

2.1.1. *Meromorphic Functions.* Fix a scheme X. If U is an open set in X, let $S(U)$ denote the set of elements in $\mathscr{O}_X(U)$ whose images in $\mathscr{O}_{X,x}$ are nonzerodivisors for all $x \in U$. Then $S(U)$ is a multiplicative set, so we can form the localization $M_X(U) = S(U)^{-1}\mathscr{O}_X(U)$. Then it is clear that M_X is a presheaf, so we can associate a sheaf $\mathscr{M}_X$ to it. We call this the *sheaf of meromorphic functions* on X.

Note there is a natural morphism of sheaves $\mathscr{O}_X \to \mathscr{M}_X$. which is a monomorphism because of the nonzerodivisor condition.

2.1.2. *Example.* If k is a field and $X = \operatorname{Spec} k[x]$, then $\mathscr{O}_X(U)$ is the ring of rational functions on an open set U in X. The image of any nonzero $f \in \mathscr{O}_X(U)$ in $\mathscr{O}_{X,x} = k[x]_\mathfrak{p}$ (x corresponds to a prime $\mathfrak{p} \subset k[x]$) is a nonzerodivisor for any x, since the localization of an integral domain is again an integral domain, so $M_X(U)$ is the fraction field of $\mathscr{O}_X(U)$, which is clearly $k(x)$. As such, $M_X = \mathscr{M}_X$ is just the constant sheaf $k(x)$, which is also isomorphic to $\mathscr{O}_{X,\xi} = k[x]_{(0)}$, where ξ is the generic point (0).

In fact, for any integral scheme X, $\mathscr{M}_X$ is the constant sheaf associated to $\mathcal{O}_{X,\xi}$, by the same argument.

2.1.3. *Invertible Elements.* To a sheaf of rings $\mathscr{F}$ on X, we can construct the sheaf $\mathscr{F}^*$ of invertible of elements, which is a sheaf of abelian groups, by defining

$$\mathscr{F}^*(U) = \{s \in \mathscr{F} : st = 1_U \text{ for some } t \in \mathscr{F}(U)\}$$

This is a presheaf: note if $st = 1_U$ in $\mathscr{F}(U)$ and $V \subseteq U$, then $s|_V t|_V = 1_V$. Furthermore, one shows easily that this is a sheaf of abelian groups.

The natural monomorphism of sheaves $\mathscr{O}_X \to \mathscr{M}_X$ restricts to a morphism $\mathscr{O}_X^* \to \mathscr{M}_X^*$. By taking the cokernel in $\mathsf{Mod}_{\mathscr{O}_X}$, we obtain the short exact sequence

$$0 \to \mathscr{O}_X^* \to \mathscr{M}_X^* \to \mathscr{M}_X^*/\mathscr{O}_X^* \to 0$$

2.1.4. *Cartier Divisors.* By applying the global sections functor $\Gamma(X, -)$, we obtain

$$0 \to \Gamma(X, \mathscr{O}_X^*) \to \Gamma(X, \mathscr{M}_X^*) \to \Gamma(X, \mathscr{M}_X^*/\mathscr{O}_X^*) \to H^1(\mathscr{M}_X^*/\mathscr{O}_X^*) \to \cdots$$

Let $\mathrm{Ca}(X) := \Gamma(X, \mathscr{M}_X^*/\mathscr{O}_X^*)$. We call this the group of Cartier divisors on X. Using the explicit construction of the sheafification functor, we see that a Cartier divisor can be described by a collection $\{(U_i, f_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open cover of X and $f_i \in \mathscr{M}_X^*(U_i)$, such that

$$f_i/f_j \in \mathscr{O}_X^*(W)$$

for some $W \subseteq U_i \cap U_j$ whenever $i \neq j$. Note that such a description is not necessarily unique: one can add or remove sets from the collection (subject to compatibility on intersections) as long as the U_i continue to cover X.

2.1.5. *Principal Divisors.* If a Cartier divisor is in the image of $\Gamma(X, \mathscr{M}_X^*) \to \mathrm{Ca}(X)$, then it is called *principal*. Note a principal divisor can be described with the singleton collection $\{(X, f)\}$ for $f \in \mathscr{M}_X^*(X)$. The group of Cartier divisors mod principal divisors is denoted $\mathrm{CaCl}(X)$. If two Cartier divisors represent the same element in $\mathrm{CaCl}(X)$, they are called *linearly equivalent*.

2.2. **Weil Divisors.** The definition of a divisor was motivated by the need to develop an intersection theory on the codimension 1 subschemes (subvarieties) of a scheme (algebraic variety over a field k). Under sufficient conditions, a divisor group with good structure can be constructed by taking the free abelian group on the irreducible codimension 1 substructures on a scheme or variety X. We now spell out those conditions and develop the theory of Weil divisors on a scheme.

For this section, let X be a Noetherian integral scheme.

2.2.1. *Prime Divisors.* Denote by $X^{(1)}$ the set of closed integral subschemes of codimension 1, or equivalently, their generic points. Elements of $X^{(1)}$ are called *prime divisors*. Then the set of Weil divisors $\mathrm{Div}(X)$ is the free abelian group on $X^{(1)}$. We typically write such sums as $\sum_Z n_Z Z$, where Z runs over the prime divisors.

2.2.2. *Principal Divisors.* As in the case of Cartier divisors, we can define a notion of a principal divisor. For a Cartier divisor, a principal divisor corresponds to a meromorphic (rational) function on X. One way of seeing the relationship between Cartier and Weil divisors says that the coefficients of a Weil divisor correspond to multiplicities of zeros and poles along the prime divisors with respect to a rational function.

Note X is integral, so $\mathscr{M}$ is the constant sheaf associated to $\mathscr{O}_{X,\xi} \cong \mathrm{Frac}(\mathscr{O}_X(X))$, where Frac denotes the fraction field. Call this $K(X)$, the field of rational functions on X. For any $f \in K(X)$, we intend to define a principal Weil divisor $\mathrm{div}(f)$.

By integrality, each prime divisor $Z \subset X$ has a generic point ξ_Z whose closure is all of Z. We define the multiplicity (order) $\mathrm{ord}_Z : \mathscr{O}_X^*(X) \to \mathbf{Z}$ by

$$\mathrm{ord}_Z(g) = \ell_{\mathscr{O}_{X,\xi_Z}}(\mathscr{O}_{X,\xi_Z}/(g)),$$

where $\ell_R(M)$ denotes the length of the R-module M, and g denotes the image of g in the stalk $\mathscr{O}_{X,\xi_Z}$ by abuse of notation. For clarity and legibility, we typically write $\mathscr{O}_{X,Z}$ instead of $\mathscr{O}_{X,\xi_Z}$.

2.2.2.1. **Lemma.** *For any prime divisor $Z \subset X$, ord_Z is a well-defined group homomorphism.*

Proof. Fix $g \in \mathscr{O}_X^*(X)$. Note $\mathscr{O}_{X,Z}$ is Noetherian of dimension 1 since Z is of codimension 1. If $(g)\mathscr{O}_{X,Z}$ is prime, then it is maximal, so $\mathscr{O}_{X,Z}/(g)$ is a field, and thus has Krull dimension 0. If not, then $\mathscr{O}_{X,Z}/(g)$ clearly has dimension 0, because (0) is not a prime ideal in the quotient. But then $\mathrm{Spec}\,\mathscr{O}_{X,Z}/(g)$ is finite and discrete, so $\mathscr{O}_{X,Z}/(g)$ is an Artinian ring.

Now note that $\mathscr{O}_{X,Z}/(g)$ is an Artinian $\mathscr{O}_{X,Z}$-module (since ideals in $\mathscr{O}_{X,Z}/(g)$ are in 1-1 correspondence with $\mathscr{O}_{X,Z}$-sub-modules) and is Noetherian as well, so by the Akizuki-Hopkins-Levitzki theorem, $\mathscr{O}_{X,Z}/(g)$ has a composition series, and thus has finite length.

One shows easily that ℓ is additive on short exact sequences, so to show ord_Z is a group homomorphism, one only needs to consider the sequence

$$0 \to (g)\mathscr{O}_{X,Z}/(gh) \to \mathscr{O}_{X,Z}/(gh) \to \mathscr{O}_{X,Z}/(h) \to 0$$

and note that $(g)\mathscr{O}_{X,Z}/(gh) \cong \mathscr{O}_{X,Z}/(h)$. ∎

We can extend ord_Z to a homomorphism $\mathscr{M}_X^*(X) \to \mathbf{Z}$ by defining

$$\mathrm{ord}_Z(g/h) = \mathrm{ord}_Z(g) - \mathrm{ord}_Z(h).$$

Then, for $f \in K(X)^*$, we define the principal divisor

$$\mathrm{div}(f) := \sum_Z \mathrm{ord}_Z(f)Z,$$

where the sum is taken over all prime divisors. To see that this is well defined, note that there exists some nonempty Zariski open affine subset $U = \mathrm{Spec}\, A$ on which f restricts to a regular function in $\mathscr{O}_X^*(U)$. Note that $X \setminus U$ is a proper closed subset. Since X is Noetherian, there can only be finitely many prime divisors in $X \setminus U$. So we can treat the case where f is regular. If $Y \subseteq U$ is a prime divisor on X, then $\mathrm{ord}_Y(f) > 0$ if and only if Y is contained in the closed subset $\mathrm{Spec}\, A/fA$, which is again proper closed, so contains finitely many prime divisors. Otherwise, $\mathrm{ord}_Y(f) = 0$.

We denote the group of principal divisors by $P(X)$. We then define

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/P(X).$$

Like in the case of Cartier divisors, we say that two Weil divisors are *linearly equivalent* if they represent the same class in $\mathrm{Cl}(X)$ and write $D \sim D'$.

2.2.3. *Example.* If X is an algebraic curve over $\mathbf{C}$, then Weil divisors correspond exactly to closed points of X. In this case, the principal Weil divisor associated to a meromorphic function on X really is given by the order of the zeros and poles at each point.

2.3. **Relationship between Cartier and Weil divisors.** Let X be a locally Noetherian integral scheme, so that Weil divisors are defined.

2.3.1. There is a natural map $\mathrm{Ca}(X) \to \mathrm{Div}(X)$ as follows. Fix a Cartier divisor $C = \{(U_i, f_i)\}$, and consider a prime Weil divisor Z. Then pick an i for which $U_i \cap Z \neq 0$, and define

$$n_Z = \mathrm{ord}_Z(f_i).$$

Then the map is given by

$$\{(U_i, f_i)\} = C \mapsto \sum_Z n_Z Z \in \mathrm{Div}(X).$$

This is well-defined: if we pick j with $U_j \cap Z \neq 0$, then $f_i/f_j \in \mathscr{O}_X(U_i \cap U_j)^*$, so the image of f_i/f_j in any stalk is invertible, so

$$\mathrm{ord}_Z(f_i/f_j) = 0,$$

so $\mathrm{ord}_Z(f_i) = \mathrm{ord}_Z(f_j)$.

2.3.2. *Remark.* If, in addition, the stalks $\mathscr{O}_{X,Z}$ are regular rings for all prime divisors $Z \subset X$, then their unique maximal ideals are principal, and thus they are discrete valuation rings with valuation val_Z. In that case, if $g \in K(X)$, then one can show

$$\mathrm{ord}_Z(g) = \mathrm{val}_Z(g).$$

This situation occurs, in particular, when X is smooth.

In fact, we will now show that in this case, the Weil divisors and Cartier divisors (and the corresponding class groups) agree.

2.3.3. **Theorem.** *Let X be a locally Noetherian integral scheme, whose local rings are unique factorization domains (we say X is locally factorial). Then the map defined in 2.3.1 is an isomorphism, and takes the principal Cartier divisors to principal Weil divisors.*

Proof. We start with a Weil divisor, which consists of prime divisors and multiplicities, and are looking for rational functions to properly represent them. The idea of the proof then is to restrict the divisors to divisors on the *local scheme* $X_x = \mathrm{Spec}\, \mathscr{O}_{X,x}$ (whose underlying set is a subset of X, and whose function field is still $K(X)$), then to show that these restricted divisors are principal, which gives us a rational function. It then remains to show compatibility relations between the found functions. The key here is that a ring is a unique factorization domain if and only if every prime ideal of height 1 is principal.

We construct an inverse map $\mathrm{Div}(X) \to \mathrm{Ca}(X)$. Fix any point $x \in X$. Prime divisors on $\mathrm{Spec}\, \mathscr{O}_{X,x}$ correspond to prime ideals of height 1, which are all principal. So each prime divisor (corresponding to) $\mathfrak{p}$ can be written (f) for some $f \in \mathscr{O}_{X,x}$. Note $\mathrm{ord}_{\mathfrak{p}}(f) = 1$ because $\mathscr{O}_{X_x,\mathfrak{p}}$ is a discrete valuation ring with maximal ideal (f). But if $\mathfrak{p}'$ is another distinct prime divisor, then $\mathfrak{p}' \neq \mathfrak{p}$, then $f \notin \mathfrak{p}'$, for otherwise we would have $\mathfrak{p} \subseteq \mathfrak{p}'$, which contradicts the fact that $\mathrm{ht}(\mathfrak{p}') = 1$, therefore f is a unit in $\mathscr{O}_{X_x,\mathfrak{p}'}$, so $\mathrm{ord}_{\mathfrak{p}'}(f) = 0$. We conclude that

$$\mathrm{Cl}(\mathrm{Spec}\, \mathscr{O}_{X,x}) = 0,$$

i.e. every divisor is principal.

Consider a Weil divisor $D \in \mathrm{Div}(X)$. For any $x \in X$, we may consider the restricted Weil divisor D_x on X_x given by

$$D_x = \sum_{x \in X_x \cap X^{(1)}} n_x \overline{\{x\}}$$

This is principal, i.e. there is an $f_x \in K(X)$ with $\mathrm{div}(f_x) = D_x$.

Note that $\mathrm{div}(f_x)$ also defines a divisor on all of X. Since $\mathrm{div}(f_x)$ and D have the same restriction to X_x, they differ only on codimension 1 subschemes outside of X_x, of which there are finitely many; call their union W_x. Then $U_x = X \setminus W_x$ is open, and $\mathrm{div}(f_x) = D$ on U_x (the restriction of a divisor to an open set U just restricts the sum to $X^{(1)} \cap U$).

Then $\{(U_x, f_x)\}$ is a Cartier divisor: to see this, note that on $U_x \cap U_y$, we have $\mathrm{div}(f_x/f_y) = 0$. If $W \subseteq U_x \cap U_y$ is any affine, then f_x/f_y is a unit on all stalks of W, and is thus not contained in any prime ideal of $\mathscr{O}_X(W)$, so is a unit in $\mathscr{O}_X(W)$. Thus, by using the sheaf axiom on an open cover of $U_x \cap U_y$ by affines, we see that $f_x/f_y \in \mathscr{O}_X(U_x \cap U_y)^*$. This does not depend on the choice of f_x: another choice would yield an equivalent Cartier divisor.

These constructions are inverse to one another. Starting with $\{(U_i, f_i)\}$, we form the corresponding Weil divisor. Then for a point $x \in X$, the principal divisor on the local scheme can be taken to be induced by f_i. Conversely, given a Weil divisor $\sum_Z n_Z Z$, we form the corresponding Cartier divisor. Then for any prime divisor Z, pick a point $x \in Z$. Then $\mathrm{val}_Z(f_x) = n_Z$ by construction, and we're done.

Finally, the principal Cartier divisor corresponding to f is sent to $\sum_Z \mathrm{val}_Z(f)Z$, which is exactly the principal Weil divisor. □

2.3.4. A *variety* is a Noetherian integral separated scheme over an algebraically closed field. A variety is *smooth* if all of its local rings are regular. Theorem 2.3.3 works, for example, in the case of smooth varieties, or for regular schemes, more generally. In these cases, we may speak of the divisor group of a scheme, or the class group, without specifying any namesake.

2.4. **Invertible Sheaves and Line Bundles.** Invertible sheaves, equivalently (geometric) line bundles, are another type of gadget used commonly in intersection theory, which under certain conditions will be equivalent to the group of Cartier divisors.

2.4.1. *Invertible Sheaves.* A sheaf $\mathscr{L}$ of $\mathscr{O}_X$-modules is called *invertible* (or *locally free of rank 1*) if there exists an open cover $\{U_i\}_{i\in I}$ of X such that for all i there exists an isomorphism

$$\mathscr{L}|_{U_i} \cong \mathscr{O}_X|_{U_i}$$

as $\mathscr{O}_X|_{U_i}$-modules.

An equivalent characterization is that there exists another locally free sheaf $\mathscr{L}'$ such that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$ is isomorphic to $\mathscr{O}_X$ as an $\mathscr{O}_X$-module.

2.4.2. *Line Bundles.* Every invertible sheaf on X realizes the sheaf of sections of a certain *line bundle* (a vector bundle of rank 1) $\pi : E \to X$. Recall that a line bundle is given by a scheme E and a morphism $E \to X$, along with an open cover $\{U_i\}$ of X and isomorphisms $\psi_i : \pi^{-1}(U_i) \to \mathbf{A}^1_{U_i}$ satisfying the compatibility condition that $\theta_{ij} = \psi_j \circ \psi_i^{-1}$ is a linear automorphism of $\mathbf{A}^1_V$ for every open affine $V \subseteq U_i \cap U_j$ and every i, j (in other words, θ_i is the identity on $\mathscr{O}_V(V)$, and takes x to cx for $c \in \mathscr{O}_V(V)$). The collection $\{U_i, \psi_i\}$ will be called a *trivializing cover*.

