$G \to$ p-adic analytic group; $L/\mathbb{Q}_p \to$ fin., ring of int. $\mathcal{O}$, & res. field k.

Proof of L 4.1: Suppose $f \in L[X]$ is the minimal poly. of ϕ over L, & take L' to be the splitting field of f.

Thm: (Peter-Schneider) The functor

$$\mathrm{Mod}_{fg}(K[[G]]) \xrightarrow{\sim} \mathrm{Ban}_G^{adm}(K)$$
$$M \longmapsto M^d$$

is an anti-equivalence of categories.

Now, recall that admissibility of Π is equivalent to having Θ^d being a finitely gen. module over $\mathcal{O}[[H]]$, for any pro-p subgroup H of G.

Also, if M is a f.g. $L[[H]] := L \otimes \mathcal{O}[[H]]$-mod, then $M_{L'}$ is a f.g. $L'[[H]]$-mod.

$\Rightarrow \Pi_{L'}$ is adm., & $\Pi_{L'} \in \mathrm{Ban}_G^{adm}(L')$.

Π is abs. irr. $\Rightarrow \Pi_{L'}$ is irr.

Lemma: Any non-zero G-equi. cont. lin. map b/w two topologically irr. adm. K-Banach space reps. of G is an iso.

$\Rightarrow$ Any non-zero G-equi. cont. lin. map $\varphi : \Pi_{L'} \to \Pi_{L'}$ is an iso.

$\Rightarrow$ Can find $\lambda \in L'$ $\ni$ $f(\lambda) = 0$ and $\phi \otimes id - \lambda$ kills $\Pi_{L'}$.

$\mathrm{Gal}(L'/L) \curvearrowright \Pi_{L'}$ via $\sigma(v \otimes \mu) = v \otimes \sigma(\mu)$ $\forall \mu \in L'$.

Choose a non-zero $v \in \Pi$, then $\phi(v) \in \Pi \Rightarrow \sigma(\lambda)v = \lambda v$ $\forall \sigma \in G(L'/L)$

$\Rightarrow \lambda \in L$, $\phi = \lambda$.

H.P.

Proof of L 4.2: Suppose Π is not abs. irr. $\Rightarrow \exists$ L'/L fin. Gal. $\ni$ $\Pi_{L'}$ contains a proper closed G-inv. subspace Σ. We have,

$$\mathrm{End}^{cont}_{L'[G]}(\Pi_{L'}) \simeq \mathrm{Hom}^{cont}_{L[G]}(\Pi, \Pi_{L'}) \simeq \mathrm{End}^{cont}_{L[G]}(\Pi)_{L'} \simeq L'.$$

$\therefore$ It is sufficient to prove that $\mathrm{End}^{cont}_{L'[G]}(\Pi_{L'})$ contains a non-triv. idempotent. We do this by showing that $\Pi_{L'}$ is semisimple (but not irr.).

Consider $\tau : \bigoplus_{\sigma \in G(L'/L)} \sigma(\Sigma) \longrightarrow \Pi_{L'}$

This map is non-zero cont. G-equiv. L-linear.
$\Pi_{L'}$ is adm. $\Rightarrow$ Any descending chain of closed G-inv. subspaces becomes constant $\Rightarrow$ Can assume Σ is irr. and adm.
Σ is irr. and adm. $\Rightarrow$ $\sigma(\Sigma)$ is irr. & adm. $\forall \sigma \in G(L'/L)$.
Thus, τ is an isomorphism $\Rightarrow$ $\Pi_{L'}$ is semisimple.

H.P.

Proof of L4.3: Suppose $\pi \hookrightarrow J$ is an inj. envelope of π in $\mathrm{Mod}_G^{sm}(k)$. Since J is inj., $\mathrm{Hom}_G(*, J)$ is exact.
~~Staments~~ Statements of Lemma are equi. to:

$$\mathrm{Hom}_G(\Theta \otimes_{\mathcal{O}} k, J) \neq 0 \Leftrightarrow \mathrm{Hom}_G(\Xi \otimes_{\mathcal{O}} k, J) \neq 0 \quad **$$

& $\Theta \otimes_{\mathcal{O}} k$ is of fin. length $\Leftrightarrow$ $\Xi \otimes_{\mathcal{O}} k$ is of fin. length
$\Rightarrow$ $\dim \mathrm{Hom}_G(\Theta \otimes k, J) = \dim \mathrm{Hom}_G(\Xi \otimes k^{ss}, J)$.

Now, if π is a subquotient of some smooth rep. K, then $\mathrm{Hom}_G(K, J) \neq 0$.
Conv., for ~~some~~ any non-zero $\phi : K \longrightarrow J$, $\pi \subset \mathrm{im}\,\phi$, as $\pi \hookrightarrow J$ is essential.
If K is of fin. length, π occurs in K with mult. $\dim \mathrm{Hom}_G(K, J)$.
** By an analogous theorem of Serre for fin. grps, & the exactness of $\mathrm{Hom}(*, J)$.

