Banach L-representations

Talk 3 - Local p-adic Langlands program

May 2020

Banach L-representations

Talk 3 - Local p-adic Langlands program

May 2020

Sample frame title

In this talk, we will be discussing L-Banach space representations of p-adic analytic groups when p is invertible.

L-Banach space representation

We assume for the rest of this talk that G be a p-adic analytic group, L a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, and residue field k.

L-Banach space representation

We assume for the rest of this talk that G be a p-adic analytic group, L a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, and residue field k.

Definition

1. An L-Banach space representation Π of G is an L-Banach space Π with an action of G by continous linear automorphisms such that the action is continous.

L-Banach space representation

We assume for the rest of this talk that G be a p-adic analytic group, L a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, and residue field k.

Definition

1. An L-Banach space representation Π of G is an L-Banach space Π with an action of G by continous linear automorphisms such that the action is continous.
2. A Banach space is called unitary if there exists a G-invariant norm defining the topology on Π.

L-Banach space representation

We assume for the rest of this talk that G be a p-adic analytic group, L a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, and residue field k.

Definition

1. An L-Banach space representation Π of G is an L-Banach space Π with an action of G by continous linear automorphisms such that the action is continous.
2. A Banach space is called unitary if there exists a G-invariant norm defining the topology on Π.

The existence of such a norm is equivalent to the existence of an open bounded G-invariant $\mathcal{O}$-lattice Θ in Π.

L-Banach space representation

We assume for the rest of this talk that G be a p-adic analytic group, L a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, and residue field k.

Definition

1. An L-Banach space representation Π of G is an L-Banach space Π with an action of G by continous linear automorphisms such that the action is continous.
2. A Banach space is called unitary if there exists a G-invariant norm defining the topology on Π.

The existence of such a norm is equivalent to the existence of an open bounded G-invariant $\mathcal{O}$-lattice Θ in Π.

Admissible representations

Definition

A unitary L-Banach space representation is admissible if $\Theta \otimes_{\mathcal{O}} k$ is an admissible smooth representation of G.

Admissible representations

Definition

A unitary L-Banach space representation is admissible if $\Theta \otimes_{\mathcal{O}} k$ is an admissible smooth representation of G.

This is equivalent to saying that for every open subgroup H of G, $(\Theta \otimes_{\mathcal{O}} k)^H$ is finite dimensional.

Admissible representations

Definition

A unitary L-Banach space representation is admissible if $\Theta \otimes_{\mathcal{O}} k$ is an admissible smooth representation of G.

This is equivalent to saying that for every open subgroup H of G, $(\Theta \otimes_{\mathcal{O}} k)^H$ is finite dimensional.

Theorem

Suppose G has an open pro-p subgroup P. A representation $\pi \in Rep_G$ is admissible if and only if π^P is finite dimensional.

Admissible representations

Definition

A unitary L-Banach space representation is admissible if $\Theta \otimes_{\mathcal{O}} k$ is an admissible smooth representation of G.

This is equivalent to saying that for every open subgroup H of G, $(\Theta \otimes_{\mathcal{O}} k)^H$ is finite dimensional.

Theorem

Suppose G has an open pro-p subgroup P. A representation $\pi \in Rep_G$ is admissible if and only if π^P is finite dimensional.

Hence, it is enough to check the admissibility condition for a single pro-p subgroup of G.

Admissible representations

Definition

A unitary L-Banach space representation is admissible if $\Theta \otimes_{\mathcal{O}} k$ is an admissible smooth representation of G.

This is equivalent to saying that for every open subgroup H of G, $(\Theta \otimes_{\mathcal{O}} k)^H$ is finite dimensional.

Theorem

Suppose G has an open pro-p subgroup P. A representation $\pi \in Rep_G$ is admissible if and only if π^P is finite dimensional.

Hence, it is enough to check the admissibility condition for a single pro-p subgroup of G.

Irreducible representations

The definition of admissibility does not depend on Θ. By [], it is equivalent to showing that $\Theta^d := Hom_{\mathcal{O}}(\Theta, \mathcal{O})$ is a finitely generated module over $\mathcal{O}[[H]]$, for any pro-p subgroup H of G.