A *section* of a line bundle $\pi : E \to X$ is a morphism $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$. For any open $U \subset E$, we define

$$\mathscr{L}_E(U) = \{s : U \to E : \pi \circ s = \mathrm{id}_U\}.$$

The restriction map $s \mapsto s|_V$ for $V \subseteq U$ makes s into a presheaf, and in fact s is a sheaf, by the usual gluing of morphisms. We call $\mathscr{L}_E$ the *sheaf of sections* on E.

2.4.3. *Isomorphisms.* An isomorphism of line bundles $\varphi : (\pi, E) \to (\pi', E')$ is given by an isomorphism of schemes $\varphi : E \to E'$ such that

(1) $\pi' \circ \varphi = \pi$

(2) If $\{U_i, \psi_i\}$ and $\{U'_j, \psi'_j\}$ are trivializing covers of E and E' respectively, then the collection

$$\{U_i, \psi_i\} \cup \{U'_j, \psi'_j \circ \varphi\}$$

is a trivializing cover of E.

2.4.4. **Lemma.** *For any line bundle $\pi : E \to X$, $\mathscr{L}_E$ is an invertible $\mathscr{O}_X$-module.*

Proof. First we show that $\mathscr{L}_E$ has the natural structure of an $\mathscr{O}_X$-module. If $s : U \to E$ is a section, we take an open affine $V = \operatorname{Spec} R \subset U$ that is contained in one of the U_i. Then $\psi_i \circ s|_V : V \to \mathbf{A}^1_V$ corresponds to $\widetilde{s} : R[x] \to R$, and there is a natural action of R on $\operatorname{Hom}_R(R[x], R)$, and we get an $\mathscr{O}_V$-module structure. The condition that the ψ_i preserves the coordinate ring of an affine is enough to show that this uniquely determines an $\mathscr{O}_U$-module structure, if we cover U with open affines V contained in some U_i. Thus we have a well-defined module structure.

Note for any open affine $V = \operatorname{Spec} R$ contained in a U_i, the identification of

$$\mathscr{L}_E(V) \cong \operatorname{Hom}_R(R[x], R) \cong R$$

shows that $\mathscr{L}_E$ is locally free of rank 1, hence is invertible. □

2.4.5. Conversely, given an invertible sheaf $\mathscr{L}$, we can construct a line bundle whose sheaf of sections is the dual of $\mathscr{L}$. Thus, the notion of invertible sheaves and line bundles are dual to one another.

2.4.6. *Picard Group.* The *Picard group* of a scheme X, denoted $\operatorname{Pic}(X)$ is given by the set of isomorphism classes of invertible sheaves on X, with the group operation $\otimes$. As noted in 2.4.1, this indeed gives a group structure, with identity $\mathscr{O}_X$.

2.4.7. There is a close relationship between Cartier divisors and invertible sheaves: namely, given a Cartier divisor on any scheme X, we can associate to it an invertible subsheaf, and under certain conditions, this will essentially give an equivalence between the two notions.

Suppose $\{(U_i, f_i)\}$ represents the Cartier divisor D. Then to each U_i, we define a subsheaf of $\mathscr{M}_{U_i}$ on U_i by (for $V \subseteq U_i$)

$$\mathscr{L}(D)_i(V) = \frac{1}{f_i|_V}\mathscr{O}_X(V)$$

One checks that this forms a sheaf on U_i, and by [EH00] I-14, glues together to a subsheaf $\mathscr{L}(D)$ of $\mathscr{M}_X$ on X.

2.4.8. **Proposition.** *The aforementioned map $\mathscr{L}(-) : \operatorname{Ca}(X) \to \{$subsheaves of $\mathscr{M}_X\}$ is an injective group homomorphism, under which the image of a principal divisor is isomorphic to $\mathscr{O}_X$ as an $\mathscr{O}_X$-module (not necessarily equal as a subsheaf of $\mathscr{M}$).*

Proof. Suppose $D \in \mathrm{Ca}(X)$ is represented by $\{(U_i, f_i)\}$ such that $\mathscr{L}(D) = \mathscr{O}_X$ as a subsheaf of $\mathscr{M}_X$. Then in all $\mathscr{O}_X(U_i)$, f_i is a unit. Since a Cartier divisor is a section of $\mathscr{M}_X^*/\mathscr{O}_X^*$, so is it clear that D is trivial.

Fix $D, E \in \mathrm{Ca}(X)$. By taking refinements, we can assume they are represented by the same open cover $\{U_i\}$ so that $D = \{(U_i, f_i)\}$ and $E = \{(U_i, g_i\})$. Then $D + D'$ is represented by $\{(U_i, f_i g_i)\}$. The corresponding invertible sheaf is defined on U_i by $(1/f_i g_i)\mathscr{O}_{U_i}$. But this clearly gives the same sheaf as the tensor product $(1/f_i)\mathscr{O}_{U_i} \otimes_{\mathscr{O}_{U_i}} (1/g_i)\mathscr{O}_{U_i}$, which shows that

$$\mathscr{L}(D + E) \cong \mathscr{L}(D) \otimes_{\mathscr{O}_X} \mathscr{L}(E).$$

If $f \in \mathscr{M}_X^*(X)$, then $\mathscr{L}(\mathrm{div}(f))$ is just $(1/f)\mathscr{O}_X$, which is isomorphic (abstractly) to $\mathscr{O}_X$ by the $\mathscr{O}_X$-module morphism $1 \mapsto f$. □

2.4.9. **Corollary.** *The homomorphism* $\mathrm{Ca}(X) \to \mathrm{Pic}(X)$ *is injective, and surjective when* X *is integral.*

Proof. Injectivity is clear from 2.4.8. If X is integral, then it suffices to show that every invertible sheaf on X is isomorphic to a subsheaf of $\mathscr{M}_X$, which in this case is just the constant sheaf corresponding to $K(X)$.

Consider the diagram

$$\begin{tikzcd}
\mathscr{L}|_{U_i} \arrow[r] \arrow[d, "\sim"'] & \mathscr{L}|_{U_i} \otimes_{\mathscr{O}_{U_i}} \mathscr{M}_X|_{U_i} \arrow[d, "\sim"] & \\
\mathscr{O}_X|_{U_i} \arrow[r, hook] & \mathscr{O}_X|_{U_i} \otimes_{\mathscr{O}_{U_i}} \mathscr{M}_X|_{U_i} \arrow[r, "\sim"] & \mathscr{M}_X|_{U_i}
\end{tikzcd}$$

We see that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M}_X$ is locally constant, and thus is globally constant, since X is irreducible. Thus, we see that $\mathscr{L} \hookrightarrow \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M}_X \xrightarrow{\sim} \mathscr{M}_X$, exhibiting $\mathscr{L}$ as a subsheaf of the constant sheaf $\mathscr{M}_X$. □

2.4.10. *Equivalences.* We finally see that if X is a Noetherian locally factorial integral scheme, then 2.3.3, 2.4.4, 2.4.5, and 2.4.9 tell us that

Cartier divisors, Weil divisors, invertible sheaves, line bundles

are all equivalent notions. Most of the theory we will concern ourselves with will land us in this case.

2.5. **Ampleness.** The notion of ampleness comes up frequently when dealing with intersection theory. We introduce it here. Heuristically, an invertible sheaf $\mathscr{L}$ on X is ample if a high enough tensor power $\mathscr{L}^{\otimes n}$ has enough global sections to embed X into a projective space.

2.5.1. *Definitions.* Given schemes X, Y and an invertible sheaf $\mathscr{L}$, we say $\mathscr{L}$ is *very ample* relative to Y if there exists a closed immersion $\pi : X \hookrightarrow \mathbf{P}_Y^N$ for some $N > 0$, such that

$$\mathscr{L} = \pi^* \mathscr{O}_{\mathbf{P}_Y^N}(1),$$

the tautological hyperplane bundle on $\mathbf{P}^N$.

An invertible sheaf $\mathscr{L}$ on a scheme X is called *ample* if for all coherent sheaves $\mathscr{F}$ on X, there exists $n > 0$ such that $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections. Equivalently, one can show that a sheaf is ample if there exists an integer $n > 0$ such that $\mathscr{L}^n$ is very ample.

2.5.2. *Ample Divisors.* Ampleness of a divisor is that of its corresponding invertible sheaf.

We will need a few lemmas involving ampleness for when we later study intersection theory on surfaces.

2.5.3. **Lemma.** *Let X be a scheme of finite type over a field K. Then if $\mathscr{L}$ is very ample and $\mathscr{F}$ is generated by global sections, then $\mathscr{F} \otimes \mathscr{L}$ is very ample.*

Proof. Ampleness of $\mathscr{L}$ gives an immersion $\phi_{\mathscr{L}} : X \to \mathbf{P}_K^n$ for some n, and GBGS for $\mathscr{F}$ gives a morphism $X \to \mathbf{P}_K^m$. If we take the Segre embedding $\mathbf{P}_K^n \times \mathbf{P}_K^m \hookrightarrow \mathbf{P}_K^N$, then we can define a map

$$\phi_{\mathscr{L}} \times \phi_{\mathscr{F}} : X \to \mathbf{P}_K^n \times \mathbf{P}_K^m \hookrightarrow \mathbf{P}_K^N,$$

which is an immersion because $\phi_{\mathscr{L}}$ is, and satisfies the property that the pullback of $\mathscr{O}(1)$ on $\mathbf{P}_K^N$ is $\mathscr{L} \otimes \mathscr{F}$. □

2.5.4. **Lemma.** *Let X be a scheme of finite type over a field. Then if $\mathscr{F}$ is any invertible sheaf and $\mathscr{L}$ is ample, there exists some $n > 0$ such that $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is very ample.*

Proof. By the first definition of ampleness, there exists some $k > 0$ such that $\mathscr{F} \otimes \mathscr{L}^{\otimes k}$ is generated by global sections. By the equivalent definition of ampleness, there exists some $l > 0$ such that $\mathscr{L}^{\otimes l}$ is very ample. Thus by 2.5.3, $\mathscr{F} \otimes \mathscr{L}^{\otimes k+l}$ is very ample. □

2.6. **Sheaf of Differentials.** In differential geometry, differential forms are a basic object of study, and they lead to rich and sophisticated theory. Classically, one defines the tangent space at a point in a manifold, which extends to the notion of a tangent bundle. Then the sheaf of differential forms is essentially given by the dual of the tangent bundle (cotangent bundle).

In algebraic geometry, we will define the notion of the sheaf of differentials intrinsically. To motivate the construction, we consider the affine case.

2.6.1. *Derivations.* Suppose A is a commutative ring, B is an A-algebra, and M is a B-module. Then an A-derivation of B into M is a map $d : B \to M$ satisfying the usual axioms of a derivation: namely that

$$d(b + b') = db + db',$$
$$d(bb') = bdb' + b'db,$$

and

$$da = 0 \text{ for all } a \in A.$$

Then the *module of relative differential forms* of B over A is a B module $\Omega_{B/A}$ and an A-derivation $d : B \to \Omega_{B/A}$ satisfying the universal property that if $d' : B \to M$ is a derivation, then there exists a unique B-module homomorphism $f : B \to M$ such that the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_{B/A} \arrow[d, "f"] \\ & M \end{tikzcd}$$

Thus, $\Omega_{B/A}$ can be thought of as the universal target of a derivation. More precisely, $\Omega_{B/A}$ represents the functor $\mathrm{Der}_A(B, -) : \mathsf{Mod}_B \to \mathsf{Ab}$, so that there is a natural isomorphism

$$\mathrm{Der}_A(B, M) \cong \mathrm{Hom}_B(\Omega_{B/A}, M)$$

2.6.2. **Lemma.** *$\Omega_{B/A}$ exists, and is unique up to unique isomorphism.*

Proof. We give a simple construction of $\Omega_{B/A}$, and another one which will motivate the general definition for schemes.

The first construction is to define $\Omega_{B/A} = F_B/I$, where F_B is the free B-module on the set of symbols $\{\delta b : b \in B\}$, and I is the submodule generated by the expressions defining a derivation (additivity, Leibniz rule, and $\delta a = 0$). Then if $d : B \to M$ is a derivation, one can define the B-module homomorphism $f : F_B \to M$ by $f(\delta b) = db$ (and extending B-linearly), and this clearly takes I to 0.

The second construction, which extends to the general definition, is as follows. Consider the "diagonal" map

$$\Delta : B \otimes_A B \to B, b \otimes b' \mapsto bb'$$

and define $I = \ker \Delta$. Note $B \otimes_A B$ is a B-module via multiplication on the left. Note I/I^2 then inherits the structure of a B-module. In particular, if $\sum b_i \otimes b_i' \in I$, then $\sum b_i b_i' = 0$, so $\sum c b_i b_i' = 0$ for $c \in B$. We then define

$$d : B \to I/I^2, b \mapsto b \otimes 1 - 1 \otimes b.$$

It is straightforward to show that d is an A-derivation. Furthermore, if $d' : B \to M$ is an A-derivation, then define the B-module homomorphism $f : B \times B \to M$ by $f(b, b') = b'd'(b)$. One checks that this is A-bilinear, and takes I^2 to 0. ∎

The following technical lemma will be useful.

2.6.3. **Lemma.** *If B is an A-algebra, $I \subseteq B$ is an ideal, then*

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B (B/I) \to \Omega_{(B/I)/A} \to 0$$

is an exact sequence of B/I-modules, where $\delta(b) = db \otimes 1$.

Proof. Omitted. □

2.6.4. *Sheaf of Differentials.* Motivated by the second proof of Lemma 2.6.2, we make the following definition. If $X \to Y$ is a scheme in Sch_Y, then let $\Delta_X : X \to X \times_Y X$ be the diagonal map. Let $\Delta(X)$ denote the scheme-theoretic image of X in $X \times_Y X$, which is isomorphic to X via Δ, and is closed in some open $W \subseteq X \times_Y X$. We define $\mathscr{I}$ (an $\mathscr{O}_W$-module) by the exact sequence

$$0 \to \mathscr{I} \to \mathscr{O}_W \to i_*\mathscr{O}_{\Delta(X)}$$

where i is the inclusion $i : \Delta(X) \hookrightarrow W$.

Note $\mathscr{I}/\mathscr{I}^2$ is a quasi-coherent $\mathscr{O}_W$-module killed by $\mathscr{I}$, so there exists some quasi-coherent $\mathscr{O}_X$-module $\Omega_{X/Y}$ whose pushforward under $X \to \Delta(X) \hookrightarrow W$ is exactly $\mathscr{I}/\mathscr{I}^2$. This is called the *sheaf of differentials of X over Y*.

2.6.5. *Affine Differentials.* If $X = \operatorname{Spec} B$ is an affine scheme over $Y = \operatorname{Spec} A$, then the diagonal map $X \to X \times_Y X$ corresponds to the diagonal map $B \otimes_A B \to B$, which is surjective, hence $X \to X \times_Y X$ is a closed immersion. Note the inclusion $X \xrightarrow{\sim} \Delta(X) \hookrightarrow X \times_Y X$ again corresponds to the diagonal morphism $B \otimes_A B \to B$, whose kernel is the ideal I from the proof of 2.6.2, which shows that the sheaf $\mathscr{I}$ given in 2.6.4 is $\widetilde{I}$. Thus we see that $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$.

In view of this, given any scheme $f : X \to Y$, one can take an affine $V \subseteq Y$ and affine $U \subseteq f^{-1}(V)$, compute the affine sheaves of differentials, and glue them together to get $\Omega_{X/Y}$. Furthermore, the derivations $d : B \to \Omega_{B/A}$ glue together to give a map $\mathscr{O}_X \to \Omega_{X/Y}$ which is a derivation on the local rings.

2.6.6. **Corollary.** *If $X \to Y$ is a morphism of schemes and $i : Z \hookrightarrow X$ is a closed subscheme of X whose sheaf of ideals is $\mathscr{I}$, then there is an exact sequence of $\mathscr{O}_Z$-modules*

$$\mathscr{I}/\mathscr{I}^2 \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0.$$

Note $\mathscr{I}$ is an $\mathscr{O}_X$-module so $\mathscr{I}/\mathscr{I}^2$ is naturally an $\mathscr{O}_X/\mathscr{I} \cong \mathscr{O}_Z$-module.

Proof. This is clear in light of 2.6.5 and 2.6.3. □

2.7. **Canonical Divisors.** The sheaf of differentials will be pertinent to our study of intersection theory because they allow us to define the *canonical divisor*, which is a crucial ingredient in the definition of Serre duality, and thus is involved with the Riemann-Roch theorem. We will define it now.

2.7.1. *Smooth Curves.* The canonical divisor will be defined only for smooth varieties (see 2.3.4). The following theorem motivates this restriction.

2.7.2. **Theorem.** *A variety X over an algebraically closed field k is smooth if and only if $\Omega_{X/k}$ is a locally free sheaf of dimension $n = \dim X$. Furthermore, if X is smooth and $Y \subset X$ is an irreducible closed subscheme of X defined by a sheaf of ideals $\mathscr{I}$, then Y is smooth if and only if*

(1) $\Omega_{Y/k}$ is a locally free $\mathscr{O}_Y$-module, and

(2) the sequence in 2.6.6 is exact on the left side as well.

