EXT GROUPS BETWEEN IRREDUCIBLE REPRESENTATIONS

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Contents

1. Introduction

Let $L/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, and residue field k. Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let

$$\mathrm{Mod}_G^{\mathrm{l.adm}}(k) = \left\{ \begin{array}{c} \text{category of locally admissible} \\ \text{smooth representations of } G \end{array} \right\}.$$

Definition 1.0.1. A block $\mathfrak{B}$ for $\mathrm{Mod}_G^{\mathrm{l.adm}}(k)$ is an equivalence class of irreducible representations, where $\pi \sim \tau$ iff there exists a sequence of irreps $\pi = \pi_0, \pi_1, \dots, \pi_r = \tau$ with any of the three holding

(1) $\pi_i \cong \pi_{i+1}$
(2) $\mathrm{Ext}^1(\pi_i, \pi_{i+1}) \neq 0$
(3) $\mathrm{Ext}^1(\pi_{i+1}, \pi_i) \neq 0$

for all $i = 0, \dots, r-1$.

By general theory (we will see this in Pol's talk), one can decompose the category $\mathrm{Mod}_G^{\mathrm{l.adm}}(k)$ as

$$\mathrm{Mod}_G^{\mathrm{l.adm}}(k) = \coprod_{\mathfrak{B}} \mathrm{Mod}_G^{\mathrm{l.adm}}(k)[\mathfrak{B}]$$

and each $\mathrm{Mod}_G^{\mathrm{l.adm}}(k)[\mathfrak{B}]$ is antiequivalent to a category of modules (over an appropriate ring). Reducing to studying one of the subcategories will allow us to say something about the image of the Montreal functor. So we need to understand the blocks $\mathfrak{B}$, or equivalently

- The isomorphism classes of irreducible smooth k-reps of G
- Extensions between irreps.

We saw the first bullet point in the last talk. Today we will discuss the second.

2. The main theorem

Let B denote the standard Borel in G (i.e. upper triangular matrices). Recall that irreps fall into four possibilities:

(1) A character of the form $\chi \circ \det$, where $\chi\colon \mathbb{Q}_p^\times \to \overline{\mathbb{F}}_p$ is smooth.
(2) (Special series) $\mathrm{Sp} \otimes (\chi \circ \det)$, where Sp is the Steinberg representation.
(3) (Principal series) Let $0 < r \leq p-1$ and $0 \neq \lambda \in \overline{\mathbb{F}}_p$, with $(r, \lambda) \neq (p-1, \pm 1)$. Then

$$\pi(r, \lambda, \chi) = \left(\mathrm{Ind}_B^G(\mu_{\lambda^{-1}} \boxtimes \mu_\lambda \omega^r)\right) \otimes (\chi \circ \det)$$

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where $\mu_\lambda(x) = \lambda^{\mathrm{val}(x)}$ and $\omega(x) = x|x|$ modulo p.[1]

(4) (Supersingular) For $0 \le r \le p-1$ they are denoted $\pi(r, 0, \chi)$.

Let π and τ be smooth irreps that admit central characters χ_π and χ_τ respectively. Since $\mathrm{Ext}^1_G(\tau, \pi) \neq 0$ implies that $\chi_\pi = \chi_\tau$, we may as well assume that $\chi_\pi = \chi_\tau = \zeta$. Furthermore, we have

$$\mathrm{Ext}^1(\tau \otimes \chi, \pi \otimes \chi) = \mathrm{Ext}^1(\tau, \pi)$$

so we can assume that π is of the form $1, \mathrm{Sp}, \pi(r, \lambda) := \pi(r, \lambda, 1)$ (here 1 denotes the trivial character).

Theorem 2.0.1 (Paškūnas, Colmez, Emerton)**.** *Let $p \ge 5$.*

(1) If π is supersingular, then $\mathrm{Ext}^1_G(\tau, \pi) \neq 0$ if and only if $\tau \cong \pi$.
(2) If $\pi \cong 1$, then $\mathrm{Ext}^1_G(\tau, \pi) \neq 0$ if and only if $\tau \cong 1, \mathrm{Sp}, \pi(p-3, 1, \omega)$ (the latter being in the principal series).
(3) If $\pi \cong \mathrm{Sp}$, then $\mathrm{Ext}^1(\tau, \pi) \neq 0$ if and only if $\tau \cong 1, \mathrm{Sp}$.
(4) If $\pi \cong \pi(r, \lambda)$ is principal series, then $\mathrm{Ext}^1_G(\tau, \pi) \neq 0$ if and only if

$$\tau \cong \begin{cases} \pi(r,\lambda) & \text{if } (r,\lambda) = (p-2, \pm 1) \\ \pi(r,\lambda), \mathrm{Sp} \otimes (\omega^{-1}\mu_{\pm 1} \circ \det) & \text{if } (r,\lambda) = (p-3, \pm 1) \\ \pi(r,\lambda), \pi(s, \lambda^{-1}, \omega^{r+1}) & \text{otherwise, where } \begin{array}{c} 0 \le s \le p-2 \\ s \equiv p-3-r \pmod{p-1} \end{array} \end{cases}$$