Irreducible representations

The definition of admissibility does not depend on Θ. By [], it is equivalent to showing that $\Theta^d := Hom_{\mathcal{O}}(\Theta, \mathcal{O})$ is a finitely generated module over $\mathcal{O}[[H]]$, for any pro-p subgroup H of G.

Definition

1. An L-Banach space representation Π is irreducible if it does not contain a proper closed G-invariant subspace.

Irreducible representations

The definition of admissibility does not depend on Θ. By [], it is equivalent to showing that $\Theta^d := Hom_{\mathcal{O}}(\Theta, \mathcal{O})$ is a finitely generated module over $\mathcal{O}[[H]]$, for any pro-p subgroup H of G.

Definition

1. An L-Banach space representation Π is irreducible if it does not contain a proper closed G-invariant subspace.
2. An L-Banach space representation Π is absolutely irreducible if $\Pi \otimes_L L'$ is irreducible for every finite extension L' of L.

Irreducible representations

The definition of admissibility does not depend on Θ. By [], it is equivalent to showing that $\Theta^d := Hom_{\mathcal{O}}(\Theta, \mathcal{O})$ is a finitely generated module over $\mathcal{O}[[H]]$, for any pro-p subgroup H of G.

Definition

1. An L-Banach space representation Π is irreducible if it does not contain a proper closed G-invariant subspace.
2. An L-Banach space representation Π is absolutely irreducible if $\Pi \otimes_L L'$ is irreducible for every finite extension L' of L.

Lemma 4.1

Let Π be an absolutely irreducible and admissible unitary L-Banach space representation of G, and let $\phi \in End^{cont}_{L[G]}(\Pi)$. If the algebra $L[\phi]$ is finite dimensional over L, then $\phi \in L$.

Lemma 4.1

Let Π be an absolutely irreducible and admissible unitary L-Banach space representation of G, and let $\phi \in End_{L[G]}^{cont}(\Pi)$. If the algebra $L[\phi]$ is finite dimensional over L, then $\phi \in L$.

Lemma 4.2

Let Π be an irreducible admissible unitary L-Banach space representation of G. If $End_{L[G]}^{cont}(\Pi) = L$, then Π is absolutely irreducible.

Lemma 4.1

Let Π be an absolutely irreducible and admissible unitary L-Banach space representation of G, and let $\phi \in End^{cont}_{L[G]}(\Pi)$. If the algebra $L[\phi]$ is finite dimensional over L, then $\phi \in L$.

Lemma 4.2

Let Π be an irreducible admissible unitary L-Banach space representation of G. If $End^{cont}_{L[G]}(\Pi) = L$, then Π is absolutely irreducible.

Lemma 4.3

Let Π be a unitary L-Banach space representation of G, let Θ and Ξ be open bounded G-invariant lattices in Π, and let π be an irreducible smooth k-representation of G. Then π is a subquotient of $\Theta \otimes_{\mathcal{O}} k$ if and only if it is a subquotient of $\Xi \otimes_{\mathcal{O}} k$. Moreover, if $\Theta \otimes_{\mathcal{O}} k$ is a G-representation of finite length, then so is $\Xi \otimes_{\mathcal{O}} k$, and their semi simplifications are isomorphic.

Lemma 4.3

Let Π be a unitary L-Banach space representation of G, let Θ and Ξ be open bounded G-invariant lattices in Π, and let π be an irreducible smooth k-representation of G. Then π is a subquotient of $\Theta \otimes_{\mathcal{O}} k$ if and only if it is a subquotient of $\Xi \otimes_{\mathcal{O}} k$. Moreover, if $\Theta \otimes_{\mathcal{O}} k$ is a G-representation of finite length, then so is $\Xi \otimes_{\mathcal{O}} k$, and their semi simplifications are isomorphic.