Proof. Omitted. □

2.7.3. *Canonical Sheaf.* If X is a smooth variety over a field k of dimension n, then the *canonical sheaf* is defined by

$$\omega_X = \bigwedge^n \Omega_{X/k},$$

the nth exterior power of $\Omega_{X/k}$. Locally this is just $\bigwedge^n \mathscr{O}_U^n \cong \mathscr{O}_U$, which shows that ω_X is an invertible sheaf. Now since X is integral and smooth, there is a corresponding Weil (or Cartier) divisor called the *canonical divisor*, typically denoted K_X. This will appear in the statement of the Riemann-Roch theorem, and

2.7.4. *Canonical Divisor.* As X is smooth and integral (X is a variety), the equivalences in 2.4.10 show us that to ω_X is associated a divisor (Cartier or Weil), which we denote by K_X. This plays an important role in Serre duality and the Riemann-Roch theorem for surfaces.

2.8. **Adjunction Formula.** As we are concerned with subschemes and subvarieties of a sufficiently nice scheme X on which to do intersection theory, we would like to see the relationship between the canonical divisor of a scheme and that of its subschemes. Fortunately, there is a close relationship, given by the following theorem, known as the "adjunction formula".

2.8.1. **Theorem.** *If $i_Y : Y \hookrightarrow X$ is a smooth subvariety of codimension r with sheaf of ideals $\mathscr{I}_Y$, then*

$$\omega_Y \cong (i_Y)^*\omega_X \otimes_{\mathscr{O}_Y} \bigwedge^r \operatorname{Hom}(\mathscr{I}_Y/\mathscr{I}_Y, \mathscr{O}_Y)$$

Proof. By taking the highest exterior powers of the terms in the exact sequence

$$0 \to \mathscr{I}_Y/\mathscr{I}_Y \to i^*\Omega_{X/k} \to \Omega_{Y/k} \to 0$$

given in Theorem 2.7.2, we get an isomorphism

$$(i_Y)^*\omega_X \cong \bigwedge^r (\mathscr{I}_Y/\mathscr{I}_Y^2) \otimes_{\mathscr{O}_Y} \omega_Y.$$

Since exterior products commute with taking the dual of an $\mathscr{O}_Y$-module, we find that

$$(i_Y)^*\omega_X \otimes_{\mathscr{O}_Y} \bigwedge^r \operatorname{Hom}(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathscr{O}_Y) \cong \omega_Y.$$

□

2.8.2. *Divisors.* If $r = 1$, then $i_Y : Y \hookrightarrow X$ is a divisor, and thus there is an invertible $\mathscr{O}_X$-module $\mathscr{L}$ associated to Y, such that $\mathscr{L}^{-1}$ is the sheaf of ideals of Y. In this case, $\mathscr{I}_Y/\mathscr{I}_Y^2$ is just the restriction of $\mathscr{I}_Y \cong \mathscr{L}^{-1}$ to Y, so we find that

$$\omega_Y \cong (i_Y)^*\omega_X \otimes_{\mathscr{O}_Y} \mathrm{Hom}((i_Y)^*\mathscr{L}^{-1}, \mathscr{O}_Y) \cong (i_Y)^*\omega_X \otimes_{\mathscr{O}_Y} (i_Y)^* \mathrm{Hom}(\mathscr{L}^{-1}, \mathscr{O}_X) \cong (i_Y)^*\omega_X \otimes_{\mathscr{O}_Y} (i_Y)^*(\mathscr{L}),$$

so that

$$\omega_Y \cong (i_Y)^*(\omega_X \otimes_{\mathscr{O}_X} \mathscr{L}).$$

2.9. **Cohomology.** We will need to use the tools of cohomology in order to formulate some auxiliary results, particularly the Riemann-Roch theorem. We will quickly review a definition of sheaf cohomology, and state some basic properties we will need in our proofs.

2.9.1. *Definition.* The cohomology functors are functors $H^i(X, -) : \mathsf{Mod}_{\mathscr{O}_X} \to \mathsf{Mod}_{\mathscr{O}_X(X)}$ for each $i \geq 0$, defined as follows. Given a sheaf $\mathscr{F} \in \mathsf{Mod}_{\mathscr{O}_X}$, we first take an injective resolution

$$0 \to \mathscr{F} \to I_0 \to I_1 \to \dots.$$

Applying the global sections functor and removing $\mathscr{F}$, we get

$$0 \to I_0(X) \xrightarrow{d_0} I_1(X) \xrightarrow{d_1} \dots.$$

Then $H^i(X, \mathscr{F}) := \ker d_i / \operatorname{im} d_{i-1}$. Note $H^0(X, \mathscr{F}) = \mathscr{F}(X)$.

2.9.2. *Coherent Sheaves.* When $\mathscr{F}$ is a coherent sheaf, one can show that $H^i(X, \mathscr{F})$ is a finitely generated $\mathscr{O}_X(X)$-module. In particular, if $X = \operatorname{Spec} k$ for a field k, then $H^i(X, \mathscr{F})$ is a finite dimensional vector space. We omit the proof here, instead referring the reader to [Har77].

2.9.3. *Long Exact Sequence.* To any short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$$

of $\mathscr{O}_X$-modules, there is an associated long exact sequence

$$0 \to H^0(X, \mathscr{F}') \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}/\mathscr{F}') \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{F}) \to \dots$$

This sequence is often used to show the vanishing of cohomology groups.

2.9.4. *Serre Duality.* In particular, we will need this result only in the case where X is a projective smooth variety over an algebraically closed field k of dimension $\dim X = n$. There is then a natural isomorphism, for each $i \geq 0$,

$$H^i(X, \mathscr{F}) \cong H^{n-i}(X, \mathscr{F}^\vee \otimes_{\mathscr{O}_X} \omega_X),$$

where $\mathscr{F}$ is an $\mathscr{O}_X$-module, $\mathscr{F}^\vee$ is its dual module, and ω_X is the canonical sheaf as defined in 2.7.3.

2.9.5. *Euler Characteristic.* Classically, the Euler characteristic was introduced as an alternating-sum invariant of polyhedra involving the faces and edges. There is a generalization to coherent sheaves: given a coherent $\mathscr{O}_X$-module $\mathscr{F}$, we define

$$\chi(\mathscr{F}) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathscr{F})$$

We mention the following useful lemma.

2.9.6. **Lemma.** *The Euler characteristic is additive on short exact sequences of coherent sheaves.*

3. Intersection Theory on Surfaces

3.1. **Curves.** Here we recall some facts about curves that we will need in order to work with surfaces. Since prime divisors on on surface are essentially irreducible curves, we will need some fundamental definitions and in particular, the Riemann-Roch theorem.

Fix an algebraically closed field $k = \overline{k}$.

3.1.1. *Definition.* A *curve* $C \to \operatorname{Spec} k$ is a smooth projective variety of dimension 1.

3.1.2. *Divisors.* In this case, Cartier and Weil divisors are the same. Since prime divisors are codimension 1 subvarieties, in the case of curves they simply correspond to the closed points of the curve. In this setting, we can define the *degree* of the divisor to be the sum of the coefficients of the divisor:

$$\deg(\sum_P n_P P) = \sum_P n_P.$$

A divisor D is called *effective* if $D > 0$.

3.1.3. *Remark.* If we consider a curve C over a non-algebraically closed field, then the degree of a prime divisor generalizes nicely. If $P \in C$ is a point, then $\deg(P) = [\kappa(P) : k]$, where $\kappa(P)$ is the residue field of P in C, which is a finite extension of k. If k is algebraically closed, then $\kappa(P) = k$ always, and we recover the previous definition. This will be important later when we discuss divisors on a curve over a number field.

3.1.4. **Theorem** (Riemann-Roch for Curves)**.** *If D is a divisor on a curve X, then*

$$\chi(\mathscr{L}(D)) = \deg(D) + 1 - g_X.$$

where $g_X = \dim H^1(X, \mathscr{O}_X)$ is the genus of X.

Proof. [Har77] IV.1.3 □

3.2. **Surfaces.** We now study intersection theory of surfaces, and prove the Hodge index theorem, which is a particular case of the Hodge standard conjecture in the case of an algebraic surface.

3.2.1. *Definition.* As in the case of curves, a *surface* is a smooth projective variety of dimension 2 over an algebraically closed field k. Again, $\mathrm{Cl}(X) \cong \mathrm{CaCl}(X)$.

3.2.2. *Intersection of Divisors.* On a surface, elements of $\mathrm{Cl}(X)$ are finite $\mathbf{Z}$-linear combinations of (possibly singular) curves. Since these are the only divisors we need consider, we will be concerned with their intersection. For this, we need a few lemmas.

Ideally, curves C, D will intersect *transversally* at a point $p \in C \cap D$, which intuitively means that they have linearly independent tangent lines at p, and precisely says that if f, g define C, D locally at point p, then the ideal generated by f_p, g_p in $\mathscr{O}_{X,p}$ is the maximal ideal $\mathfrak{m}_p$. Alternatively, it means that the curves do not share a common 1-dimensional component. In particular, note that since X is Noetherian, the intersection of transversally intersecting curves is finite.

Intersection theory on a surface is summarized by the following theorem.

3.2.3. **Theorem.** *There is a unique intersection product* $i : \mathrm{Cl}(X) \times \mathrm{Cl}(X) \to \mathbf{Z}$ *satisfying the following axioms*

(1) $i(C, D) = \#(C \cap D)$ *if* C, D *are curves intersecting transversally.*

(2) $i(C, D) = i(D, C)$

(3) $i_C : D \mapsto i(C, D)$ *is a group homomorphism for all* $C \in \mathrm{Cl}(X)$*.*

3.2.4. *Remark.* Note in particular that intersection with any principal divisor is always 0.

3.2.5. *Moving Lemma.* Note that for transversally intersecting curves, the intersection product is uniquely defined, so it remains to see what happens for curve with bad (nontransversal) intersection. Classically, this corresponds to the case

$$\dim \operatorname{span}(T_p(C) + T_p(D)) = 1$$

as opposed to 2 in the transversal case.

Fortunately, for curves on a surface there is a "moving lemma" of sorts, which lets us swap out a curve with bad intersection for a linearly equivalent one with transversal intersection without changing the intersection product. This crucially relies on the fact that X is projective, or equivalently, that there exists a very ample divisor on X. Therefore, we can use Bertini's theorem to our advantage.

3.2.6. **Lemma.** *For irreducible curves* $C_1, \ldots, C_n$ *on a surface* X*, and a very ample divisor* D*, then there exists a very ample divisor* D' *such that* $D' \sim D$ *and each* C_i *intersects* D' *transversally.*

Proof. [Har77] V.1.2. This basically makes use of Bertini's theorem, which says that most hyperplanes have regular intersection with a subvariety of projective space. □

Proof of 3.2.3: First we define the intersection product for very ample divisors, then extend to the general case.

If C, D are very ample, then by Lemma 3.2.6, we can pick nonsingular $D' \sim D$ and nonsingular $C' \sim C$ such that C' and D' intersect transversally. In this case, we define

$$i(C, D) = \#(C' \cap D').$$

To show that this is well-defined, we first show that the sequence

$$0 \to (i'_C)^* \mathscr{L}(-D') \to \mathscr{O}_{C'} \to (i_{C' \cap D'})_* \mathscr{O}_{C' \cap D'} \to 0$$

is exact, where $i_{C'} : C' \hookrightarrow D'$ and $i_{C' \cap D'} : C' \cap D' \hookrightarrow C'$. Suppose at a point $x \in C'$ that C', D" are locally defined by $f, g \in \mathscr{O}_{X,x}$. Localizing at x, we get the sequence

$$0 \to g\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x}/(f) \to \mathscr{O}_{X,x}/(f) \to \mathscr{O}_{X,x}/(f, g) \to 0$$

Note $\mathscr{O}_{X,x}$ is a UFD and C' and D' intersect transversally, so f, g are coprime. Therefore, we can replace the first term in the sequence to find

$$0 \to g\mathscr{O}_{X,x}/(f) \to \mathscr{O}_{X,x}/(f) \to \mathscr{O}_{X,x}/(f, g) \to 0,$$

which is exact, so the original sequence is exact. Since pullback preserves the tensor product, this shows that $(i'_C)^* \mathscr{L}(D')$ is the ideal sheaf of $C' \cap D'$. Since the intersection is transversal, its degree is $\#(C' \cap D')$. Thus if we choose any other $D'' \sim D$, we'll have $\mathscr{L}(D'') \cong \mathscr{L}(D')$, so we'll have

$$\#(C' \cap D'') = \#(C' \cap D').$$

By the same argument, we can pick another $C'' \sim C$ and nothing will change. Note by 2.5.3, the sum of two very ample divisors is again very ample. Furthermore, the degree is additive on divisors, so the intersection pairing is additive on very ample divisors.

Now let C, D be arbitrary divisors in $\mathrm{Div}(X)$. Fix an ample divisor H on X. By 2.5.4, there exists n such that $C + nH, D + nH, nH$ are all very ample. Then using Lemma 3.2.6, choose nonsingular curves

(1) $C' \sim C + nH$
(2) $D' \sim D + nH$ transversal to C'.
(3) $E' \sim nH$ transversal to D'.
(4) $F' \sim nH$ transversal to C' and E'

Note $C \sim C' - E'$ and $D \sim D' - F'$. Now define

$$i(C, D) = i(C', D') - i(C', F') - i(E', D') + i(E', F').$$

One can show that this is well defined, and in fact this shows uniqueness, since this gives an explicit expression for the intersection number, since each of the terms in the definition is actually just $\#(C' \cap D')$ since they are nonsingular and meet transversally. $\square$

3.2.7. *Riemann-Roch.* There is a corresponding Riemann-Roch theorem for surfaces that is useful in the proof of the Hodge index theorem.

3.2.7.1. **Theorem.** *If $D \in \mathrm{Div}(X)$, then*

$$\chi(\mathscr{L}(D)) = \frac{1}{2} i(D, D+K) + \chi(\mathcal{O}_X),$$

where K is the canonical divisor.

Proof. [Har77] V.1.6 □

3.2.8. **Corollary.** *If H is ample and $D \in \mathrm{Div}(X)$ with $i(D,H) > 0$ and $i(D,D) > 0$, then for all $n >> 0$, $nD \sim E$ where E is effective.*

Proof. Since $i(D,H) > 0$, for all $n >> 0$, we have $ni(D,H) > n_0$, where n_0 is as in [Har77] V.1.7. Therefore, $\dim_k H^0(X, \Omega_X - \mathscr{L}(D)^{\otimes n}) = 0$. Since $\dim_k H^1(\mathscr{L}(D)^{\otimes n}) \geq 0$, we have by Riemann-Roch that

$$\dim_k H^0(X, \mathscr{L}(D)^{\otimes n}) = \frac{1}{2} n^2 i(D,D) - \frac{1}{2} n i(D,K) + \chi(\mathscr{O}_X).$$

But $i(D,D) > 0$, so as n becomes large, we have $\dim_k H^0(X, \mathscr{L}(D)^{\otimes n}) > 0$. So nD is effective if n is large enough. □

3.3. Hodge Index Theorem.

3.3.1. *Numerical Equivalence.* The Hodge index theorem is a statement about definiteness of the nondegenerate bilinear form given by intersection of divisors on surfaces. However, the intersection product on the class group is not necessarily nondegenerate. Given a curve, one can always find another divisor whose intersection is nonzero, but given two nonzero divisors in $\mathrm{Cl}(X)$, it is not necessarily true that their intersection product is nonzero. For this reason, we define a subgroup of $\mathrm{Cl}(X)$

$$\mathrm{Cl}^0(X) = \{D \in \mathrm{Cl}(X) : i(D,E) = 0 \text{ for all } E \in \mathrm{Cl}(X)\},$$

and then define

$$\mathrm{Num}(X) = \mathrm{Cl}(X)/\,\mathrm{Cl}^0(X).$$

Two divisors D, E are called *numerically equivalent* (we write $D \sim_{\mathrm{num}} E$ if their difference is contained in $\mathrm{Cl}^0(X)$.

3.3.2. **Theorem** (Hodge Index Theorem)**.** *Let X be a smooth projective surface. Let H be an ample divisor on X and let $D \in \mathrm{Div}(X)$ be a divisor such that $D \not\sim_{\mathrm{num}} 0$, but such that $i(D,H) = 0$. Then $i(D,D) < 0$.*

Proof. Suppose $i(D,D) > 0$. By Lemma 2.5.4, we can choose $n > 0$ such that $H' = D + nH$ is very ample. Furthermore, note that

$$i(D, H') = i(D,D) + ni(D,H) = i(D,D) > 0,$$

so by Corollary 3.2.8, there exists $m > 0$ such that mD is an effective divisor (mod linear equivalence). Thus, $i(mD, H) > 0$, so $i(D, H) > 0$, a contradiction.