Remark. One can consider an involution on triples (r, λ, χ) given by

$$(r, \lambda, \chi) \mapsto (s, \lambda^{-1}, \chi \cdot \omega^{r+1})$$

where $0 \le s \le p-2$ with $s \equiv p-3-r$ modulo $p-1$. This essentially corresponds to switching the order of the characters in the induction and twisting by the "modulus character" $\omega \otimes \omega^{-1}$. Then each case in part (4) corresponds to a different behaviour of this involution: the first case corresponds to the fixed points of this involution, the second corresponds to the case when there is a degenerate triple in the orbit, and the last case is when both triples correspond to principal series representations.

Corollary 2.0.2. *Let $p \ge 5$. The category $\mathrm{Mod}^{\mathrm{l.adm}}_G(k)$ has the following blocks:*

- $\mathfrak{B} = \{\pi\}$, *π supersingular*
- $\mathfrak{B} = \{\mathrm{Ind}_B^G(\delta_1 \otimes \delta_2\omega^{-1}), \mathrm{Ind}_B^G(\delta_2 \otimes \delta_1\omega^{-1})\}$ *where $\delta_2\delta_1^{-1} \neq \omega^{\pm 1}, 1$*
- $\mathfrak{B} = \{\mathrm{Ind}_B^G(\delta \otimes \delta\omega^{-1})\}$
- $\mathfrak{B} = \{1, \mathrm{Sp}, \mathrm{Ind}_B^G(\omega \otimes \omega^{-1})\} \otimes (\delta \circ \det)$.

Remark. To be precise here we actually need to take k to be algebraically closed. In general, each block of $\mathrm{Mod}^{\mathrm{l.adm}}_G(k)$ contains one of the above after a finite extension of k.

3. Emerton's strategy

Let $T \subset B$ denote the standard torus. Emerton has defined an ordinary parts functor

$$\mathrm{Ord}_B \colon \mathrm{Mod}^{\mathrm{adm}}_G(k) \to \mathrm{Mod}^{\mathrm{adm}}_T(k)$$

which is right adjoint to $\chi_1 \otimes \chi_2 \mapsto \mathrm{Ind}_B^G(\chi_2 \otimes \chi_1)$.[2] In fact, he shows that you can derive this functor, and one has a derived adjunction formula:

$$R\,\mathrm{Hom}_G\left(\mathrm{Ind}_B^G(\psi^w), \pi\right) = R\,\mathrm{Hom}_T\left(\psi, R\,\mathrm{Ord}_B\,\pi\right).$$

This gives rise to the spectral sequence:

$$E_2^{i,j} \colon \mathrm{Ext}^i_T\left(\psi, R^j\,\mathrm{Ord}_B\,\pi\right) \Rightarrow \mathrm{Ext}^{i+j}_G\left(\mathrm{Ind}_B^G\,\psi^w, \pi\right)$$

and the 5-term exact sequence

$$0 \to \mathrm{Ext}^1_T(\psi, \mathrm{Ord}_B\,\pi) \to \mathrm{Ext}^1_G(\mathrm{Ind}_B^G\,\psi^w, \pi) \to \mathrm{Hom}_T(\psi, R^1\,\mathrm{Ord}_B\,\pi) \to \mathrm{Ext}^2_T(\psi, \mathrm{Ord}_B\,\pi) \to \cdots$$

[1] One can also define $\pi(0, \pm 1, \chi)$ and $\pi(p-1, \pm 1)$, but these are not irreducible. In fact (1) and (2) appear as subquotients of these representations.

[2] It is right adjoint to the usual functor Ind_B^G, hence the twist in this definition. We will write $(\chi_1 \otimes \chi_2)^w = \chi_2 \otimes \chi_1$.