For a unitary L-Banach space representation Π of G, with Θ an open bounded G-invariant lattice in Π, its Schikhof dual is denoted by

$$\Theta^d := Hom_{\mathcal{O}}(\Theta, \mathcal{O})$$

equipped with the topology of pointwise convergence. If $\Theta \otimes_{\mathcal{O}} k$ is a G-representation of finite length, then we denote its semi-simplification (which is independent of Θ, by the above Lemma) by

$$\bar{\Pi} := (\Theta \otimes_{\mathcal{O}} k)^{ss}$$

For a compact open subgroup H of G, let $Mod_G^{proaug}(\mathcal{O})$ denote the category of profinite $\mathcal{O}[[H]]$-modules with an action of $\mathcal{O}[G]$ such that the two actions are the same when restricted to $\mathcal{O}[H]$.

For a compact open subgroup H of G, let $Mod_G^{proaug}(\mathcal{O})$ denote the category of profinite $\mathcal{O}[[H]]$-modules with an action of $\mathcal{O}[G]$ such that the two actions are the same when restricted to $\mathcal{O}[H]$.

Lemma 4.4

Θ^d is an object of $Mod_G^{proaug}(\mathcal{O})$.

For a compact open subgroup H of G, let $Mod_G^{proaug}(\mathcal{O})$ denote the category of profinite $\mathcal{O}[[H]]$-modules with an action of $\mathcal{O}[G]$ such that the two actions are the same when restricted to $\mathcal{O}[H]$.

Lemma 4.4
Θ^d is an object of $Mod_G^{proaug}(\mathcal{O})$.

Lemma 4.5
Suppose Π is irreducible and admissible. Let $\phi : M \to \Theta^d$ be a non-zero morphism in $Mod_G^{proaug}(\mathcal{O})$. Then, there exists an open bounded G-invariant lattice Ξ in Π such that $\Xi^d = \phi(M)$.

Definitions

1. Let $Mod_G^{sm}(\mathcal{O})$ be the category of smooth representations of G on $\mathcal{O}$-torsion modules.

Definitions

1. Let $Mod_G^{sm}(\mathcal{O})$ be the category of smooth representations of G on $\mathcal{O}$-torsion modules.
2. Let $Mod_G^{lfin}(\mathcal{O})$ be the full subcategory of $Mod_G^{sm}(\mathcal{O})$ consisting of locally finite representations.

Definitions

1. Let $Mod_G^{sm}(\mathcal{O})$ be the category of smooth representations of G on $\mathcal{O}$-torsion modules.
2. Let $Mod_G^{lfin}(\mathcal{O})$ be the full subcategory of $Mod_G^{sm}(\mathcal{O})$ consisting of locally finite representations.
3. Let $Mod_G^{?}(\mathcal{O})$ be a full subcategory of $Mod_G^{lfin}(\mathcal{O})$ closed under subquotients and arbitrary direct sums in $Mod_G^{lfin}(\mathcal{O})$.

Definitions

1. Let $Mod_G^{sm}(\mathcal{O})$ be the category of smooth representations of G on $\mathcal{O}$-torsion modules.
2. Let $Mod_G^{lfin}(\mathcal{O})$ be the full subcategory of $Mod_G^{sm}(\mathcal{O})$ consisting of locally finite representations.
3. Let $Mod_G^{?}(\mathcal{O})$ be a full subcategory of $Mod_G^{lfin}(\mathcal{O})$ closed under subquotients and arbitrary direct sums in $Mod_G^{lfin}(\mathcal{O})$.
4. Let $\mathcal{C}(\mathcal{O})$ be a full subcategory of $Mod_G^{proaug}(\mathcal{O})$ that is equivalent to the dual of $Mod_G^{?}(\mathcal{O})$ via Pontryagin duality.