Suppose $i(D, D) = 0$. We will then construct another divisor $D' \not\sim_{\text{num}} 0$ satisfying $i(D', H) = 0$ and $i(D', D') > 0$, then refer to the first case. then since $D \not\sim_{\text{num}} 0$, there exists some E such that $i(D, E) \neq 0$. Now define $E' = i(H, H)E - i(E, H)H$. Then $i(E', H) = 0$, and

$$i(D, E') = i(H, H)i(D, E) - i(E, H)i(D, H) = i(H, H)i(D, E) > 0.$$

If $D' = nD + E$, then $i(D', H) = 0$, and

$$i(D', D') = n^2 i(D, D) + 2ni(D, E) + i(E, E) = 2ni(D, E) + i(E, E).$$

Thus for some good choice of n, $i(D', D') > 0$, and we get a contradiction by the first part of the proof. □

3.3.3. *Definiteness.* The Hodge index theorem should be thought of as a statement about the intersection pairing on $\mathrm{Num}(X)$. More precisely, note that $\mathrm{Num}(X)$ is defined precisely so that $\mathrm{Num}(X) \times \mathrm{Num}(X) \to \mathbf{Z}$ is nondegenerate. One can show that $\mathrm{Num}(X)$ is a finitely generated abelian group. Thus, by nondegeneracy, one can diagonalize the pairing on $\mathrm{Num}(X)_{\mathbf{R}} = \mathrm{Num}(X) \otimes_{\mathbf{Z}} \mathbf{R}$. The Hodge index theorem says that the resulting matrix will look like

$$\begin{pmatrix} + & & & \\ & - & & \\ & & \ddots & \\ & & & - \end{pmatrix}.$$

The first row corresponds to the choice of hyperplane section (i.e. choice of ample divisor), and the negativity of the remaining rows comes from the Hodge index theorem.

Later on, when we discuss the Hodge standard conjecture, we will see how the negative-definiteness of the intersection pairing away from H generalizes (conjecturally) to higher dimensions.

4. Higher Dimensional Intersection Theory

4.1. **Chow Groups.** The divisor group of a scheme parametrizes the substructures of codimension 1. In the case of surfaces, intersecting curves was the entire theory, so we could use the language of divisors. However, to develop a more general intersection theory, we need to consider higher codimension subvarieties. For this, we introduce the Chow groups.

For this section, an *algebraic scheme* will always be a scheme of finite type over an algebraically closed field k. If it is integral, it will be called an *algebraic variety*.

4.1.1. *Cycles.* The notion of a divisor is generalized to the notion of a cycle. If X is an algebraic scheme, then let $Z^p(X)$ denote the free abelian group on the set of codimension p closed integral subvarieties. Note a codimension p integral subvariety can be identified with its generic point whose local ring has dimension p. Thus $Z^p(X)$ is the free abelian group on the set

$$\{\xi \in X : \dim \mathscr{O}_{X,\xi} = p\}$$

4.1.2. *Rational Equivalence.* Consider $W \in Z^{p-1}(X)$. Then W is integral, and $\mathrm{Div}(W) = Z^p(X)$ immediately by definition. Then as in the notation introduced in 2.2.2, we define $R^p(X)$ to be the subgroup of $Z^p(X)$ generated by

$$\bigcup_{W \in Z^{p-1}(X)} P(W).$$

In other words, given a $(p-1)$-cycle W and a rational function $f \in K(W)$, the principal divisor $\mathrm{div}(f)$ is a generator for $R^p(X)$.

4.1.3. *Chow Group.* The pth Chow group is then defined to be

$$\mathrm{CH}^p(X) = Z^p(X)/R^p(X).$$

Note $\mathrm{CH}^p(X) = 0$ for $p < 0$ and $p > n$.

4.1.4. *Remark.* Since $Z^1(X)$ is the set of divisors, and $R^1(X)$ is the set of principal divisors, we have $\mathrm{CH}^1(X) = \mathrm{Cl}(X)$. Furthermore, note $R^0(X) = 0$ by definition and X itself is the only irreducible 0-codimensional integral subvariety of X, so

$$\mathrm{CH}^0(X) = Z^0(X) \cong \mathbf{Z}.$$

4.1.5. *Affine Space.* Consider $\mathbf{A}_k^n = \operatorname{Spec} k[x_1, \ldots, x_n]$. An integral hypersurface Y in $\mathbf{A}_k^n$ is cut out by a regular polynomial $f \in k[x_1, \ldots, x_n]$, so $Y = \mathrm{div}(f)$, which shows that

$$\mathrm{CH}^1(X) = 0.$$

If we take a point $(c_1, \ldots, c_n)$ in $\mathbf{A}_k^n$, it is the divisor of the function on any line containing that point which vanishes uniquely at that point. So

$$\mathrm{CH}^n(X) = 0.$$

4.2. **Intersection Pairing.** Having defined the Chow groups, we now have a natural setting in which to do intersection theory. Whereas in the case of surfaces, we had only to define a map $\mathrm{Cl}(X) \times \mathrm{Cl}(X) \to \mathbf{Z}$, in the general case we will construct maps

$$\mathrm{CH}^p(X) \times \mathrm{CH}^q(X) \to \mathrm{CH}^{p+q}(X),$$

which will involve more sophisticated versions of the intersection multiplicities.

Fulton's book [Ful98] outlines a general intersection theory for varieties that are not necessarily projective, or even quasi-projective, but it is quite technically involved, and requires a few auxiliary structures in order to define the intersection of two cycles. To state the standard conjectures on cycles, as formulated by Grothendieck, it will suffice to look at projective varieties over a field. In this particular case we define an intersection theory more simply, by first defining intersection multiplicities for the case of proper intersection of subvarieties, and then using a moving lemma, as in the case of surfaces.

4.2.1. *Proper Intersection.* In the case of surfaces, we defined a notion of transversal intersection, which required that the intersection of two lines was simply the sum of points.

We say that two subvarieties $V \in Z^p(X)$ and $W \in Z^q(X)$ *intersect properly* if for every (there are finitely many) irreducible component Z of $V \cap W$, we have

$$\operatorname{codim}_X(Z) = \operatorname{codim}_X(V) + \operatorname{codim}_X(W)$$

This definition is justified by the following lemma.

4.2.2. **Proposition.** *Let X be a smooth projective variety and let V, W be two closed subvarieties. Then for every irreducible component Z of $V \cap W$,*

$$\operatorname{codim}_X(Z) \leq \operatorname{codim}_X(V) + \operatorname{codim}_X(W)$$

Proof. This proposition can be shown locally, since dimension can be computed at the stalk of a closed points of a variety. The variety X is separated, so $\Delta : X \to X \times_k X$ is a closed immersion, so locally we can assume $X \times_k X = \operatorname{Spec}(A)$, with $X \cong \Delta(X)$ a closed subset. Since X and $X \times_k X$ are smooth, we can assume that

$$X = \operatorname{Spec}(A/(f_1, \ldots, f_d))$$

where $d = \dim(X)$ and $f_1, \ldots, f_d$ is a regular sequence in A. Now note that $V \times_k W$ is a variety, and thus irreducible, so $V \times_k W$ is $\operatorname{Spec}(A/\mathfrak{p})$ for some prime $\mathfrak{p}$ in A of height $c_V + c_W$ where $c_V = \operatorname{codim}_X(V)$, and $c_W = \operatorname{codim}_X(W)$. Thus

$$V \cap W \cong V \times_k W \cap \Delta(X) \cong \operatorname{Spec}(A/(\mathfrak{p} + (f_1, \ldots, f_d))).$$

An irreducible component Z of $V \cap W$ corresponds to a minimal prime $\mathfrak{q}$ over $\mathfrak{p} + (f_1, \ldots, f_d)$. Since adding an f_i to $\mathfrak{p} + (f_1, \ldots, f_{i-1})$ can increase the height (of a minimal prime over it) by at most one, we have

$$\operatorname{ht} \mathfrak{q} \leq \operatorname{ht} \mathfrak{p} + d.$$

But by substituting $\operatorname{ht} \mathfrak{q} = 2d - \dim(Z)$ and $\operatorname{ht} \mathfrak{p} = c_V + c_W$, we get

$$c_Z \leq c_V + c_W$$

as desired. □

4.2.3. *Remark.* Proper intersection emphasizes a desired stability in our definition of intersection. For example, in a 3-dimensional variety, the intersection of two surfaces will *generically* give us a union of curves, which is to say that if we shift the surfaces by a small amount, we will typically still get a union of curves.

On the other hand, suppose we intersect two curves in 3-dimensional space. Then although they may have points in common, shifting them by a small amount will render them disjoint. The domain and target of the intersection pairing reflect the notion of proper intersection.

4.2.4. *Intersection Multiplicities.* This treatment is due to Serre in [Ser58]. Suppose V, W are closed subvarieties of X with codimension p and q respectively, with $p+q \leq n$. Suppose further that they intersect properly. Then at some irreducible component Z in $V \cap W$ we define the following intersection multiplicity

$$e(X, V, W, Z) = \sum_{n=0}^{\dim(X)} \ell_{\mathscr{O}_{X,Z}}(\mathrm{Tor}_n^{\mathscr{O}_{X,Z}}(\mathscr{O}_{V,Z}, \mathscr{O}_{W,Z}))$$

Then we define

$$V \cdot W = \sum_Z e(X, V, W, Z) Z$$

where the sum runs over all irreducible components of $V \cap W$. Extend bilinearly to define a map

$$Z^p \times Z^q \to Z^{p+q}.$$

To show that this gives us an intersection product on the Chow groups, we need the following proposition.

4.2.5. **Proposition.** *If $\alpha \in R^p(X)$ and $\beta \in Z^q(X)$, then $\alpha \cdot \beta \in R^{p+q}(X)$.*

Proof. [Sta17] Theorem 25.2 □

4.2.6. **Corollary.** *The intersection pairing descends to a map*

$$\mathrm{CH}^p(X) \times \mathrm{CH}^q(X) \to \mathrm{CH}^{p+q}(X).$$

4.3. **Moving Lemma.** To extend the intersection pairing to arbitrary non-proper intersection, we need the following lemma.

4.3.1. **Lemma.** *Let X be a smooth projective variety. Let $\alpha \in Z^p(X)$ and $\beta \in Z^q(X)$. Then there exists an $\alpha' \in Z^p(X)$ such that $[\alpha'] = [\alpha]$ in $\mathrm{CH}^p(X)$ (rationally equivalent) and such that α' and β intersect properly.*

Proof. [Sta17] Lemma 24.3 □

This concludes our discussion of intersection pairings on the Chow groups. We will now discuss their relevance to the standard conjectures.

5. Standard Conjectures

5.1. **Weil Cohomology Theories.** Although Chow groups provide us with a complete description of the intersection theory on a smooth projective variety, in general they are huge, complex objects that are difficult to work with.

However, a lot of information can be gleaned about the structure of algebraic varieties and their intersection by attaching simpler objects to the varieties that are compatible with the intersection pairing structure, and then studying these objects.

Weil cohomology theories are often useful in this endeavor. A Weil cohomology theory associates to a variety a graded K-algebra for some field K in which each graded component has finite dimension, as well as maps between the graded components that reflect some of the structure of the underlying intersection theory. Although lots of information will be lost in the "linearization", working with vector spaces over a field of characteristic 0 can be much simpler than working with Chow groups.

5.1.1. *Definition.* We follow a note of de Jong [dJ07]. Fix a field K of characteristic 0 and an algebraically closed field k. A *Weil cohomology theory* is given by the following data:

(1) A contravariant functor

$$H^* : \{\text{smooth projective varieties over } k\} \to \{\text{graded commutative } K\text{-algebras}\}.$$

The grading is denoted

$$H^*(X) = \bigoplus H^n(X),$$

where the H^n are functors taking values in Vect_K. Multiplication in this algebra is denoted $\alpha \smile \beta$. We write $f^* : Y \to X$ for $H(f)$, for $f : X \to Y$.

(2) For each X of dimension n, a "trace" isomorphism

$$\mathrm{Tr}_X : H^{2n}(X) \xrightarrow{\sim} K$$

(3) A group homomorphism $\mathrm{cl}_X : Z^p(X) \to H^{2p}(X)$ called the *cycle class map*.

The axioms are as follows:

(1) Each $H^i(X)$ is a finite dimensional vector space, and $H^i(X) = 0$ for $i < 0$ and if $i > 2n$, where $n = \dim(X)$.

(2) If p_X and p_Y are the projections from $X \times_k Y$, then the K-algebra map

$$H^*(X) \otimes_K H^*(Y) \to H^*(X \times_k Y), \alpha \otimes \beta \mapsto p_X^*(\alpha) \smile p_Y^*(\beta)$$

is an isomorphism.

(3) (Poincaré Duality) Suppose X has dimension n. For $0 \le i \le 2n$,

$$\mathrm{Tr}_X \circ \smile : H^i(X) \otimes_K H^{2n-i}(X) \to K$$

is a perfect bilinear pairing.

(4) The trace commutes with products: in other words, for $\alpha \in H^{2n}(X)$ and $\beta \in H^{2n}(Y)$,

$$\mathrm{Tr}_{X \times_k Y}(p_X^*(\alpha) \smile p_Y^*(\beta)) = \mathrm{Tr}_X(\alpha)\,\mathrm{Tr}_Y(\beta).$$

(5) The map cl_X commutes with products: in other words, if X and Y are smooth projective varieties over k and $V \subseteq X$ and $W \subseteq Y$ are closed integral subvarieties, then

$$\mathrm{cl}_{X \times_k Y}(V \times_k W) = p_X^*(\mathrm{cl}_X(V)) \smile p_Y^*(\mathrm{cl}_Y(W))$$

(6) If $f : X \to Y$ is a morphism of smooth projective varieties, $Z \subseteq X$ is a closed integral subvariety, and $\alpha \in H^{2\dim(Z)}(Y)$, then

$$\mathrm{Tr}_X(\mathrm{cl}_X(Z) \smile f^*(\alpha)) = d\,\mathrm{Tr}_Y(\mathrm{cl}_Y(f(Z)) \smile \alpha)$$

where d is the degree of the morphism $f : Z \to f(Z)$.

(7) If $f : X \to Y$ is as above, then suppose $W \subseteq Y$ is a closed integral subvariety such that the components of $f^{-1}(W)$ have pure dimension $\dim(W) + \dim(X) - \dim(Y)$, and either f is flat in a neighborhood of W or $f^{-1}(W)$ is generically smooth. Then

$$f^*(\mathrm{cl}_Y(W)) = \sum_{i=1}^{r} \ell_{\mathscr{O}_{X,W_i}}(\mathscr{O}_{Z,W_i})\,\mathrm{cl}_X(W_i)$$

where the W_i are the irreducible components of $f^{-1}(W)$.

(8) In the case of a point, where $x\,\mathrm{Spec}\,k$, then $\mathrm{cl}_x(x) = 1$, and $\mathrm{Tr}_x(1) = 1$.

5.1.2. *Pushforward.* Functoriality of H^* gives us pullback maps, but one can use the Poincaré duality to obtain pushforward maps as well.

Fix X, Y smooth projective varieties of dimensions n, m respectively. Given $f : X \to Y$, we get a map $f^* : H^*(Y) \to H^*(X)$ compatible with the grading. This induces a dual map $(f^*)^t : \mathrm{Hom}_K(H^*(X), K) \to \mathrm{Hom}_K(H^*(Y), K)$. The following commutative diagram defines the pushforward:

$$\begin{tikzcd}
H^i(X) \arrow[r, "\sim"] \arrow[d, "f_*"] & \mathrm{Hom}_K(H^{2n-i}(X), K) \arrow[d, "(f^*)^t"] \\
H^{i-2(n-m)}(Y) \arrow[r, "\sim"] & \mathrm{Hom}_K(H^{2n-i}(Y), K)
\end{tikzcd}$$

After unraveling the definition, one sees that given a cycle $\alpha \in H^i(X)$, the pushforward $f_*(\alpha)$ is the unique element in $H^{i-2(n-m)}$ such that for all β we have

$$\mathrm{Tr}_Y(f_*(\alpha) \smile \beta) = \mathrm{Tr}_X(\alpha \smile f^*(\beta)).$$

5.1.3. *Intersection Pairing.* One nice property of a Weil cohomology theory is that it is compatible with the intersection pairing, in the following sense.

5.1.3.1. **Proposition.** *If X is a smooth projective variety, H^* is a Weil cohomology theory, and $V \in Z^p(X)$, $W \in Z^q(X)$ are closed integral subvarieties that intersect properly, then*

$$\mathrm{cl}_X(V \cdot W) = \mathrm{cl}_X(V) \smile \mathrm{cl}_X(W).$$

Proof. Suppose $V \cdot W = \sum_i n_i Z_i$. Letting $\Delta : X \to X \times_k X$, we have

$$\Delta^{-1}(V \times W) = \bigcup_i Z_i.$$

Then by axiom 7,

$$\mathrm{cl}_X(V \cdot W) = \sum_i n_i \,\mathrm{cl}_X(Z_i) = \Delta^*(\mathrm{cl}_{X\times_k X}(V \times_k W)).$$

By axiom 5,

$$\Delta^*(\mathrm{cl}_{X\times_k X}(V \times_k W)) = \Delta^*(p_X^*(\mathrm{cl}_X(V)) \smile p_X^*(\mathrm{cl}_Y(W))).$$

Finally, by functoriality,

$$\Delta^*(p_X^*(\mathrm{cl}_X(V)) \smile p_X^*(\mathrm{cl}_Y(W))) = (p_X \circ \Delta)^*(\mathrm{cl}_X(V) \smile \mathrm{cl}_X(W)) = \mathrm{cl}_X(V) \smile \mathrm{cl}_X(W).$$

□

5.2. **Statement of the Conjectures.** Having defined the notion of a Weil cohomology theory, we can now state the standard conjectures.

Grothendieck originally included these conjectures in his formulation of the category of pure motives. If the standard conjectures are true, then the category of pure motives will be a semisimple abelian category. The conjectures were formulated in the 1960s, but since then, not much progress has been made in proving them in full generality.

