# Ext groups between irreps

## §1 Introduction

Let $L/\mathbb{Q}_p$ be a finite extension.
$\mathcal{O} \subseteq L$ ring of integers.

$k$ residue field.
$G = GL_2(\mathbb{Q}_p)$

$$\mathrm{Mod}_G^{l.adm}(k) = \left\{ \begin{array}{c} \text{Smooth loc.} \\ \text{admissible} \\ k\text{-representations} \\ \text{of } \cancel{\text{the}}\ G \end{array} \right\}$$

Def: A block $\mathcal{B}$ for $\mathrm{Mod}_G^{l.adm}(k)$ is an equivalence class of irreducible objects, where $\pi \sim \tau$ iff $\exists$ sequence of irreps

$$\pi = \pi_0, \pi_1, \ldots, \pi_r = \tau$$

~~such that~~
with any of the three holding:

(1) $\pi_i \simeq \pi_{i+1}$

(2) $\mathrm{Ext}^1_G(\pi_i, \pi_{i+1}) \neq 0$

(3) $\mathrm{Ext}^1_G(\pi_{i+1}, \pi_i) \neq 0$

for ~~all~~ $i = 0, \ldots, r-1$

By general theory (see this in Pol's talk) we can decompose $\mathrm{Mod}_G^{l.adm}(k)$ as

$$\mathrm{Mod}_G^{l.adm}(k) = \bigsqcup_{\mathfrak{B}} \mathrm{Mod}_G^{l.adm}(k)[\mathfrak{B}]$$

and each subcat. $\mathrm{Mod}_G^{l.adm}(k)[\mathfrak{B}]$ is anti-equivalent to ~~a~~ the category of modules (over an appropriate ring). So we need to understand

- Iso classes of irreps ✓ (last time)
- Extensions between irreps (today).

## §2 Main theorem

Let $B \subset G$ the standard Borel (i.e. upper triangular matrices). Recall ~~irreps~~ there are 4 possibilities for irreps

1) A character $(\chi \circ \det)$
$$\chi : \mathbb{Q}_p^\times \to \overline{\mathbb{F}_p}^\times$$

2) (Special series) $Sp \otimes (\chi \circ \det)$
$$Sp = \text{Steinberg rep.}$$

$0 \neq \lambda \in \overline{\mathbb{F}_p}$

3) (Principal series) Let $0 < r \leq p-1$

$(r, \lambda) \neq (p-1, \pm 1)$

$$\pi(r, \lambda, \chi) \cong \left( \mathrm{Ind}_B^G (\mu_{\lambda^{-1}} \otimes \mu_\lambda \omega^r) \right) \otimes (\chi \circ \det)$$

where

$$\mu_\lambda(x) = \lambda^{\mathrm{val}(x)}$$

$$\omega(x) = x|x|_p \mod p$$

4) (Supersingular) denoted $\pi(r, 0, \chi)$

$0 \leq r \leq p-1$

Let $\pi$ and $\tau$ be smooth irreps that admit central characters $\chi_\pi, \chi_\tau$

Since $\mathrm{Ext}^1(\tau, \pi) \neq 0$

$$\Rightarrow \quad \chi_\pi = \chi_\tau$$

Assume $\chi_\pi = \chi_\tau = \zeta : Z \longrightarrow \overline{\mathbb{F}_p}^\times$

Furthermore,

$$\mathrm{Ext}_G^1(\tau \otimes (\chi \circ \det), \pi \otimes (\chi \circ \det))$$

$$= \mathrm{Ext}_G^1(\tau, \pi)$$

So we can assume that $\pi$ is either $1$, $\mathrm{Sp}$, $\pi(r, \lambda) := \pi(r, \lambda, 1)$ or supersingular.

Theorem (Paškūnas, Colmez, Emerton):
Let $p \geq 5$.

1) If $\pi$ is supersingular, then

$$\mathrm{Ext}^1_G(\tau, \pi) \neq 0 \quad \text{iff} \quad \tau \cong \pi$$

2) If $\pi \simeq 1$, then $\mathrm{Ext}^1_G(\tau, \pi) \neq 0$
iff

$$\tau \simeq 1, \mathrm{Sp}, \underbrace{\pi(p-3, 1, \omega)}_{\text{principal series}}$$

3) If $\pi \simeq \mathrm{Sp}$, then

$$\mathrm{Ext}^1_G(\tau, \pi) \neq 0 \quad \text{iff} \quad \tau \simeq 1, \mathrm{Sp}$$