Example 3.1. Let π be supersingular. Then one has $R^i \operatorname{Ord}_B \pi = 0$ for all i, because π is not a subquotient of any (non-trivial) parabolic induction. Therefore the above exact sequence says that

$$\operatorname{Ext}^1_G\left(\operatorname{Ind}_B^G \psi^w, \pi\right) = 0$$

for any smooth admissible representation ψ of T. This implies that

$$\operatorname{Ext}^1_G(\tau, \pi) = 0 \qquad \tau \text{ principal series}$$

and also $\operatorname{Ext}^1_G(\operatorname{Sp} \otimes (\chi \circ \det), \pi) = 0$ because we have the quotient

$$\operatorname{Ind}_B^G(\chi \otimes \chi) \to \operatorname{Sp} \otimes (\chi \circ \det).$$

Example 3.2. Take $\psi = \delta_1 \boxtimes \delta_2$ and $\xi = \chi_1 \boxtimes \chi_2$. Let $\pi = \operatorname{Ind}_B^G \xi^w$. Then Emerton [Eme10, Theorem 4.2.12] shows

- $\operatorname{Ord}_B \pi = \xi^w = \chi_2 \boxtimes \chi_1$
- $R^1 \operatorname{Ord}_B \pi = \chi_1 \omega^{-1} \boxtimes \chi_2 \omega$
- $R^i \operatorname{Ord}_B \pi = 0$ for $i \geq 2$.

We also note that $\operatorname{Hom}_T(\alpha, \beta) \neq 0$ iff $\alpha = \beta$, and $\operatorname{Ext}^1_T(\alpha, \beta) \neq 0$ iff $\alpha = \beta$. The above sequence implies that

$$\operatorname{Ext}^1_G(\operatorname{Ind}_B^G \psi^w, \pi) \neq 0 \quad \text{if and only if} \quad \begin{array}{l} (\delta_1, \delta_2) = (\chi_2, \chi_1) \quad \text{or} \\ (\delta_1, \delta_2) = (\chi_1 \omega^{-1}, \chi_2 \omega) \end{array}$$

4. Paškūnas' strategy

Let I and I_1 denote the Iwahori and pro-p-Iwahori subgroups respectively, i.e.

$$I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \qquad I_1 = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}$$

and let Z denote the centre of G. For a smooth character $\zeta \colon Z \to \overline{\mathbb{F}}_p^\times$, we define the (non-commutative) Hecke algebra

$$\mathcal{H} = \mathcal{H}_\zeta := \operatorname{End}\left(c\text{-}\operatorname{Ind}_{ZI_1}^G \zeta\right).$$

For a smooth representation π with central character ζ, one has $\pi^{I_1} \cong \operatorname{Hom}_G(c\text{-}\operatorname{Ind}_{ZI_1}^G \zeta, \pi)$. Vigneras [Vig04] shows that the functor

$$\begin{array}{rcl} \mathcal{I} \colon \operatorname{Rep}_{G,\zeta} & \to & \operatorname{Mod}_{\mathcal{H}} \\ \pi & \mapsto & \pi^{I_1} \end{array}$$

where $\operatorname{Mod}_{\mathcal{H}}$ is the category of *right* $\mathcal{H}$-modules, is a bijection on irreducible objects. It has a left adjoint

$$\mathcal{T}(M) = M \otimes_{\mathcal{H}} c\text{-}\operatorname{Ind}_{ZI_1}^G \zeta$$

and the counit $\mathcal{T}\mathcal{I} \to \operatorname{id}$ is a natural isomorphism. In fact, Ollivier [Oll09] has shown that $\mathcal{I}$ and $\mathcal{T}$ are quasi-inverse to each other, when you restrict to representations of G which are generated by their I_1-invariants. This gives a spectral sequence

$$E_2^{i,j} \colon \operatorname{Ext}^i_{\mathcal{H}}(\mathcal{I}(\tau), R^j\mathcal{I}(\pi)) \Rightarrow \operatorname{Ext}^{i+j}_{G,\zeta}(\tau, \pi)$$

and a five-term exact sequence similar to the previous section.

Example 4.1. An explicit calculation in [Paš07, Proposition 10.2] shows that if $\tau \not\cong \pi$ then

$$\operatorname{Ext}^1_{G,\zeta}(\tau, \pi) \cong \operatorname{Ext}^1_G(\tau, \pi) \cong \operatorname{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), R^1\mathcal{I}(\pi)).$$

Suppose that $\pi = \pi(r, 0)$ is supersingular, with $0 < r < p - 1$. Then one can show that

$$R^1\mathcal{I}(\pi) \cong \mathcal{I}(\pi) \oplus \mathcal{I}(\pi)$$

which implies that $\operatorname{Ext}^1_G(\tau, \pi) = 0$ for all $\tau \not\cong \pi$.

References

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