5.2.1. *Lefschetz Operator.* If $X \subseteq \mathbf{P}^n$ is a smooth projective variety and $H \subseteq \mathbf{P}^n$ is a hyperplane, then $W = H \cap X$ defines a hyperplane section in X. Then define

$$L : H^p(X) \to H^{p+2}(X), \quad \alpha \mapsto \alpha \smile \mathrm{cl}(W).$$

We will need the following theorem to formulate the first conjectures:

5.2.2. **Theorem** (Strong Lefschetz)**.** *The iterated map*

$$L^{n-p} : H^p(X) \to H^{2n-p}(X)$$

is an isomorphism.

With this, we can define the *Lefschetz operator* as the map Λ in the following diagram

$$\begin{tikzcd} H^p(X) \arrow[r, "L^{n-p}"] \arrow[d, "\Lambda"'] & H^{2n-p} \arrow[d, "L"] \\ H^{p-2}(X) \arrow[r, "L^{n-p+1}"'] & H^{2n-p+2} \end{tikzcd}$$

This makes sense because the top and bottom row are isomorphisms by 5.2.2.

5.2.3. **Conjecture** (Conjecture B)**.** *The Lefschetz operator Λ is algebraic. In other words, there exists a cycle $V \subset X \times_k X$ of dimension $n-1$ such that*

$$\Lambda(\alpha) = p_{2,*}(p_1^*(\alpha) \smile \mathrm{cl}_X(V))$$

for all $\alpha \in H^i(X)$.

The second standard conjecture is a generalization of the Hodge Index Theorem, and concerns the positive (or negative, depending on the dimension)-definiteness of an intersection pairing on certain cohomology classes induced by algebraic cycles.

Specifically, the intersection pairing will be defined on the image of the map

$$\mathrm{cl}_X^{\mathbf{Q}} : Z^p(X) \otimes_{\mathbf{Z}} \mathbf{Q} \to H^{2p}(X), \alpha \otimes q \mapsto q \, \mathrm{cl}_X(\alpha).$$

The image is a sub-$\mathbf{Q}$-vector space of $H^{2p}(X)$, which we call $C^p(X)$. By Proposition 5.1.3.1, the cup product restricts nicely to the $C^p(X)$, and we get maps

$$C^p(X) \otimes_{\mathbf{Q}} C^q(X) \xrightarrow{\smile} C^{p+q}(X).$$

An element of $C^n(X)$ is a $\mathbf{Q}$-linear combination of elements of the form $\mathrm{cl}_X(x)$, where $x \in X$ is a closed point. But by the pushforward formula in 5.1.2,

$$\mathrm{Tr}_x(\mathrm{cl}_x(x)) = \mathrm{Tr}_X(\mathrm{cl}_X(x)),$$

but one can show easily that $\mathrm{Tr}_x(\mathrm{cl}_x(x)) = 1$. Thus,

$$\mathrm{Tr}_X |_{C^n(X)} : C^n(X) \to \mathbf{Q}.$$

Furthermore, the cup product restricts nicely to the $C^p(X)$

5.2.4. **Conjecture** (Hodge Standard Conjecture)**.** *Fix a hyperplane section $W = H \cap X$ in X and define the map L as before. Fix $j \leq \dim(X)/2$. Then consider sub-$\mathbf{Q}$-vector space $C^j(X)$ of $H^{2j}(X)$ defined by*

$$C^j_{\mathrm{Pr}} = \{\alpha \in \mathrm{im}(\mathrm{cl}_X : Z^j(X) \to H^{2j}(X)) : L^{n-2j+1}(\alpha) = 0\}$$

Then the intersection pairing $C^j_{\mathrm{Pr}(X)} \times C^j_{\mathrm{Pr}(X)} \to K$ on $C^j(X)$ given by

$$(\alpha, \beta) \mapsto (-1)^j \, \mathrm{Tr}_X(\alpha \smile \beta \smile \mathrm{cl}_X(W)^{\smile n-2j})$$

is positive definite.

5.2.5. *Characteristic 0.* In fact, the Hodge standard conjecture is known for fields of characteristic 0, using classical methods from Hodge theory of complex algebraic varieties.

5.2.6. *Remark.* If $\dim(X) = 2$ and $j = 1$, then this is just the Hodge index theorem for surfaces. To see this, note that hyperplane sections correspond to ample divisors, so application of L is just intersection with a very ample divisor. Note further that the trace just gives the intersection product when restricted to divisors.

But the key to the generalization is the fact that numerical equivalence (see 3.3.1) is the same as homological equivalence for divisors: this is to say that $\mathrm{Num}(X) \cong H^2(X)$ via the cycle class map.

Does this hold more generally? Let

$$\mathrm{CH}^p_0(X) = \{\alpha \in \mathrm{CH}^p(X) : \deg(\alpha, \beta) = 0 \text{ for all } \beta \in \mathrm{CH}^{n-p}(X)\}.$$

Then two cycles are called *numerically equivalent* if their difference lies in $\mathrm{CH}^p_0(X)$.

5.2.7. **Conjecture** (Conjecture D)**.** *Numerical equivalence and homological equivalence of cycles is the same equivalence relation.*

In fact, if the Hodge standard conjecture is true, one can show that that conjecture D and conjecture B are the same.

6. Arithmetic Surfaces

We now turn our attention to the arithmetic reformulation of the standard conjectures. To begin, we will discuss Arakelov's original motivation [Ara74] for developing arithmetic intersection theory on models for algebraic curves over finite extensions of $\mathbf{Q}$, and develop an analogous version of the Hodge index theorem in this case. Then we will develop a more modern formulation of arithmetic intersection, developed primarily by Gillet and Soulé [GS94].

6.1. **Arithmetic Surfaces.**

6.1.1. *Definitions.* For our purposes, the definition of curves and surfaces over an arbitrary scheme is similar to the definition over an algebraically closed field.

6.1.2. *Models.* If $C \to \operatorname{Spec} K$ is a smooth curve (see Definition 3.1.1), then a *smooth model* for C is a smooth projective scheme $S \to \operatorname{Spec} \mathcal{O}_K$ of relative dimension 1 such that the following is a pullback diagram:

$$\begin{array}{ccc} C & \longrightarrow & \operatorname{Spec} K \\ \downarrow & & \downarrow \\ S & \longrightarrow & \operatorname{Spec} \mathcal{O}_K \end{array}$$

Equivalently, the fiber of the map $S \to \operatorname{Spec} \mathcal{O}_K$ over the zero ideal in $\mathcal{O}_K$ is isomorphic to C. For example, if $K = \mathbf{Q}$, then smooth models are smooth projective surfaces over $\operatorname{Spec} \mathbf{Z}$. Note that $\operatorname{Spec} \mathbf{Z}$ has dimension 1, so C is a 2-dimensional scheme. This motivates the following definition: an *arithmetic surface* is a smooth projective scheme over $S \to \operatorname{Spec} \mathcal{O}_K$ of relative dimension 1 for some number field K. Furthermore, C will always be geometrically connected and geometrically irreducible.

We will take for granted the fact that there always exists a smooth model as defined above. A proof can be found in Chapter 10 of [Liu06].

6.1.3. *Analogy.* By 2.4.10, the notions of line bundles, Cartier divisors, Weil divisors and invertible sheaves are all the same. However, for arithmetic objects, to get a good notion of intersection theory we need to add more structure.

One way to see this is to consider the following: given a curve C in an affine space over an algebraically closed field, one can always define a map

$$\operatorname{Div}(C) \to \mathbf{Z}.$$

However, this map does not descend to a map on $\mathrm{Cl}(C) \to \mathbf{Z}$; for example, take a rational function which has a single zero, but no poles. To get a map $\mathrm{Cl}(C) \to \mathbf{Z}$, we need to consider completions of such curves in a projective space. Then we get the following lemma.

6.1.3.1. **Lemma.** *If C is a smooth projective curve over an algebraically closed field k and $f \in K(C)$, then*

$$\deg \operatorname{div}(f) = 0.$$

Proof. This is essentially a reflection of the fact that a rational function on a projective curve is given by a quotient of homogeneous polynomials of the same degree in the coordinate ring. For more precise details, see for example [Har77] Corollary 6.10. ☐

Lemma 6.1.3.1 is correctly interpreted as the analog of a product formula in arithmetic. To see this, note that

$$\deg \operatorname{div}(f) = \sum_{P \in C} v_P(f),$$

where $v_P(f)$ is the order of vanishing of f at P. The product formula for a number field K (written using a sum) states for $a \in K$ that

$$\sum_{\iota : K \hookrightarrow \mathbf{C}} -\log|\iota(a)| + \sum_{\mathfrak{p}} v_{\mathfrak{p}}(a) = 0$$

where the $\mathfrak{p}$ are closed points in $\operatorname{Spec} \mathcal{O}_K$, and the $v_{\mathfrak{p}}$ are the $\mathfrak{p}$-adic absolute values. This analogy is part of a larger analogy between number fields and function fields for curves over finite fields.

However, if we merely consider divisors on $\operatorname{Spec} \mathcal{O}_K$, we don't get the product formula, because the prime divisors in $\operatorname{Spec} \mathcal{O}_K$ correspond to closed points, which are nonzero prime ideals in $\mathcal{O}_K$, so we are missing points corresponding to the infinite places. Arakelov's idea was to build these "missing points" into the intersection theory directly, rather than adding them to the scheme structure of $\operatorname{Spec} \mathcal{O}_K$, or to an arithmetic surface over $\operatorname{Spec} \mathcal{O}_K$.

6.1.4. *Divisors over ∞.* If $[K : \mathbf{Q}] = n$, then there are n embeddings $\iota_k : K \hookrightarrow \mathbf{C}$ preserving $\mathbf{Q}$. For each of these embeddings $\iota_1, \ldots, \iota_n$, we can form the pullback

$$\begin{tikzcd} S_k = C \times_{\operatorname{Spec} K, \iota_k} \operatorname{Spec} \mathbf{C} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbf{C} \arrow[d, "\operatorname{Spec} \iota_k"] \\ C \arrow[r] & \operatorname{Spec} K \end{tikzcd}$$

or equivalently, if $\iota_k : \mathcal{O}_K \hookrightarrow \mathbf{C}$,

$$\begin{tikzcd} S_k = S \times_{\operatorname{Spec} \mathcal{O}_K, \iota_k} \operatorname{Spec} \mathbf{C} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbf{C} \arrow[d, "\operatorname{Spec} \iota_k"] \\ S \arrow[r] & \operatorname{Spec} \mathcal{O}_K \end{tikzcd}$$

This gives us a 1-dimensional irreducible smooth scheme over Spec $\mathbf{C}$, which by Serre's GAGA theorem, corresponds to a 1-dimensional irreducible compact smooth complex manifold, i.e. a compact Riemann surface. Arakelov's novel idea was to treat these Riemann surfaces as though they were (prime) *divisors over infinity*. To justify this terminology, note the following topological lemma:

6.1.4.1. **Lemma.** *If $D \in \mathrm{Cl}(S)$ is an irreducible (Weil) divisor, then its image under $\pi : S \to \operatorname{Spec} \mathcal{O}_K$ is either all of $\operatorname{Spec} \mathcal{O}_K$ or a singleton $\{\mathfrak{p}\}$ where $\mathfrak{p}$ is a closed point.*

Proof. Note π is continuous and D is irreducible, so $\pi(D)$ is irreducible. Note π is projective $\implies$ proper $\implies$ closed, so $\pi(D)$ is an irreducible closed subset. Thus, it is either a point (if it's 0-dimensional) or the entire space, since $\operatorname{Spec} \mathcal{O}_K$ is integral, thus irreducible. ☐

If the image of D is $\operatorname{Spec} \mathcal{O}_K$, we say that D is *horizontal*, and if the image is a point, we say that D is *vertical*. Note that the fiber over a point in $\operatorname{Spec} \mathcal{O}_K$ may not necessarily be irreducible, but the "fibers over infinity" in this context are simply the S_k.

With this in mind, our intuition dictates that if S_k were indeed part of S, and if the infinite places (embeddings) corresponded to actual points in $\operatorname{Spec} \mathcal{O}_K$, then in principle we would expect to have

$$\pi(S_k) = \{\infty_k\}.$$

6.1.5. *Hermitian Metrics.* As a general rule, Arakelov theory heavily incorporates the complex geometry induced by the complex embeddings $K \hookrightarrow \mathbf{C}$. For example, we will need the existence of a Hermitian metric h_k on each S_k, with corresponding (real) (1,1)-form called the "volume form" $\mu_k = i/2(h_k - \overline{h_k})$. Furthermore, we will stipulate that

$$\int_{S_k} \mu_k = 1.$$

6.1.6. *Divisors over a prime.* Consider the set of places of $\mathcal{O}_K$. Given any prime $\mathfrak{p}$ in $\mathcal{O}_K$, we can form a scheme $S_p \to \operatorname{Spec} \kappa(\mathfrak{p}) = \mathcal{O}_K/\mathfrak{p}$, called the "reduction mod $\mathfrak{p}$" of S at $\mathfrak{p}$, by taking the following pullback.

$$\begin{array}{ccc} S_{\mathfrak{p}} & \longrightarrow & \operatorname{Spec} \kappa(\mathfrak{p}) \\ \downarrow & & \downarrow \\ S & \longrightarrow & \operatorname{Spec} \mathcal{O}_K \end{array}$$

This gives us a family $\{S_{\mathfrak{p}}\}_{P_f}$, where P_f is the set of finite places of $\mathcal{O}_K$. But by defining the Riemann surfaces as we did above, we now have a family $\{S_{\mathfrak{p}}\}_{P_f} \cup \{S_k\}_{1,\ldots,n}$, which is essentially indexed by $P = P_f \cup P_\infty$ is the set of all places on $\mathcal{O}_K$.

6.1.7. *Arakelov Divisors.* The group of *Arakelov divisors* is defined as

$$\mathrm{Div}_{\mathrm{Ar}}(S) = \mathrm{Div}(S) \oplus \mathbf{R}[S_1] \oplus \cdots \oplus \mathbf{R}[S_n],$$

where the $[S_k]$ are formal symbols corresponding to the Riemann surfaces S_k. We sometimes write

$$\mathrm{Div}_{\infty}(S) = \mathbf{R}[S_1] \oplus \cdots \oplus \mathbf{R}[S_k].$$

Notice that valuations corresponding to the finite places have value group $\mathbf{Z}$, while the valuations corresponding to the finite places have value group dense in $\mathbf{R}$. This is reflected in the definition of the Arakelov divisor.

6.1.8. *Principal Arakelov Divisors.* Since we introduce infinite divisors, we have to redefine the notion of a principal divisor to reflect the structure of the Arakelov divisor group.

To motivate our construction, we first consider the Arakelov divisor group of $\operatorname{Spec} \mathcal{O}_K$ itself. Since $\operatorname{Spec} \mathcal{O}_K$ is 1-dimensional, its divisors are simply its closed points. We define

$$\mathrm{Div}_{\mathrm{Ar}}(K) := \mathrm{Div}(\operatorname{Spec} \mathcal{O}_K) \oplus \mathbf{R}[S_1] \oplus \cdots \oplus \mathbf{R}[S_k]$$

as above. Since we were motivated by the product formula, it makes sense to define the *Arakelov degree* $\mathrm{Div}_{\mathrm{Ar}}(K) \to \mathbf{R}$ by

$$\deg_{\mathrm{Ar}}(\sum_{\mathfrak{p}} n_{\mathfrak{p}}\mathfrak{p} + \sum_{i=1}^{n} r_k[S_k]) = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log(\#\mathcal{O}_K/\mathfrak{p}) + \sum_{i=1}^{k} r_k$$

In this case, principal divisors are induced by elements of K itself. For $a \in K$,

$$\mathrm{div}_{\mathrm{Ar}}(a) = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(a)[\mathfrak{p}] + \sum_{k=1}^{n} -\log|\iota_k(a)|[S_k]$$

(note in this case S_k is just a point, the 0-dimensional complex manifold corresponding to $\operatorname{Spec} \mathbf{C}$). Then the group of principal divisors is denoted $P_{\mathrm{Ar}}(K)$, and the Arakelov class group is defined by

$$\mathrm{Cl}_{\mathrm{Ar}}(K) = \mathrm{Div}_{\mathrm{Ar}}(K)/P_{\mathrm{Ar}}(K).$$

By the product formula, $\deg_{\mathrm{Ar}} \mathrm{div}_{\mathrm{Ar}}(a) = 0$.

Any rational function $f \in K(S)$ induces a meromorphic function $f_k : S_k \to \mathbf{C}$ for each k, using GAGA. This allows us to define the *principal Arakelov divisor* of f as follows:

$$\mathrm{div}_{\mathrm{Ar}}(f) = \mathrm{div}(f) + \sum_{k=1}^{n} \left(\int_{S_k} -\log|f_k|\mu_k \right) [S_k].$$

Then the *Arakelov class group* is given by

$$\mathrm{Cl}_{\mathrm{Ar}}(S) = \mathrm{Div}_{\mathrm{Ar}}(S)/P_{\mathrm{Ar}}(S),$$

where $P_{\mathrm{Ar}}(S)$ is the subgroup of principal divisors.