4) If $\pi \simeq \pi(r, \lambda)$ $\quad 0 \neq \lambda$, $0 < r \leq p-1$ $\quad (r, \lambda) \neq (p-1, \pm 1)$

then $\mathrm{Ext}^1_G(\tau, \pi) \neq 0$ iff

$$\tau \simeq \begin{cases} \pi(r, \lambda) & (r, \lambda) = (p-2, \pm 1) \\ \pi(r, \lambda), \ \mathrm{Sp} \otimes (\omega^{-1} \mu_{\pm 1} \circ \det) & (r, \lambda) = (p-3, \pm 1) \\ \pi(r, \lambda), \ \underbrace{\pi(s, \lambda^{-1}, \omega^{r+1})}_{\text{principal series.}} & \text{otherwise, } 0 \leq s \leq p-2, \ s \equiv p-3-r \mod p-1 \end{cases}$$

Remark: Consider involution

$$(r, \lambda, \chi) \mapsto (s, \lambda^{-1}, \chi \omega^{r+1})$$

$s \equiv p-3-r \mod p-1$

corresponds to

$$\mathrm{Ind}\, \chi_1 \otimes \chi_2 \rightsquigarrow \mathrm{Ind}\, \chi_2 \omega \otimes \chi_1 \omega^{-1}$$

Corollary: Let $p \geq 5$. The cat $\mathrm{Mod}_G^{l\text{-adm}}(k)$ has the following blocks:

1) $\mathcal{B} = \{\pi\}$, $\pi$ supersingular.

2) $\mathcal{B} = \{\mathrm{Ind}_B^G \delta_1 \otimes \delta_2 \omega^{-1}, \mathrm{Ind}_B^G \delta_2 \otimes \delta_1 \omega^{-1}\}$

$$\delta_2 \delta_1^{-1} \neq \omega^{\pm 1}, 1.$$

3) $\mathcal{B} = \{\mathrm{Ind}_B^G \delta \otimes \delta \omega^{-1}\}$

4) $\mathcal{B} = \{1, \mathrm{Sp}, \mathrm{Ind}_B^G(\omega \otimes \omega^{-1})\} \otimes (\delta \circ \det)$.

## §3 Emerton's strategy

Let $T \subset B$ be the std torus.
Emerton has defined an "ordinary parts" functor

$$\mathrm{Ord}_B : \mathrm{Mod}_G^{adm}(k) \longrightarrow \mathrm{Mod}_T^{adm}(k)$$

which is right adjoint to

$$\chi_1 \otimes \chi_2 \mapsto \mathrm{Ind}_B^G \chi_2 \otimes \chi_1 \simeq \mathrm{Ind}_B^G (\chi_1 \otimes \chi_2)^{\omega}$$

(write $(\chi_1 \otimes \chi_2)^{\omega} = \chi_2 \otimes \chi_1$)

In fact, one can derive this further & we get derived adjunction:

$$R\mathrm{Hom}_G(\mathrm{Ind}_B^G \psi^\omega, \pi) = R\mathrm{Hom}_T(\psi, R\mathrm{Ord}_B \pi)$$

Get a spectral sequence

$$E_2^{i,j} : \mathrm{Ext}_T^i(\psi, R^j\mathrm{Ord}_B \pi)$$

$$\Rightarrow \mathrm{Ext}_G^{i+j}(\mathrm{Ind}_B^G \psi^\omega, \pi)$$

$\rightsquigarrow$ get long exact sequence

$$0 \to \mathrm{Ext}_T^1(\psi, \mathrm{Ord}_B \pi) \to \mathrm{Ext}_G^1(\mathrm{Ind}_B^G \psi^\omega, \pi)$$

$$\to \mathrm{Hom}_T(\psi, R^1\mathrm{Ord}_B \pi) \to \mathrm{Ext}_T^2(\psi, \mathrm{Ord}_B \pi) \to \cdots$$

<u>Example 1</u> Let $\pi$ be supersingular.