H.P.

Proof of L4.4: There is a topological isomorphism:
Lemma (Paškūnas): Let $(E, \|\cdot\|)$ be an L-Banach space. Assume $\|E\| \subset |L|$. Let E^0 be the unit ball in E, and let $M := \mathrm{Hom}_A(E^0, A)$. Then there exists a canonical topo. iso.

$$M \otimes_A k \simeq (E^0 \otimes_A k)^\vee$$

So, we have, $\Theta^d \otimes_{\mathcal{O}} \mathcal{O}/\varpi^n \mathcal{O} \simeq (\Theta/\varpi^n \Theta)^\vee \quad \forall n \geq 1$
where ϖ is a uniformizer in $\mathcal{O}$.

Also, $\forall n \geq 1$, $\Theta/\varpi^n\Theta$ is a smooth rep. of G in $\mathrm{Mod}_G^{sm}(\mathcal{O})$,
hence $(\Theta/\varpi^n\Theta)^\vee$ is an obj. of $\mathrm{Mod}_G^{pro\,aug}(\mathcal{O})$.
$\Rightarrow$ $\Theta^d \simeq \varprojlim \Theta^d/\varpi^n\Theta^d \simeq \varprojlim (\Theta/\varpi^n\Theta)^\vee$
is an obj. of $\mathrm{Mod}_G^{pro\,aug}(\mathcal{O})$

H.P.

Proof of L4.6: Clearly, (ii) $\Rightarrow$ (i).
(i) $\Rightarrow$ (ii): Any two open bounded lattices are commensurable and $\mathfrak{C}(\mathcal{O})$ is closed under subquotients.

H.P.

Proof of L4.8: ~~Band~~ $\mathrm{Ban}_G^{adm}(L)$ being abelian follows from the Thm stated in proof of L4.1. Take Π to be an obj. of $\mathrm{Ban}^{adm}_{\mathfrak{C}(\mathcal{O})}$, & let Θ be an open bounded G-inv. lattice in Π. Then by L4.6, Θ^d is an obj. of $\mathfrak{C}(\mathcal{O})$, and since $\mathfrak{C}(\mathcal{O})$ is a full subcategory of $\mathrm{Mod}_G^{pro\,aug}(\mathcal{O})$ closed under subquotients, any subquotient of Θ^d in $\mathrm{Mod}_G^{pro\,aug}(\mathcal{O})$ lies in $\mathfrak{C}(\mathcal{O})$.
Dually, any subquotient of Π in $\mathrm{Ban}_G^{adm}(L)$ lies in $\mathrm{Ban}^{adm}_{\mathfrak{C}(\mathcal{O})}$.
$\therefore$ Since $\mathrm{Ban}_G^{adm}(L)$ is abelian, so is $\mathrm{Ban}^{adm}_{\mathfrak{C}(\mathcal{O})}$.

Proof of L4.9: Since any two open bounded lattices in Π are commensurable, the defn of $m(\Pi)$ ~~does~~ is ind. of choice of Θ.
Take an exact sequence $0 \to \Pi_1 \to \Pi_2 \to \Pi_3 \to 0$ in $\mathrm{Ban}^{adm}_{\mathfrak{C}(\mathcal{O})}$.
Let Θ be an open bounded G-inv. lattice in Π_2.
Π_i are adm. $\Rightarrow$ $\Pi_1 \cap \Theta$ ~~is~~ & $\mathrm{im}(\Theta)$ in Π_3 are open bounded G-inv. lattices. $\therefore$ We get an exact seq. $0 \to \Theta_1 \to \Theta_2 \to \Theta_3 \to 0$.
Dually, $0 \to \Theta_3^d \to \Theta_2^d \to \Theta_1^d \to 0$ in $\mathfrak{C}(\mathcal{O})$.
$\widetilde{P}$ is proj. in $\mathfrak{C}(\mathcal{O})$ $\Rightarrow$ Exact seq. of right $\widetilde{E}$-mod.
$0 \to \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \Theta_3^d) \to \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \Theta_2^d) \to \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \Theta_1^d) \to 0$
Remains exact on tensoring with L.

H.P.

~~Cor~~ Proof of C4.10: Π is adm. $\Rightarrow$ any descending chain of closed G-inv. subspaces becomes stationary. Follows from exactness of m.

H.P.

Proof of L4.11: Here, we take $\mathrm{Mod}^{?}_{G}(\mathcal{O}) = \mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})$.

An obj. M of $\mathrm{Mod}^{\mathrm{pro\,aug}}_{G}(\mathcal{O})$ is an obj. of $\mathfrak{C}(\mathcal{O}) \Leftrightarrow M = \varprojlim M_i$ where the lim. is over all quotients in $\mathrm{Mod}^{\mathrm{pro\,aug}}_{G}(\mathcal{O})$ of fin. length and Z acts on M via ζ^{-1}.

Π is adm. $\Rightarrow \theta/\varpi^n\theta$ is adm. smooth $\forall n \geq 1$.