6.2. **Intersection of Divisors.** Following our treatment of surfaces over an algebraically closed field, we wish to define a symmetric intersection pairing

$$i_{\mathrm{Ar}} : \mathrm{Cl}_{\mathrm{Ar}}(S) \times \mathrm{Cl}_{\mathrm{Ar}}(S) \to \mathbf{R}$$

on arithmetic surfaces, which will now have to take the S_k (divisors at infinity) into account. There are four cases we need consider:

(1) *Finite divisors at finite places.* This is relatively straightforward: if C, D are irreducible finite divisors (i.e. have zero $[S_k]$ components), and P is a closed point in S, then there exists $f, g \in \mathcal{O}_{S,\mathfrak{p}}$ that locally define C and D. Furthermore, the residue field $\kappa(P)$ is a finite field, so we can define

$$i_P(C, D) = \ell_{\mathcal{O}_{S,P}}(\mathcal{O}_{S,P}/(f, g)) \cdot [\kappa(P) : \kappa(\pi(P))],$$

which is well defined by 2.2.2.1. Then for a closed point $[\mathfrak{p}] \in \operatorname{Spec} \mathcal{O}_K$, we define

$$i_{\mathfrak{p}}(C, D) = \sum_{P \in \pi^{-1}([\mathfrak{p}])} i_P(C, D)$$

(2) *Finite divisors at infinite places.* Assuming either of the divisors are vertical, it is clear that we should have no intersection at an infinite place. On the other hand, if both are horizontal, we should have some notion of intersection number. This involves the theory of Green functions, which we will treat below.

(3) *Infinite divisors* It is conceptually clear that $i(S_k, S_{k'})$ should be 0 for two infinite divisors. It is also clear that $i(S_k, D)$ should be 0 for D a vertical divisor.

If D is a horizontal divisor, then D is the closure of a point ξ_D in the generic fiber C. Note the residue field $\kappa(\xi_D)$ is a finite field extension of K, so at any S_k, we define

$$i(S_k, D) = \deg_K(D) := [\kappa(\xi_D) : K].$$

To motivate this definition, notice that ξ_D is defined by a map $\operatorname{Spec} \kappa(\xi_D) \to X$, and there are $m := [\kappa(\xi_D) : K]$ embeddings of $\kappa(\xi_D) \hookrightarrow \mathbf{C}$ extending $K \hookrightarrow \mathbf{C}$, which gives m conjugate points on S_k.

It remains to treat the case of finite divisors at infinite places. Given two irreducible horizontal divisors D, E, we can again choose points P, Q in the generic fiber X whose closures give all D, E respectively. These points have residue fields L, M which have degree l, m over K, and thus there are l, m embeddings of $L, M \hookrightarrow \mathbf{C}$ extending an embedding $K \hookrightarrow \mathbf{C}$, which give us l, m points on S_k. Call these points

$$P_1, \ldots, P_l \text{ and } Q_1, \ldots, Q_m.$$

Then we define (for the kth embedding $K \hookrightarrow \mathbf{C}$)

$$i_k(D, E) = \sum_{i,j} i_{S_k}(P_i, Q_j),$$

where i, j run over all $1, \ldots, l$ and $1, \ldots, m$, and i_{S_k} is an intersection product for any two points on a Riemann surface, which we will define now, using the theory of Green functions.

6.2.1. *Green Functions.* Fix a Riemann surface X. By Serre's GAGA, X corresponds to a complex variety of dimension 1, whose divisor group $\mathrm{Div}(X)$ is generated by the points in X. A *Green family* for X is a set of smooth functions $\{g_D\}_{D\in\mathrm{Div}(X)}$ subject to the following conditions:

(G1) For $D \in \mathrm{Div}(X)$, g_D is defined on $X \setminus \mathrm{supp}(D) \to \mathbf{R}$.

(G2) The association $g : \mathrm{Div}(X) \to \{\text{smooth functions}\}$ is a group homomorphism.

(G3) If $D \in \mathrm{Div}(X)$ is locally represented by Zariski open $U \subseteq X$ and f meromorphic on U, then
$$g_D + \log|f(\cdot)|$$
extends to a smooth function on all of U. This says that g_D has *logarithmic singularities* along D.

(G4) *(Harmonicity)* The following differential equation is satisfied for all divisors D:
$$dd^c g_D = (\deg D)\mu.$$
Note dd^c is the complex version of the Laplacian operator, so this really does reflect a notion of harmonicity.

(G5) *(Normalization)* For all divisors D,
$$\int_X g_D\mu = 0.$$

Furthermore, one can show that in (G3), if $X = U$ then $g_D + \log|f(\cdot)|$ will be constant, equal to
$$\int_X -\log|f|\mu.$$

To a Green family is associated the corresponding *Green function* defined as
$$g : X \times X \setminus \Delta(X) \to \mathbf{R}, \quad (P,Q) \mapsto g_P(Q).$$
We will show in a moment that in fact $g(P,Q) = g(Q,P)$.

6.2.2. *Total Intersection.* Then the intersection multiplicity of $P, Q \in X$ for $P \neq Q$ is defined as
$$i_{S_k}(P,Q) = g(P,Q)$$
for every Riemann surface S_k. Then given two finite divisors D, E, we define the total intersection number
$$i(D,E) = \sum_{\mathfrak{p}} i_{\mathfrak{p}}(D,E) + \sum_k i_{S_k}(D,E).$$
We now have a complete description of intersection for Arakelov divisors, except for self-intersection of finite divisors. To intersect a finite divisor D with itself, we simply replace D with $D + (f)$ for some rational function (f), such that $\mathrm{supp}(D + (f))$ and $\mathrm{supp}(D)$ are disjoint, then use the fact that the intersection product is invariant under linear equivalence, which we will prove in a moment.

6.2.3. *More on Green Families.* We briefly sketch the idea of the proof of existence for Néron families, following [Ara74].

Fix a Riemann surface X. To define a Green family, one only needs to define g_P for single points $P \in X$. To do so, we consider the collection of nonnegative smooth functions $\phi_P : X \to \mathbf{R}$ that take value 0 at P, and are locally expressible as

$$\phi_P(z) = |t_P(z)|u(z),$$

where $u : X \to \mathbf{R}$ is a nonzero smooth function, and $t_P : X \to \mathbf{C}$ is a meromorphic function with a simple pole at P. Furthermore, we require that ϕ_P is positive on $X \setminus \{p\}$. Then the Green function is

$$g_P(x) = -\log \phi_P(x) \text{ for all } x \in X \setminus \{P\}.$$

Arakelov then uses (G4) and (G5) to uniquely specify such functions.

6.2.3.1. **Proposition.** *The above described family of functions is a Green family.*

Proof. We only need to prove (G3). If a divisor D is locally represented by (U, f), then $D = \sum_i n_i P_i - \sum_j m_j Q_j$, where $n_i, m_j > 0$, where P_i are the zeros of f in U with multiplicity n_i, and Q_j are the poles with multiplicity $-m_i$. Thus

$$f(z) = \frac{\prod_i (z - P_i)^{n_i}}{\prod_j (z - Q_j)^{m_j}} h(z),$$

where $h(z)$ is nonzero and holomorphic on U. Therefore,

$$\lambda_D(z) + \log|f(z)| = c(z) \log \left[\frac{\prod_j \phi_{Q_j}(z)^{m_j}}{\prod_i \phi_{P_i}(z)^{n_i}} \frac{\prod_i |z - P_i|^{n_i}}{\prod_j |z - Q_j|^{m_j}} \right],$$

for some smooth $c(z)$. But since the ϕ_{P_i} and ϕ_{Q_j} have simple zeros at their respective points, this logarithm will extend to all of U. □

As promised,

6.2.3.2. **Proposition.** *A Green function is symmetric on $X \times X \setminus \Delta(X)$.*

Proof. Fix two distinct points $P, Q \in X$, and pick two small disks U_P, U_Q around P, Q. Then consider the integral

$$\int_{X \setminus (U_P \cup U_Q)} (\log \phi_P \Delta \log \phi_Q - \log \phi_Q \Delta \log \phi_P) dx dy.$$

We want to look at the behavior as the radius of disks go to 0. So using Green's second identity, this integral becomes

$$\int_{\partial U_P \cup \partial U_Q} \log \phi_P \frac{\partial \log \phi_Q}{\partial n} - \log \phi_Q \frac{\partial \log \phi_P}{\partial n} ds$$

where $\partial/\partial n$ is the normal derivative. But this is equal to

$$\int_{\partial U_P} \log \phi_P \frac{\partial \log \phi_Q}{\partial n} ds - \int_{\partial U_Q} \log \phi_Q \frac{\partial \log \phi_P}{\partial n} ds - \int_{\partial U_P} \log \phi_Q \frac{\partial \log \phi_P}{\partial n} ds + \int_{\partial U_Q} \log \phi_P \frac{\partial \log \phi_Q}{\partial n} ds$$

Taking into account properties of the ϕ, we can assume without loss of generality that the first two integrals are of the form

$$\int_{C_r} \log |z| \frac{\partial}{\partial n} \log |z - c| ds,$$

where C_r is a circle of radius r about 0 in $\mathbf{C}$, and $c \in \mathbf{C}$ is arbitrary. But

$$\int_{C_r} \log |z| \frac{\partial}{\partial n} \log |z - c| ds \leq \log(r) M \int_{C_r} r dt \to 0 \text{ as } r \to 0,$$

where $\partial/\partial n(\log |z - c|)$ is bounded by some constant M for small r. The last two integrals are essentially of the form

$$\int_{C_r} \log |z - c| \frac{\partial}{\partial n} \log |z| ds,$$

for some $c \in \mathbf{C}$ which becomes

$$\int_0^{2\pi} \log |re^{i\theta} - c| \frac{1}{r} r d\theta \to 2\pi \log |c| \text{ as } r \to 0.$$

Therefore, the whole integral approaches, as $r \to 0$,

$$2\pi(\log \phi_P(Q) - \log \phi_Q(P)).$$

On the other hand, Arakelov's two conditions show that

$$\int_{X \setminus (U_P \cup U_Q)} (\log \phi_P \Delta \log \phi_Q - \log \phi_Q \Delta \log \phi_P) dx dy = 2\pi \int_{X \setminus (U_P \cup U_Q)} (\log \phi_P - \log \phi_Q) d\mu = 0.$$

(for this, we use the following equivalent formulation of the harmonicity condition:

$$\frac{1}{2\pi} \Delta \log \phi_P dx dy = -\mu,$$

where Δ is the usual Laplacian) □

6.3. **Arithmetic Hodge Index Theorem.** Following [Hri85], we now discuss the Hodge index theorem for arithmetic surfaces. The classical Hodge index theorem is a statement about the signature of the bilinear intersection form on the set of divisor classes of a surface. The arithmetic version has the same flavor, but now there are more divisors.

To see this, consider an arithmetic surface $\pi : S \to \operatorname{Spec} \mathcal{O}_K$ which is a model for $C = S_K$. Note we have

$$\operatorname{Div}_{\mathrm{Ar}}(S) = \operatorname{Div}_{\mathrm{hor}}(S) \oplus \operatorname{Div}_{\mathrm{ver}}(S) \oplus \operatorname{Div}_{\infty}(S),$$

which correspond to the horizontal, vertical, and infinite prime divisors. There is an isomorphism

$$\operatorname{Div}_{\mathrm{hor}}(S) \cong \operatorname{Div}(C),$$

which in the forward direction is given by taking the generic point (which is the same as intersecting with the fiber of S over the generic point in $\operatorname{Spec}\mathcal{O}_K$) and the reverse direction is the topological closure. So we see that arithmetic surfaces contain the intersection theory of the curves that they model, but also contain information coming from the vertical divisors and infinite divisors, so we will need to discover how they interact with the intersection pairing.

In particular, we will use Néron's work on the intersection theory of a curve of the form $C \to \operatorname{Spec} K$ (with K a number field) without proof, and we will give a more detailed treatment of the vertical and infinite divisors.

6.3.1. *Degree.* Consider the generic point isomorphism $\xi : \mathrm{Div}_{\mathrm{hor}}(S) \to \mathrm{Div}(C)$, and recall from 6.2 that for a prime horizontal divisor D we have

$$\deg_K(D) := [\kappa(\xi_D) : K].$$

Thus, we have a degree homomorphism $\deg_K : \mathrm{Div}_{\mathrm{hor}}(S) \cong \mathrm{Div}(C) \to \mathbf{Z}$. We will first concern ourselves with the degree 0 divisors on C, which we will denote by $\mathrm{Div}^0(C)$, or $\mathrm{Div}^0_{\mathrm{hor}}(S)$ if we are considering them as vertical divisors.

6.3.1.1. **Proposition.** *The intersection product i_{Ar} is invariant under linear equivalence of divisors.*

Proof. Fix a rational function f. Then

$$i(\mathrm{div}_{\mathrm{Ar}}(f), [S_k]) = i(\mathrm{div}(f), [S_k]) + \sum_{k=1}^{n}(\cdots)\cancel{i([S_k],[S_k])} = i(\mathrm{div}(f), [S_k]).$$

But no vertical divisor intersects $[S_k]$, and by considering the intersection product in $\mathrm{Div}(C)$ instead of $\mathrm{Div}_{\mathrm{hor}}(S)$, we see that

$$\deg_K \mathrm{div}(f) = 0.$$

Thus, $\mathrm{div}(f)_{\mathrm{Ar}}$ has 0 intersection with the infinite divisors. If E is a vertical divisor over a prime $\mathfrak{p}$, then by definition it has 0 intersection with the infinite divisors, then the intersection number is just $i_{\mathfrak{p}}(\mathrm{div}(f), E)$. But we can look at this locally as a curve over $\kappa(\mathfrak{p})$, whence the intersection product is 0 by classical intersection theory over a finite field.

So the remaining case is when E is a prime horizontal divisor. In this case,

$$i(\mathrm{div}_{\mathrm{Ar}}(f), E) = i(\mathrm{div}(f), E) + i(\mathrm{div}_\infty(f), E) = i(\mathrm{div}(f), E) + \sum_{k=1}^{n}(\int_{S_k} -\log|f_k|\mu_k)\deg_K(E).$$

Note we have

$$i(\mathrm{div}(f), E) = \sum_{\mathfrak{p}} i_{\mathfrak{p}}(\mathrm{div}(f), E) + \sum_{k=1}^{n} i_k(\mathrm{div}(f), E)$$

Now E splits into $\deg_K(E)$ points $\{Q_j^k\}$ after using the kth embedding $K \hookrightarrow \mathbf{C}$. Therefore,

$$i_k(\mathrm{div}(f), E) = \sum_j g_{\mathrm{div}(f)}(Q_j) = \sum_j (\int_{S_k} -\log|f_k|\mu_k)\deg_K(Q_j) - \log|f_k(Q_j)|.$$

which is equal to

$$\int_{S_k} -\log|f_k|\mu_k) \deg_K(E) - \sum_j \log|f_k(Q_j)|$$

Combining, we get

$$i(\mathrm{div}_{\mathrm{Ar}}(f), E) = \sum_{\mathfrak{p}} i_{\mathfrak{p}}(\mathrm{div}(f), E) + \sum_{k=1}^{n} \sum_j -\log|f_k(Q_j)|$$

which is 0 by the product formula. □

Our question for the moment, concerns the intersection pairing on

$$\mathrm{Div}^0_{\mathrm{Ar}}(S) = \mathrm{Div}^0_{\mathrm{hor}}(S) \oplus \mathrm{Div}_{\mathrm{ver}}(S) \oplus \mathrm{Div}_{\infty}(S).$$

6.3.2. *Infinite Divisors.* Note that $(S_k, S_{k'}) = 0$, so

$$i(\mathrm{Div}_{\infty}(S), \mathrm{Div}_{\infty}(S)) = 0.$$

Furthermore, if D is a vertical divisor, then $i(S_k, D) = 0$, so

$$i(\mathrm{Div}_{\infty}(S), \mathrm{Div}_{\mathrm{ver}}(S)) = 0.$$

Lastly, if D is a prime horizontal divisor, then $i(S_k, D) = \deg_K(D)$, so if $\deg_K(E) = 0$ for some $E = \sum_i D_i \in \mathrm{Div}_{\mathrm{hor}}(S)$, then

$$i(S_k, E) = i(S_k, \sum_i D_i) = \sum_i i(S_k, D_i) = \sum_i \deg_K(D_i) = \deg_K(E) = 0.$$

Combining, we get

$$i(\mathrm{Div}_{\infty}(S), \mathrm{Div}^0_{\mathrm{Ar}}(S)) = 0.$$

Thus, as long as we restrict our attention to divisors with degree 0 finite component, we can safely ignore the infinite divisors.