Definitions

1. Let $Mod_G^{sm}(\mathcal{O})$ be the category of smooth representations of G on $\mathcal{O}$-torsion modules.
2. Let $Mod_G^{lfin}(\mathcal{O})$ be the full subcategory of $Mod_G^{sm}(\mathcal{O})$ consisting of locally finite representations.
3. Let $Mod_G^{?}(\mathcal{O})$ be a full subcategory of $Mod_G^{lfin}(\mathcal{O})$ closed under subquotients and arbitrary direct sums in $Mod_G^{lfin}(\mathcal{O})$.
4. Let $\mathcal{C}(\mathcal{O})$ be a full subcategory of $Mod_G^{proaug}(\mathcal{O})$ that is equivalent to the dual of $Mod_G^{?}(\mathcal{O})$ via Pontryagin duality.

Note that there is an anti-equivalence of categories between $Mod_G^{sm}(\mathcal{O})$ and $Mod_G^{proaug}(\mathcal{O})$. Moreover, $Mod_G^{?}(\mathcal{O})$ has injective envelopes, thus $\mathcal{C}(\mathcal{O})$ has projective envelopes.

Lemma 4.6

For an admissible unitary L-Banach space representation Π of G, the following are equivalent:

(i) There exists an open bounded G-invariant lattice Θ in Π such that Θ^d is an object of $\mathcal{C}(\mathcal{O})$;

(ii) For every open bounded G-invariant lattice Θ in Π, Θ^d is an object of $\mathcal{C}(\mathcal{O})$.

Category of L-Banach representations

Definition

Let $Ban_G^{adm}(L)$ denote the category of admissible L-Banach space representations of G, with morphisms continous G-equivariant L-linear homomorphisms. Let $Ban_{\mathcal{C}(\mathcal{O})}^{adm}$ denote the full subcategory of $Ban_G^{adm}(L)$ with admissible L-Banach space representations of G satisfying the conditions of Lemma 4.6.

Category of L-Banach representations

Definition

Let $Ban_G^{adm}(L)$ denote the category of admissible L-Banach space representations of G, with morphisms continous G-equivariant L-linear homomorphisms. Let $Ban_{\mathcal{C}(\mathcal{O})}^{adm}$ denote the full subcategory of $Ban_G^{adm}(L)$ with admissible L-Banach space representations of G satisfying the conditions of Lemma 4.6.

Lemma 4.8

The subcategory $Ban_{\mathcal{C}(\mathcal{O})}^{adm}$ is closed under subquotients in $Ban_G^{adm}(L)$. Further, it is abelian.

Lemma 4.9

Let $\tilde{P}$ be a projective object in $\mathcal{C}(\mathcal{O})$, and let $\tilde{E} := End_{\mathcal{C}(\mathcal{O})}\tilde{P}$.

Lemma 4.9

Let $\tilde{P}$ be a projective object in $\mathcal{C}(\mathcal{O})$, and let $\tilde{E} := End_{\mathcal{C}(\mathcal{O})}\tilde{P}$. There is an exact functor:

$$Ban^{adm}_{\mathcal{C}(\mathcal{O})} \to RMod(\tilde{E}[1/p])$$

$$\Pi \mapsto m(\Pi) := Hom_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \otimes_{\mathcal{O}} L$$

Lemma 4.9

Let $\tilde{P}$ be a projective object in $\mathcal{C}(\mathcal{O})$, and let $\tilde{E} := End_{\mathcal{C}(\mathcal{O})}\tilde{P}$. There is an exact functor:

$$Ban^{adm}_{\mathcal{C}(\mathcal{O})} \to RMod(\tilde{E}[1/p])$$

$$\Pi \mapsto m(\Pi) := Hom_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \otimes_{\mathcal{O}} L$$

Corollary 4.10

For a projective object $\tilde{P}$ in $\mathcal{C}(\mathcal{O})$ and an object Π of $Ban^{adm}_{\mathcal{C}(\mathcal{O})}$, there exists a smallest closed G-invariant subspace Π_1 of Π such that $m(\Pi/\Pi_1)$ is zero.

The case $GL_2(\mathbb{Q}_p)$

Now, let us apply this to the particular case of $GL_2(\mathbb{Q}_p)$.

The case $GL_2(\mathbb{Q}_p)$

Now, let us apply this to the particular case of $GL_2(\mathbb{Q}_p)$.