Thm: (Emerton) Let V be an obj. of $\mathrm{Mod}^{\mathrm{sm}}_{G}(A)$. Then, TFAE: (for $G = GL_2\mathbb{Q}_p$)

(1). V is of fin. length, and is adm.

(2). V is fin. gen. as an $A[G]$-mod, and is adm.

(3). V is of finite length, and is Z-finite, ie. the quotient of $A[Z]$ by its annihilator is a finite A-alg.

$\Rightarrow$ Since Z acts on $\theta/\varpi^n\theta$ by a character ζ, any fin. gen. subrepresentation of $\theta/\varpi^n\theta$ is of finite length. Hence, $(\theta/\varpi^n\theta)^\vee$ is an obj. of $\mathfrak{C}(\mathcal{O})$, and then use L4.2 proof.

H.P.

Similar to L4.3 proof.

Proof of L4.13: (i) $\Leftrightarrow$ (ii) by L4.3 proof.

(iii) $\Rightarrow$ (ii): Since $\widetilde{P} \twoheadrightarrow S$ is essential.

(ii) $\Rightarrow$ (iii): $\mathfrak{C}(\mathcal{O})$ is closed under subquotients, and $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, *)$ is exact.

(iii) $\Rightarrow$ (iv):

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \theta^d) \simeq \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \varprojlim \theta^d/\varpi^n\theta^d) \simeq \varprojlim \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \theta^d/\varpi^n\theta^d)$$

Since $\widetilde{P}$ is proj, the transition maps are surj.

(iv) $\Rightarrow$ (iii): $\theta^d/\varpi\theta^d \simeq \varpi^n\theta^d/\varpi^{n+1}\theta^d$, as θ^d is $\mathcal{O}$-torsion free.

$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \theta^d/\varpi\theta^d) = 0 \Rightarrow \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \theta^d/\varpi^n\theta^d) = 0 \ \forall n \geq 1$.

H.P.

__Proof of L4.14:__ Let $m = \mathrm{Hom}_{C(O)}(\tilde{P}, \Theta^d)$ & let $M \subset \Theta^d$ be the image of $m \hat{\otimes}_{\tilde{E}} \tilde{P} \to \Theta^d$

($m \hat{\otimes}_E P := \varprojlim (m/m_i) \otimes_E (P/P_j)$)

If $M = 0$, then $m = 0$ is fin. gen.; $\therefore$ take $M \neq 0$.

__Lemma:__ $\mathrm{Hom}_{C(A)}(P, m \hat{\otimes}_E P) \simeq m$.

m_i, P_j = basis of open nbhds of 0 in m, P

Consider $m \hat{\otimes}_{\tilde{E}} \tilde{P} \twoheadrightarrow M \hookrightarrow \Theta^d$

Apply $\mathrm{Hom}_{C(O)}(\tilde{P}, *)$, and the Lemma, to get, $\mathrm{Hom}_{C(O)}(\tilde{P}, M) \simeq m$.

~~This is exactly~~

Now, Π is adm. $\Rightarrow \Theta^d$ is a fin. gen. $O[[H]]$-mod.

$\Rightarrow (\Theta^d)^\vee$ is adm. smooth.

Quotients of adm. reps. are adm. $\Rightarrow M^\vee$ is adm. smooth.

The G-socle of $M^\vee$ is a fin. direct sum of irr. reps., as every summand contributes to invariants by a pro-p subgrp of G.

$\therefore \mathrm{Hom}_G(\pi_i, M^\vee)$ is a fin. dim. k-v.s. $\forall 1 \leq i \leq n$.

Dually, $\mathrm{Hom}_{C(O)}(M, S_i)$ is a fin. dim. k-v.s., of dim. d_i.

M is a quotient of $m \hat{\otimes}_{\tilde{E}} \tilde{P}$

$\Rightarrow$ All the irr. summands appearing in its cosocle are iso. to S_i.

$\Rightarrow \mathrm{cosoc}\, M \simeq \bigoplus_{i=1}^{n} S_i^{\oplus n_i}$, $n_i = d_i / \dim(\mathrm{End}_{C(O)}(S_i))$

We choose a ~~surj~~ surj.: $a : \tilde{P}^{\oplus \ell} \twoheadrightarrow \mathrm{cosoc}\, M$ for some ℓ.

a factors through $b : \tilde{P}^{\oplus \ell} \twoheadrightarrow M$ since $\tilde{P}$ is proj. & the map $M \twoheadrightarrow \mathrm{cosoc}\, M$ is an essential epimorphism.

Apply $\mathrm{Hom}_{C(O)}(\tilde{P}, *)$ to $b \Rightarrow \tilde{E}^{\oplus \ell} \twoheadrightarrow m$.

H.P.

__Note:__ The socle of a module M is a dual notion to that of the radical.

$\mathrm{soc}(M) = \sum \{ N \mid N \text{ is a simple submodule of } M \}$

$= \bigcap \{ E \mid E \text{ is an essential submodule of } M \}$.