6.3.3. *Divisor Decomposition.* Note $P_{\mathrm{Ar}}(S) \subseteq \mathrm{Div}^0_{\mathrm{Ar}}(S)$ because C is smooth and projective, so we can look at

$$\mathrm{Cl}^0_{\mathrm{Ar}}(S) = [\mathrm{Div}^0_{\mathrm{hor}}(S) \oplus \mathrm{Div}_{\mathrm{ver}}(S) \oplus \mathrm{Div}_{\infty}(S)]/P_{\mathrm{Ar}}(S).$$

One sees easily that $\mathrm{Div}_{\infty}(S) \cap P_{\mathrm{Ar}}(S) = (\mathcal{O}_K^*)_{\infty}$, where $(\mathcal{O}_K^*)_{\infty}$ is the image of the unit group in $\mathcal{O}_K$ under the map

$$\mathcal{O}_K \to \mathrm{Div}_{\infty}(S), \;\; a \mapsto -\sum (\log|a|_k)[S_k],$$

Recall that the intersection pairing i is trivial on $P_{\mathrm{Ar}}(S)$, so

$$\mathrm{Cl}^0_{\mathrm{Ar}}(S) = [\mathrm{Div}^0_{\mathrm{hor}}(S) \oplus \mathrm{Div}_{\mathrm{ver}}(S)]/P_{\mathrm{Ar}}(S) \oplus \mathrm{Div}_{\infty}(S)/(\mathcal{O}_K^*)_{\infty}.$$

Write $\mathrm{Cl}^0_{\mathrm{fin}}(S) = [\mathrm{Div}^0_{\mathrm{hor}}(S) \oplus \mathrm{Div}_{\mathrm{ver}}(S)]/P_{\mathrm{Ar}}(S)$ and $\mathrm{Cl}_{\infty}(S) = \mathrm{Div}_{\infty}(S)/(\mathcal{O}_K^*)_{\infty}$. Then, after tensoring with $\mathbf{Q}$, we get an orthogonal splitting

$$\mathbf{Q}\,\mathrm{Cl}^0_{\mathrm{Ar}}(S) = \mathbf{Q}\,\mathrm{Cl}^0_{\mathrm{fin}}(S) \perp \mathbf{Q}\,\mathrm{Cl}_{\infty}(S).$$

Thus, for the moment we will only consider the intersection pairing on $\mathbf{Q}\,\mathrm{Cl}^0_{\mathrm{fin}}(S)$.

6.3.4. *Vertical Divisors.* Next we will treat the case of vertical divisors. Observe that

$$\mathrm{Div}_{\mathrm{ver}}(S) = \bigoplus_{(0)\neq\mathfrak{p}\subseteq\mathcal{O}_K} \mathrm{Div}_{\mathfrak{p}}(S)$$

where $\mathrm{Div}_{\mathfrak{p}}(S)$ is the free abelian group on the vertical divisors over a closed point $[\mathfrak{p}] \in \operatorname{Spec} \mathcal{O}_K$. Each $\mathrm{Div}_{\mathfrak{p}}(S)$ is finitely generated, since our schemes are Noetherian. Furthermore, the inclusion $S_{\mathfrak{p}} \hookrightarrow S$ induces a map $\phi : \mathrm{Div}(S) \to \mathrm{Div}(S_{\mathfrak{p}}) \cong \mathrm{Div}_{\mathfrak{p}}(S)$, so we define $S_{\mathfrak{p}} = \phi(\pi^{-1}([\mathfrak{p}]))$, and define $\mathrm{Fib}_{\mathfrak{p}}(S)$ to be the subgroup of $\mathrm{Div}_{\mathfrak{p}}(S)$ generated by $S_{\mathfrak{p}}$.

6.3.5. **Proposition.** *The intersection pairing descends to* $\mathbf{Q}\,\mathrm{Div}_{\mathfrak{p}}(S)/\mathbf{Q}\,\mathrm{Fib}_{\mathfrak{p}}(S)$*, and is negative definite there.*

Proof. We follow [Lan88] for this proof. If $D \in \mathrm{Div}_{\mathfrak{p}}(S)$, then by definition of the intersection pairing,

$$i(D, S_{\mathfrak{p}}) = \sum_{\mathfrak{q}} i_{\mathfrak{q}}(D, S_{\mathfrak{p}}) + \sum_{k} \cancel{i_{S_k}(D, S_{\mathfrak{p}})} = i_{\mathfrak{p}}(C, D).$$

Thus, we can treat D and $S_{\mathfrak{p}}$ as divisors in $\pi^{-1}([\mathfrak{p}])$, the reduction at $\mathfrak{p}$, and compute the intersection number locally. But in $\pi^{-1}([\mathfrak{p}])$, $S_{\mathfrak{p}}$ is the principal divisor corresponding to the image of the uniformizer $t \in \mathcal{O}_K$ under the map

$$\mathcal{O}_K \to \Gamma(\mathcal{O}_{\pi^{-1}([\mathfrak{p}])}, \pi^{-1}([\mathfrak{p}])),$$

so $i_{\mathfrak{p}}(D, S_{\mathfrak{p}}) = 0$.

For negative definiteness, we let $S_{\mathfrak{p}} = \sum_i n_i D_i$, where D_i are vertical divisors over $\mathfrak{p}$ and $n_i > 0$. Then we define a matrix (a_{ij}) by

$$a_{ij} = i(n_i D_i, n_j D_j) = n_i n_j i_{\mathfrak{p}}(D_i, D_j).$$

(1) This matrix is clearly symmetric.

(2) If $i \neq j$, then D_i and D_j don't share common components, so by definition we have $a_{ij} \geq 0$.

(3) $\sum_j a_{ij} = 0$ since $i(D, S_{\mathfrak{p}}) = 0$.

Using these properties, one can show that for a vector $x_i \in \mathbf{Q}$,

$$\sum_{i,j} a_{ij} x_i x_j = -\sum_{i<j} a_{ij}(x_i - x_j)^2 \leq 0.$$

This shows negativity. Furthermore, one can show that there exists at least one nonzero x_i term since $\pi^{-1}([\mathfrak{p}])$ is connected. Thus, the form is nondegenerate. ∎

6.3.6. *Finer Decomposition.* Next, Hriljac shows that there exists an exact sequence

$$0 \to \mathbf{Q}\,\mathrm{Cl}^0(C) \to \mathbf{Q}\,\mathrm{Cl}^0_{\mathrm{fin}}(S) \to \bigoplus_{(0)\neq\mathfrak{p}\subseteq\mathcal{O}_K} \mathbf{Q}\,\mathrm{Div}_{\mathfrak{p}}(S)/\mathbf{Q}\,\mathrm{Fib}_{\mathfrak{p}}(S) \to 0$$

given by a map $I+\Phi : \mathbf{Q}\operatorname{Cl}^0(C) \to \mathbf{Q}\operatorname{Cl}^0_{\mathrm{fin}}(S)$ whose image is orthogonal to the last term in the sequence. Thus, there is an orthogonal splitting

$$\mathbf{Q}\operatorname{Cl}^0_{\mathrm{fin}}(S) = \operatorname{im}(I+\Phi) \bigoplus_{(0)\neq\mathfrak{p}\subseteq\mathcal{O}_K} \mathbf{Q}\operatorname{Div}_{\mathfrak{p}}(S)/\mathbf{Q}\operatorname{Fib}_{\mathfrak{p}}(S).$$

We will omit the definition of the Φ operator. Details can be found in [Hri85].

By Proposition 10.1.21 in [Liu06], there are only finitely many primes $\mathfrak{p}$ such that $\mathbf{Q}\operatorname{Div}_{\mathfrak{p}}(S) \neq \mathbf{Q}\operatorname{Fib}_{\mathfrak{p}}(S)$, so the sum on the right is finite, in which each summand is finitely generated as well. By the Mordell-Weil theorem, $\operatorname{im}(I+\Phi)$ is finitely generated as well.

This orthogonally splits the horizontal divisors and the vertical divisors. We have already treated the intersection pairing on each $\mathbf{Q}\operatorname{Div}_{\mathfrak{p}}(S)/\mathbf{Q}\operatorname{Fib}_{\mathfrak{p}}(S)$, so now we need to look at the intersection pairing on the horizontal part.

6.3.7. *Néron 's Work.* We will use the following result due to Néron :

6.3.7.1. **Theorem.** *The intersection pairing on the image* $\operatorname{im}(I+\Phi)$ *is negative definite.*

6.3.8. *Hodge Index Theorem.* Now we can give the full proof of the Hodge Index Theorem. In this version we will also need to disregard the divisors numerically equivalent to 0, so we define

$$N_{\mathrm{Ar}}(S) = \{D \in \operatorname{Cl}_{\mathrm{Ar}}(S) : i(D,E) = 0 \text{ for all } E \in \operatorname{Cl}_{\mathrm{Ar}}(S)\}$$

6.3.8.1. **Theorem.** *The intersection pairing on* $\mathbf{R}\operatorname{Cl}_{\mathrm{Ar}}(S)/\mathbf{R}N_{\mathrm{Ar}}(S)$ *is nondegenerate with signature* $(+,-,\dots,-)$.

Proof. We already showed the intersection pairing on $\mathbf{Q}\operatorname{Cl}^0_{\mathrm{fin}}(S)$ is negative definite, so the same holds after replacing $\mathbf{Q}$ with $\mathbf{R}$. It remains to consider Arakelov divisors with nonzero degrees.

We can pick a class $c \in \operatorname{Cl}_{\mathrm{hor}}(S)$ which generates the nonzero degree horizontal divisor classes, and an element $e \in \operatorname{Cl}_\infty(S)$ which generates the nonzero degree infinite divisors. Then

$$\mathbf{R}\operatorname{Cl}_{\mathrm{Ar}}(S)/\mathbf{R}N_{\mathrm{Ar}}(S) = \mathbf{R}c \oplus \mathbf{R}e \oplus \mathbf{R}\operatorname{Cl}^0_{\mathrm{fin}}(S).$$

Note intersection with c defines a linear map $\mathbf{R}\operatorname{Cl}^0_{\mathrm{fin}}(S) \to \mathbf{R}$, but by nondegeneracy of the intersection form, there exists some $u \in \mathbf{R}\operatorname{Cl}^0_{\mathrm{fin}}(S)$ such that intersection with c is the same as intersection with u, so we replace c with $c-u$ to assure that

$$\mathbf{R}c \perp \mathbf{R}\operatorname{Cl}_{\mathrm{fin}}(S).$$

We already know that e is orthogonal to $\mathbf{R}\operatorname{Cl}_{\mathrm{fin}}(S)$, so it remains to orthogonalize c and e.

Note $i(c,e) \neq 0$, because $\deg_K(c) > 0$, and e is an infinite divisor. By scaling, we can assume $i(c,e) = 1$. We already know that $i(e,e) = 0$. Thus, we define

$$e_1 = c + \frac{1-i(c,c)}{2}e \qquad\qquad e_2 = c - \frac{1+i(c,c)}{2}e.$$

Then

$$i(e_1, e_1) = i(c, c) + (1 - i(c, c))i(c, e) + (\dots)\cancel{i(e, e)} = 1,$$

and

$$i(e_2, e_2) = i(c, c) - (1 + i(c, c))i(c, e) + (\dots)\cancel{i(e, e)} = -1,$$

This completes the proof. □

7. Arithmetic Intersection Theory

The generalization of Arakelov's work to higher dimensions was not immediately clear when he first defined the notion of intersection on an arithmetic surface. Given a higher dimensional $X \to \operatorname{Spec} \mathcal{O}_K$ (more precise details to follow), we can still form the pullbacks over the infinite places, but now we get complex manifolds of higher dimension. In the case of arithmetic surfaces, all of the intersection theory was contained in the Arakelov divisor group, and we just intersected curves to get intersection numbers. Furthermore, since the surfaces $S \to \operatorname{Spec} \mathcal{O}_K$ had relative dimension 1, the resulting complex manifolds were actually compact Riemann surfaces, and we associated a Green function to each point (divisor) on the Riemann surface.

However, in higher dimensions, we need to instead consider Chow groups and their intersection theory, and instead of Green functions, we will need the theory of Green currents, which are a higher dimensional generalization of the Green functions. This will allow us to define arithmetic Chow groups, which are the main objects of study in arithmetic intersection theory. This theory was developed by Gillet and Soulé in [GS90] and [GS94], as well as numerous other auxiliary papers.

For simplicity, we restrict attention to $K = \mathbf{Q}$ and study schemes over $\operatorname{Spec} \mathbf{Z}$.

7.1. **Green Currents.** Here we perform the required complex analysis, and either give or cite the main proofs of the results we need.

Fix a smooth connected complex manifold X and denote $n := \dim_{\mathbf{C}} X$. We will assume X is oriented such that $dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$ is positively oriented.

7.1.1. *Complex Differential Forms.* Let $A^{p,q}(X)$ denote the space of complex (p, q)-forms on X. Any such $\omega \in A^{p,q}(X)$ can be written

$$\sum_{|I|=p, |J|=q} f_{I,J}(z, \bar{z}) dz_I \wedge d\bar{z}_J,$$

where each $f_{I,J}$ is a C^∞-function on X. Let

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X).$$

As usual, $\partial, \bar{\partial}$ and $d = \partial + \bar{\partial}$ will denote the differentials. In addition, we define an operator $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$ so that

$$dd^c = -\frac{1}{2\pi i} \partial\bar{\partial}.$$

7.1.2. *Currents.* Let $D^{p,q}(X)$ denote the space of *(p,q)-currents* on X. This means that if $T \in D^{p,q}(X)$, then T is a map

$$T : A^{n-p,n-q}(X) \to \mathbf{C}$$

such that for any sequence $\{\omega_k\}_{k\in\mathbf{N}}$ of differential forms in $A^{n-p,n-q}(X)$ such that the support of each ω_k is contained in a fixed compact set K, then $T(\omega_k) \to_k 0$ if $\omega_k \to_k 0$, meaning that the coefficients of the ω_k and all their derivatives tend uniformly to 0 as $k \to \infty$. Again, let

$$D^n(X) = \bigoplus_{p+q=n} D^{p,q}(X).$$

There are differential maps $\partial, \bar{\partial}, d$ defined on the currents as well. The definition for ∂ is

$$(\partial T)(\alpha) = T((-1)^{p+q+1}\partial\alpha),$$

where T is a (p,q)-current. The definition is similar for $\bar{\partial}$ and d.

7.1.3. *Current Induced by a Form.* There is a map $[\cdot] : A^{p,q}(X) \to D^{p,q}(X)$ defined in the following way. If $\omega \in A^{p,q}(X)$ and $\alpha \in A^{n-p,n-q}(X)$,

$$[\omega](\alpha) = \int_X \omega \wedge \alpha.$$

This is clearly a linear functional, and one can show that it in fact defines a current.

We will need the fact that the differentials for forms and currents are compatible with one another. For this, we use Stokes' theorem, and the definition of the differential map. We do the case for the differential d. If $\omega \in A^{p,q}(X)$ with $p + q = r$, then

$$\begin{aligned}
[d\omega](\alpha) &= \int_X d\omega \wedge \alpha \\
&= \int_X d(\omega \wedge \alpha) + (-1)^{r+1} \int_X \omega \wedge d\alpha \\
&= \int_{\partial X = \varnothing} \omega \wedge \alpha + [\omega]((-1)^{r+1} d\alpha) \\
&= (d[\omega])(\alpha).
\end{aligned}$$

This is true for ∂ and $\bar{\partial}$ as well: just pay attention to the invocation of Stokes' theorem.

7.1.4. *Currents and Submanifolds.* If $i : Y \hookrightarrow X$ is a closed analytic submanifold of codimension p, then we define an associated current in $D^{p,p}(X)$ such that for $\alpha \in A^{n-p,n-p}(X)$,

$$\delta_Y(\alpha) = \int_{Y^{ns}} i^*\alpha.$$

One can show that this is a current.

7.1.5. *Green Currents.* Fix a codimension p analytic subspace $i : Y \hookrightarrow X$. Then a *Green current* for Y in X is a current $g_Y \in D^{p-1,p-1}(X)$ satisfying the equation

$$dd^c g_Y + \delta_Y = [\omega]$$

where $\omega \in A^{p,p}(X)$.

7.1.6. *Remark.* On a Riemann surface X at a point P, we defined a Green function

$$g_P : X \setminus \{P\} \to \mathbf{R},$$

which was C^∞. Let μ be the volume form on X. Then $\delta_P : C^\infty(X) \to \mathbf{R}$ is defined by $\delta_P(f) = f(P)$, which matches the definition given in 7.1.4.

7.1.7. **Lemma.** *With the notation above,*

$$dd^c[g_P] + \delta_P = [\mu].$$

Proof. For $f \in C^\infty(X)$, the statement translates to

$$f(P) + \int_X g_P \wedge d^c df = \int_X f \wedge \mu.$$

Using (G4) and $dd^c + d^c d = 0$, we see that

$$g_P \wedge d^c df - f \wedge \mu = -g_P \wedge dd^c f - f \wedge dd^c g_P.$$

But

$$\begin{aligned} -d(g_P \wedge d^c f - f \wedge d^c g_P) &= -(\cancel{dg_P \wedge d^c f} + g_P \wedge dd^c f + \cancel{df \wedge d^c g_P} + f \wedge dd^c g_P) \\ &= -g_P \wedge dd^c f - f \wedge dd^c g_P. \end{aligned}$$

Stokes' theorem shows that the integral on X is 0 away from a small disk around P, so we are taking this integral on a small disk around P. But since dd^c is the Laplacian operator, this essentially reduces to the proof of Proposition 6.2.3.2, so the value of the integral becomes $f(P)$. □

This highlights the generalization we are making. Notice that in our formulation, μ is an arbitrary form, so in some sense we have slightly more flexibility in the theory as to the choice of a Green current.

7.1.8. *First Chern Form.* But there is one differential form that is important for intersection theory, and can help us define forms of logarithmic type for divisors. This is the first Chern form of a Hermitian line bundles. Although we haven't talked about putting Hermitian metrics on line bundles, they play a crucial role in a lot of modern Arakelov theory. For example, in [Ara74], Arakelov shows that Arakelov divisors are the same as Hermitian line bundles subject to hxarmonicity and normalization conditions.

7.1.8.1. **Proposition.** *If $(L, \|\cdot\|)$ is a Hermitian line bundle on X, then there exists a smooth form*

$$c_1(L, \|\cdot\|) \in A^{1,1}(X)$$

such that if $U \subseteq X$ is open and $s \in \mathscr{L}(U)$ is nonvanishing on U, then we have

$$c_1(L, \|\cdot\|) = -dd^c \log \|s\|^2,$$

Proof. We can actually use this expression to *define* $c_1(L, \|\cdot\|)$: it suffices to show that this definition does not depend on the choice of s, and agrees on the overlaps of two open sets.