Lemma 4.11

Let $G = GL_2(\mathbb{Q}_p)$, and $\zeta : Z \to \mathcal{O}^\times$ be a continous character of Z, the center of G. Suppose Π is an admissible L-Banach space representation of G with central character ζ, and let Θ be an open bounded G-invariant lattice in Π. Let $Ban^{adm}_{G,\zeta}(L)$ denote the category of admissible L-Banach space representations of G on which Z acts by the character ζ. Then,

The case $GL_2(\mathbb{Q}_p)$

Now, let us apply this to the particular case of $GL_2(\mathbb{Q}_p)$.

Lemma 4.11

Let $G = GL_2(\mathbb{Q}_p)$, and $\zeta : Z \to \mathcal{O}^\times$ be a continous character of Z, the center of G. Suppose Π is an admissible L-Banach space representation of G with central character ζ, and let Θ be an open bounded G-invariant lattice in Π. Let $Ban^{adm}_{G,\zeta}(L)$ denote the category of admissible L-Banach space representations of G on which Z acts by the character ζ. Then,
(i) Θ^d is an object of $\mathcal{C}(\mathcal{O})$;

The case $GL_2(\mathbb{Q}_p)$

Now, let us apply this to the particular case of $GL_2(\mathbb{Q}_p)$.

Lemma 4.11

Let $G = GL_2(\mathbb{Q}_p)$, and $\zeta : Z \to \mathcal{O}^\times$ be a continous character of Z, the center of G. Suppose Π is an admissible L-Banach space representation of G with central character ζ, and let Θ be an open bounded G-invariant lattice in Π. Let $Ban^{adm}_{G,\zeta}(L)$ denote the category of admissible L-Banach space representations of G on which Z acts by the character ζ. Then,

(i) Θ^d is an object of $\mathcal{C}(\mathcal{O})$;

(ii) $Ban^{adm}_{\mathcal{C}(\mathcal{O})} = Ban^{adm}_{G,\zeta}(L)$.

Projective envelopes

Let us now try to understand the endomorphism rings $\tilde{E}$ of the projective envelopes $\tilde{P}$ in $\mathcal{C}(\mathcal{O})$.

Lemma 4.13

Let $\tilde{P}$ be a projective envelope of an irreducible object S in $\mathcal{C}(\mathcal{O})$. Suppose $\pi := S^\vee$ is a smooth irreducible k-representation of G, Π is an object of $Ban^{adm}_{\mathcal{C}(\mathcal{O})}$, and Θ is an open bounded G-invariant lattice in Π. TFAE:

Projective envelopes

Let us now try to understand the endomorphism rings $\tilde{E}$ of the projective envelopes $\tilde{P}$ in $\mathcal{C}(\mathcal{O})$.

Lemma 4.13

Let $\tilde{P}$ be a projective envelope of an irreducible object S in $\mathcal{C}(\mathcal{O})$. Suppose $\pi := S^{\vee}$ is a smooth irreducible k-representation of G, Π is an object of $Ban^{adm}_{\mathcal{C}(\mathcal{O})}$, and Θ is an open bounded G-invariant lattice in Π. TFAE:

(i) π is a subquotient of $\Theta \otimes k$;

Projective envelopes

Let us now try to understand the endomorphism rings $\tilde{E}$ of the projective envelopes $\tilde{P}$ in $\mathcal{C}(\mathcal{O})$.

Lemma 4.13

Let $\tilde{P}$ be a projective envelope of an irreducible object S in $\mathcal{C}(\mathcal{O})$. Suppose $\pi := S^\vee$ is a smooth irreducible k-representation of G, Π is an object of $Ban^{adm}_{\mathcal{C}(\mathcal{O})}$, and Θ is an open bounded G-invariant lattice in Π. TFAE:

(i) π is a subquotient of $\Theta \otimes k$;

(ii) S is a subquotient of $\Theta^d \otimes k$;

Projective envelopes

Let us now try to understand the endomorphism rings $\tilde{E}$ of the projective envelopes $\tilde{P}$ in $\mathcal{C}(\mathcal{O})$.