Then $R^i\mathrm{Ord}_B \pi = 0$ for all $i \geq 0$

($\pi$ can't be expressed as sub quotient of a non-trivial parabolic induction)

$$\Rightarrow \mathrm{Ext}_G^1(\mathrm{Ind}_B^G \psi^\omega, \pi) = 0$$

$$\Rightarrow \mathrm{Ext}_G^1(\tau, \pi) = 0 \quad \text{when } \tau \text{ is principal series}$$

$$\mathrm{Ext}_G^1(\mathrm{Sp} \otimes (\chi \circ \det), \pi) = 0 \qquad \chi \text{ smooth character.}$$

Example 2: $\qquad \pi = \mathrm{Ind}_B^G \xi^\omega$

Take $\psi = \delta_1 \boxtimes \delta_2$, $\xi = \chi_1 \boxtimes \chi_2$.
Then Emerton shows

- $\mathrm{Ord}_B \pi = \xi^\omega = \chi_2 \boxtimes \chi_1$
- $R^1\mathrm{Ord}_B \pi = \chi_1\omega^{-1} \boxtimes \chi_2\omega$
- $R^i\mathrm{Ord}_B \pi = 0$ for $i \geq 2$.

We note that $\mathrm{Hom}_T(\alpha, \beta) \neq 0$
iff $\alpha = \beta$
and $\mathrm{Ext}^1_T(\alpha, \beta) \neq 0$
iff $\alpha = \beta$.

$$\mathrm{Ext}^1_G(\mathrm{Ind}_B^G \psi^\omega, \pi) \neq 0$$

if and only if $(\delta_1, \delta_2) = (\chi_2, \chi_1)$ or

$(\delta_1, \delta_2) = (\chi_1\omega^{-1}, \chi_2\omega)$

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## §4 Paškūnas' strategy

Let $I$, $I_1$ be the Iwahori, prop-p-Iwahori respectively:

$$I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \qquad I_1 = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$$

and let $Z \subset G$ be the centre.

For a smooth character $\zeta : Z \to \overline{\mathbb{F}}_p^\times$ consider the (non-commutative) Hecke algebra

$$\mathcal{H} = \mathcal{H}_\zeta := \mathrm{End}_G\left(c\text{-}\mathrm{Ind}_{ZI_1}^G \zeta\right)$$

Vigneras shows that the functor

$$\begin{aligned} I : \mathrm{Rep}_{G,\zeta} &\longrightarrow \mathrm{Mod}_{\mathcal{H}} \\ \pi &\longmapsto \pi^{I_1} \simeq \mathrm{Hom}_G(c\text{-}\mathrm{Ind}_{ZI_1}^G \zeta, \pi) \end{aligned}$$

right $\mathcal{H}$-modules

smooth $G$-reps with $\overline{\mathbb{F}}_p$-coeffs with central char $\zeta$

is a bijection on irreducible objects. It has a left adjoint

$$\begin{aligned} T : \mathrm{Mod}_{\mathcal{H}} &\to \mathrm{Rep}_{G,\zeta} \\ M &\mapsto M \otimes_{\mathcal{H}} c\text{-}\mathrm{Ind}_{ZI_1}^G \zeta \end{aligned}$$

and $TI \xrightarrow{\sim} \mathrm{id}$ is a natural isomorphism.

Ollivier has shown that $I, T$ are quasi-inverse to each other, when you restrict to reps of $G$ which are generated by $I_1$-invariants.

Get spectral seq.

$$E_2^{i,j}: \quad \mathrm{Ext}^i_{\mathcal{H}}(\mathcal{I}(\tau), R^j\mathcal{I}(\pi)) \Rightarrow \mathrm{Ext}^{i+j}_{G,\zeta}(\tau, \pi)$$

$$0 \to \cancel{\mathrm{Hom}}\, \mathrm{Ext}^1_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(\pi)) \to \mathrm{Ext}^1_{G,\zeta}(\tau,\pi)$$

$$\to \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), R^1\mathcal{I}(\pi)) \to \mathrm{Ext}^2_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(\pi))$$

$$\to \cdots$$

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__Example 3:__ An explicit calculation

In Paškūnas' paper, shows that if $\tau \not\cong \pi$ then

$$\mathrm{Ext}^1_{G,\zeta}(\tau,\pi) \cong \mathrm{Ext}^1_G(\tau,\pi) \cong \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), R^1\mathcal{I}(\pi))$$

Assume $\pi$ is supersingular, $\pi = \pi(r,0)$
$0 < r < p-1$. One can show that

$$R^1\mathcal{I}(\pi) \cong \mathcal{I}(\pi) \oplus \mathcal{I}(\pi)$$

~~So~~ if $\tau \not\cong \pi$ then $\mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(\pi) \oplus \mathcal{I}(\pi)) \neq 0$

$$\Rightarrow \mathcal{I}(\tau) \cong \mathcal{I}(\pi)$$

$$\Rightarrow \tau \cong \pi \qquad \text{contradiction.}$$