Since L is one dimensional, if we choose another $s' \in \mathscr{L}(U)$ nonvanishing on U, then there exists a nonzero holomorphic function f on U such that $s' = fs$. But then

$$-dd^c \log \|s'\|^2 = -dd^c \log \|fs\|^2 = -dd^c \log \|f\|^2 - dd^c \log \|s\|^2,$$

and by holomorphicity and the Leibniz rule we have

$$\partial\bar{\partial} \log \|f\|^2 = 2\partial\bar{\partial} \log(f\bar{f}) = 2\partial \left[\frac{\bar{\partial}(f\bar{f})}{f\bar{f}}\right] = 2\partial \left[\frac{f\cancel{\bar{\partial}(\bar{f})}}{f\bar{f}} + \frac{\bar{\partial}(f)\bar{f}}{f\bar{f}}\right] = \frac{-2\bar{\partial}\partial(f)}{f} = 0.$$

This also proves that this definition agrees on the overlaps: if s, s' are nonzero on $\mathscr{L}(U)$ and $\mathscr{L}(U')$, then their restrictions to $U \cap U'$ differ by a nonzero holomorphic function. □

7.2. **Uniqueness.**

7.2.1. **Proposition.** *If X is a Kähler manifold, and $Y \subseteq X$ is an analytic submanifold with Green currents g_1, g_2, then*

$$g_1 - g_2 = [\eta] + \partial S_1 + \bar{\partial} S_2,$$

where $\eta \in A^{p-1,p-1}(X)$, $S_1 \in D^{p-2,p-1}(X)$, and $S_2 \in D^{p-1,p-2}(X)$.

Proof. [Sou95] This is an application of the Hodge decomposition for a Kähler manifold. □

7.3. **Logarithmic Forms.** When we defined Green functions for a divisor D in 6.2.1, we stated in axiom 3 that we wanted the currents to have logarithmic asymptotic behavior at the support of the divisor D. It turns out that besides wanting to have a close generalization of Arakelov's original theory, Green currents with similar logarithmic properties will also allow us to consistently define intersections in our arithmetic Chow rings. Thus, we will define a notion of logarithmic asymptotic behavior for currents.

7.3.1. *Forms of Logarithmic Type.* If X is a smooth connected complex manifold and $i : Y \hookrightarrow X$ is a codimension p analytic subvariety with $p > 0$, then we say a smooth differential form α on $X \setminus Y$ is of *logarithmic type along Y* if there exists a projective morphism $\pi : \widetilde{X} \to X$ such that $\pi^{-1}(Y)$ is a normal crossings divisor (n.c.d), $\pi : \widetilde{X} \setminus E \to X \setminus Y$ is smooth, and $\pi_*(\beta) = \alpha$, where β is a C^∞ form on $\widetilde{X} \setminus E$ satisfying the following property. For every $x \in \widetilde{X}$, let $z_1 \dots z_k$ be a local equation cutting out E, since it is an n.c.d. Then there exist (locally) smooth forms $\alpha_1, \dots, \alpha_k, \gamma$ on $\widetilde{X} \setminus E$ such that

$$\beta = \sum_{i=1}^{k} \alpha_i \log |z_i|^2 + \gamma.$$

7.3.2. *Resolution of Singularities.* In order to prove existence of forms of logarithmic type, we will need to use Hironaka's theorem on the resolution of singularities. Our complex geometry works with smooth manifolds, so we will occasionally need to invoke this theorem. We state it now.

7.3.3. **Theorem** ([Hir64])**.** *Let X be a scheme of finite type of dimension n over $\mathbf{C}$, and $Y \subseteq X$ proper closed X such that $X \setminus Y$ is smooth. Then there exists a proper map*

$$\pi : \widetilde{X} \to X$$

such that

(1) $\widetilde{X}$ is smooth,

(2) $X \setminus \pi^{-1}(Y) \to X \setminus Z$ is an isomorphism,

(3) $E = \pi^{-1}(Y)$ is a divisor with normal crossings, i.e. is expressible in any local chart $U \cong \mathbf{C}^n$ as the divisor of the equation $z_1 \cdots z_n$.

7.3.4. **Theorem.** *If $i : Y \hookrightarrow X$ is an irreducible analytic submanifold, then there exists a smooth form g_Y of logarithmic type on $X \setminus Y$ such that $[g_Y]$ is a Green current for Y.*

Proof. We will prove the theorem in the case that Y has codimension $p = 1$. The rest of the proof is fairly technical, and can be found in [Sou95] Section II.2.2.

For the case of a divisor ($p = 1$), this is an instance of the Poincaré-Lelong formula, which we prove now. Given a divisor Y, there exists a line bundle L on X, a Hermitian metric $\|\cdot\|$ on L, and a section s such that $|\operatorname{div} s| = Y$, where $\operatorname{div} s$ is the divisor cut out by the section s. Then we show that $g_Y = -\log \|s(\cdot)\|^2$ is the desired form.

We will show that $-\log \|s\|^2$, which is defined on $X \setminus \operatorname{div}(s)$ is a Green current for Y as follows:

$$dd^c[-\log \|s\|^2] + \delta_Y = [c_1(L, \|\cdot\|)],$$

Using 7.3.3, we can take a resolution $\pi : \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth, $\pi^{-1}(Y)$ is a normal crossings divisor. Actually, we can prove the lemma here for $\pi^{-1}(Y)$ and then use the pushforward (integration along the fibers) of forms to prove the original theorem.

So suppose Y is a n.c.d. Using a partition of unity, we will prove the identity above in a local chart $\mathbf{C}^n$, on which Y is defined by $z_1 \cdots z_n$. One shows that the desired identity is additive in this expression, so we reduce to the case where Y is cut out by z_1. Then by rearranging the expression, and noting that the leftmost term and the rightmost term differ by a sign, it suffices to show that for any ω compactly supported in $\mathbf{C}^n$, we have

$$-\int_{\mathbf{C}^n} \log |z_1|^2 dd^c(\omega) = \int_{z_1=0} \omega.$$

The integrand in the right ahnd side is undefined on $z_1 = 0$, but is fully defined if we write

$$\int_{\mathbf{C}^n} \log |z_1|^2 dd^c(\omega) = \lim_{\epsilon \to 0} \int_{|z_1| \geq \epsilon} \log |z_1|^2 dd^c(\omega).$$

Using the Leibniz rule and Stokes' theorem, this becomes

$$\lim_{\epsilon\to 0}\int_{|z_1|=\epsilon} \log|z_1|^2 d^c(\omega) + \lim_{\epsilon\to 0}\int_{|z_1|\geq\epsilon} d\log|z_1|^2 d^c(\omega).$$

But since ω is compactly supported, we can convert $d^c(\omega)$ it to polar coordinates and bound its coefficients by some large constant, so the first integral reduces to the form

$$\lim_{\epsilon\to 0} 2\log\epsilon \int_0^2 \pi\epsilon d\theta = 0.$$

Using the Lebniz rule and Stokes' theorem again on the second integral, we get

$$\lim_{\epsilon\to 0}\int_{|z_1|\geq\epsilon} d^c\log|z_1|^2 d(\omega) = \lim_{\epsilon\to 0}\int_{|z_1|\geq\epsilon} d^c\log|z_1|^2\omega - \lim_{\epsilon\to 0}\int_{|z_1|\geq\epsilon} dd^c\log|z_1|^2(\omega).$$

The second integral vanishes because $dd^c\log|z_1|^2 = 0$ on $|z_1|\geq\epsilon$. After switching to polar coordinates, one can show that the first integral is in fact $\int_{z_1=0}\omega$. □

7.3.5. *Star Product.* We briefly mention, without giving any detail (see [Sou95] Section II.3 for proofs), how to define a notion of intersection for Green forms.

Fix $Y, Z \subseteq X$ closed and irreducible that intersect properly. If g_Y is a smooth form of logarithmic type such that $[g_Y]$ is a Green current for Y, and g_Z is any Green current for Z, then we can define a *star product* $[g_Y]\star g_Z$ which is a Green current for the divisor induced by $Y\cap Z$. Let ω_Y, ω_Z be the forms corresponding to $[g_Y]$ and g_Z. If we pretend for a moment that these currents are actually forms, then the star product can be thought of as the following operation:

$$g_Y\star g_Z = g_Y\wedge\delta_Z + \omega_Y\wedge g_Z,$$

although the real definition is a bit more technically involved. One can show that the star product is associative and commutative.

Suppose we have $Y\subseteq X$ and a Green current g_Y for Y. By 7.3.4 we can always find a smooth form $\widetilde{g}_Y$ of logarithmic type, whose corresponding current differs from g_Y by a current of the form

$$g_Y - [\widetilde{g}_Y] = [\eta] + \partial S_1 + \bar{\partial} S_2$$

by 7.2.1. But then notice that $dd^c[\eta] = 0$, so addition by $[\eta]$ preserves Green currents. Thus, if we work mod ∂ and $\bar{\partial}$, all Green currents can be assumed to be of logarithmic type, so we can always define a star product of Green currents.

7.4. **Arithmetic Chow Groups.** We can now define the arithmetic Chow groups for an arithmetic variety over $\operatorname{Spec}\mathbf{Z}$.

7.4.1. *Arithmetic Variety.* For this section, we will consider regular projective flat schemes $X\to\operatorname{Spec}\mathbf{Z}$. This will be the higher dimensional analogue of an arithmetic surface that we will treat. We form the pullback

$$\begin{tikzcd}
X_{\mathbf{C}} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbf{C} \arrow[d] \\
X \arrow[r] & \operatorname{Spec} \mathbf{Z},
\end{tikzcd}$$

and using GAGA we get a smooth projective complex manifold, which by abuse of notation we will also call $X_{\mathbf{C}}$, which is Kähler by projectivity. The complex conjugation map $\iota : \mathbf{C} \to \mathbf{C}$ induces the isomorphism $F_\infty : X_{\mathbf{C}} \to X_{\mathbf{C}}$:

$$\begin{tikzcd}
X_{\mathbf{C}} \arrow[r] \arrow[dr, dotted, "F_\infty"] \arrow[ddr, bend right] & \operatorname{Spec} \mathbf{C} \arrow[dr, bend left, "\operatorname{Spec} \iota"] & \\
& X_{\mathbf{C}} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbf{C} \arrow[d] \\
& X \arrow[r] & \operatorname{Spec} \mathbf{Z},
\end{tikzcd}$$

We make some slight restrictions on the complex analysis of $X_{\mathbf{C}}$ as follows. We define

$$A^{p,p}(X) = \{\omega \in A^{p,p}(X_{\mathbf{C}}) : \omega \text{ is real and } F_\infty^* \omega = (-1)^p \omega\}$$

and

$$D^{p,p}(X) = \{T \in D^{p,p}(X_{\mathbf{C}}) : T \text{ is real and } F_\infty^* T = (-1)^p T\}$$

for $p = 1, \ldots, n = \dim X$. There is an inclusion $[\cdot] : A^{p,p}(X) \hookrightarrow D^{p,p}(X)$ as defined in 7.1.3.

7.4.2. *Arithmetic Cycles.* Most of the theory of Chow groups that we developed earlier can be developed over $\operatorname{Spec} \mathbf{Z}$, even though our treatment was over an algebraically closed field. This treatment involves advanced algebraic K-theory and we will not treat it here, but we will use Chow groups for X in our formulation of the arithmetic Chow groups.

Let $Z^p(X)$ denote the set of codimension p cycles on X. If $Z = \sum_i Z_i$ is a sum of irreducible cycles, then we can define the associated current

$$\delta_Z = \sum_i \delta_{Z_i}.$$

Then a Green current for a cycle is an element $g_Z \in D^{p-1,p-1}(X)$ for which there is a form $\omega_Z \in A^{p,p}(X)$ satisfying

$$dd^c g_Z + \delta_Z = [\omega_Z].$$

Note Green currents are additive: if $g_Z, g_{Z'}$ are green currents for Z, Z', then $g_Z + g_Z'$ is a Green current for $Z + Z'$.

We define the group of *arithmetic p-cycles* to be

$$\widehat{Z}^p(X) = \{(Z, g_Z) : Z \in Z^p(X) \text{ and } g_Z \text{ a Green current for } Z\}.$$

7.4.3. *Rational Cycles.* If $Y \subseteq X$ is a closed integral subscheme, then for any nonzero $f \in K(Y)$, we get a cycle $\operatorname{div} f$, which we mod out by in the case of classical Chow groups. But in the arithmetic case, we give it a Green current as follows.

If $f \in K(Y)$, then f induces a rational function f_∞ on $Y_{\mathbf{C}} \subseteq X_{\mathbf{C}}$. However, we developed the complex analysis on smooth manifolds, so we take a resolution of singularities $\widetilde{Y} \to Y_{\mathbf{C}}$, and note that f_∞ restricts to a rational function $\widetilde{f}_\infty$ on $\widetilde{Y}$. Note $\log|\widetilde{f}_\infty|^2$ is an integrable function on $\widetilde{Y}$, so it defines a current $[\log|\widetilde{f}_\infty|^2]$ on $D^{0,0}(\widetilde{Y})$. Letting $\widetilde{i}$ denote the map $Y \to X_{\mathbf{C}}$, we get a current in $D^{p-1,p-1}(X_{\mathbf{C}})$ by pushing forward (using integration along the fibers) $[\log|\widetilde{f}_\infty|^2]$ under $\widetilde{i}$. The resulting current is denoted $[\log|f|^2]$, and is defined by

$$[\log|f|^2](\omega) = \int_{\widetilde{Y}} \log|\widetilde{f}_\infty|^2 \cdot \widetilde{i}^*(\omega)$$

One can show that $-[\log|f|^2]$ is a Green current for $\operatorname{div} f$. This motivates the following definition: the group of *rational arithmetic p-cycles* $\widehat{R}^p(X)$ is the subgroup of $\widehat{Z}^p(X)$ generated by the divisors of the form $(\operatorname{div} f, -[\log|f|^2])$ for all $f \in K(Y)$ for all closed integral subvarieties $Y \subseteq X$, and by elements of the form $(0, \partial(u) + \bar{\partial}(v))$ whee $u \in D^{p-2,p-1}(X)$ and $v \in D^{p-1,p-2}$. The latter such element is ignored because logarithmic forms will be defined up to addition of elements of that form. Then the pth arithmetic Chow group is defined to be

$$\widehat{\mathrm{CH}}^p(X) = \widehat{Z}^p(X)/\widehat{R}^p(X).$$

7.5. **Intersection Pairing.** One can show, using algebraic K-theory, that there is an intersection pairing

$$\mathbf{R}\,\mathrm{CH}^p(X) \to \mathbf{R}\,\mathrm{CH}^q(X) \to \mathbf{R}\,\mathrm{CH}^{p+q}(X),$$

where the $\mathbf{R}$ indicates that we tensor with $\mathbf{R}$. But in fact, there is a pairing

$$\mathbf{R}\widehat{\mathrm{CH}}^p(X) \to \mathbf{R}\widehat{\mathrm{CH}}^q(X) \to \mathbf{R}\widehat{\mathrm{CH}}^{p+q}(X).$$

In the case of proper intersection, this will be exactly

$$i([(Y, g_Y)], [(Z, g_Z)]) = [(i(Y, Z), g_Y \star g_Z)].$$

With this, we can state the standard conjectures.

7.6. **Arithmetic Standard Conjectures.** The first conjecture concerns degeneracy of the intersection pairing. If X is an arithmetic variety of relative dimension n over $\operatorname{Spec}\mathbf{Z}$, then there is a natural notion of arithmetic degree, which turns out to be a map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^{n+1}(X)_{\mathbf{R}} \to \mathbf{R}, \quad (Z, g) \mapsto \log(\#\mathscr{O}_Z(Z)) + \frac{1}{2}\int_{X_{\mathbf{C}}} g.$$

Using the intersection pairing, we get an intersection product

$$\widehat{\mathrm{CH}}^p(X) \times \widehat{\mathrm{CH}}^{d+1-p}(X) \to \widehat{\mathrm{CH}}^{d+1}(X) \to \mathbf{R}$$

7.6.1. **Conjecture.** *The intersection product is nondegenerate.*

The second conjecture is a version of the Lefschetz conjecture. Given a line bundle H on X with a smooth Hermitian metric, we can define an arithmetic cycle $\widehat{c}_1(H) \in \widehat{\mathrm{CH}}^1(X)_{\mathbf{R}}$ by taking a nonzero rational section s of H and defining

$$\widehat{c}_1(H) = [(\operatorname{div} s, -\log \|s\|^2)].$$

Then denote by L intersection with $\widehat{c}_1(H)$. Then the second conjecture is as follows.

7.6.2. **Conjecture.** *If H is ample on X, one can choose the smooth Hermitian metric so that*

(1) The map

$$L^{d+1-2p} : \widehat{\mathrm{CH}}^p(X)_{\mathbf{R}} \to \widehat{\mathrm{CH}}^{d+1-p}$$

is an isomorphism.

(2) If $x \in \widehat{\mathrm{CH}}(X)_{\mathbf{R}}$ is nonzero and $L^{d+2-2p}(x) = 0$, then

$$(-1)^p \widehat{\deg}(x \cdot L^{d+1-2p}(x)) > 0.$$

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