Lemma 4.13

Let $\tilde{P}$ be a projective envelope of an irreducible object S in $\mathcal{C}(\mathcal{O})$. Suppose $\pi := S^{\vee}$ is a smooth irreducible k-representation of G, Π is an object of $Ban^{adm}_{\mathcal{C}(\mathcal{O})}$, and Θ is an open bounded G-invariant lattice in Π. TFAE:

(i) π is a subquotient of $\Theta \otimes k$;
(ii) S is a subquotient of $\Theta^d \otimes k$;
(iii) $Hom_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \Theta^d \otimes k) \neq 0$;

Projective envelopes

Let us now try to understand the endomorphism rings $\tilde{E}$ of the projective envelopes $\tilde{P}$ in $\mathcal{C}(\mathcal{O})$.

Lemma 4.13

Let $\tilde{P}$ be a projective envelope of an irreducible object S in $\mathcal{C}(\mathcal{O})$. Suppose $\pi := S^{\vee}$ is a smooth irreducible k-representation of G, Π is an object of $Ban^{adm}_{\mathcal{C}(\mathcal{O})}$, and Θ is an open bounded G-invariant lattice in Π. TFAE:

(i) π is a subquotient of $\Theta \otimes k$;

(ii) S is a subquotient of $\Theta^d \otimes k$;

(iii) $Hom_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \Theta^d \otimes k) \neq 0$;

(iv) $Hom_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \neq 0$

Lemma 4.14

Suppose $\tilde{P}$, S, π and Θ are as in Lemma 4.13. Then,
(i) If $Hom_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \neq 0$, then π is an admissible representation of G.

Lemma 4.14

Suppose $\tilde{P}$, S, π and Θ are as in Lemma 4.13. Then,
(i) If $Hom_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \neq 0$, then π is an admissible representation of G.
(ii) $End_{\mathcal{C}(\mathcal{O})}(S) \cong End_G(\pi)$ is a finite field extension of k.

Lemma 4.14

Suppose $\tilde{P}$, S, π and Θ are as in Lemma 4.13. Then,
(i) If $Hom_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \neq 0$, then π is an admissible representation of G.
(ii) $End_{\mathcal{C}(\mathcal{O})}(S) \cong End_G(\pi)$ is a finite field extension of k.

Lemma 4.15

Let $\tilde{P}$ be a projective envelope of an irreducible object S in $\mathcal{C}(\mathcal{O})$ with $d := dim_k(End_{\mathcal{C}(\mathcal{O})}(S))$ finite. Let M be in $\mathcal{C}(\mathcal{O})$, such that $M_k := M \otimes k$ is of finite length in $\mathcal{C}(\mathcal{O})$. Then, $Hom_{\mathcal{C}(\mathcal{O})}(\tilde{P}, M)$ is a free $\mathcal{O}$-module of rank equal to the multiplicity with which S occurs as a subquotient of M_k multiplied by d.

Let $S_1, ..., S_n$ be irreducible pairwise non-isomorphic objects of $\mathcal{C}(\mathcal{O})$ such that $End_{\mathcal{C}(\mathcal{O})}(S_i)$ is finite dimensional over k, for $1 \leq i \leq n$. Let $\tilde{P}$ be a projective envelope of $S := \bigoplus S_i$ and let $\tilde{E} := End_{\mathcal{C}(\mathcal{O})}(\tilde{P})$. The, $\tilde{E}/rad\tilde{E} \cong \prod End_{\mathcal{C}(\mathcal{O})}(S_i)$, where $rad(E)$ denotes the Jacobson radical of E. For $1 \leq i \leq n$, let $\pi_i := S_i^\vee$ be a smooth irreducible representation of G.

The End

Proposition 4.17

Let Π be an object in $Ban^{adm}_{\mathcal{C}(\mathcal{O})}$, and let Θ be an open bounded G-invariant lattice in Π. Then, $Hom_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \Theta^d)$ is a finitely generated module over $\tilde{E}$.