Mod p and integral p-adic representations of $\mathrm{GL}_n(\mathbb{Q}_p)$

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1 Pro-p groups

(1.1) Definition (Profinite groups)**.** For us a ***profinite group*** (resp. a pro-p group) is a compact Hausdorff topological group with a fundamental system of neighbourhoods consisting of normal subgroups of finite index (resp. index p) (the ***profinite topology***).

A ***locally profinite group*** is a Hausdorff topological group such that every neighbourhood of the identity is contained in an open compact subgroup, in particular, there is a fundamental system of neighbourhoods of compact open subgroups. A locally profinite group is profinite iff it is compact.

(1.2) Definition (Profinite topology)**.** The ***profinite topology*** on a group is a topology on the underlying set of the group defined in the following equivalent ways:

- It has as a basis of open subsets all left (equiv. right) cosets of subgroups of finite index.
- It has as a basis of opens subsets all cosets of normal subgroups of finite index.

Under this, any group becomes a topological group.

(1.3) Examples. We have the following examples of pro-p groups:

- $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.
- $\mathrm{GL}_n(\mathbb{Z}_p)$ has an open subgroup $U = \mathrm{I} + pM_n(\mathbb{Z}_p)$ which is a pro-p group. Further, each $K_r = 1 + p^r M_n(\mathbb{Z}_p)$ is pro-p ($K(r)/K(r+1) \cong M_n(\mathbb{F}_p)$ under $1 + p^s A \mapsto \overline{A}$).
- A pro-p group G is a p-adic analytic group (group with the structure of an analytic manifold over $\mathbb{Q}_p$ such that group multiplication and inversion are analytic functions) iff it is of finite rank ($r > 0$ such that any closed subgroup has a topological generating set with no more than r elements). This also implies that a pro-p group is p-adic analytic iff it is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ for some n.

(1.4) Example. $\mathrm{GL}_n(\mathbb{Q}_p)$ is a topological group with a unique maximal compact $K = \mathrm{GL}_n(\mathbb{Z}_p)$ (up to conjugation). Inside K, we have a filtration $K(r) = 1 + p^r\mathcal{M}_n(\mathbb{Z}_p) \supset K(r+1) \supset \cdots$ which form a fundamental system of neighbourhoods of $1 \in \mathrm{GL}_n(\mathbb{Q}_p)$ so $\mathrm{GL}_n(\mathbb{Q}_p)$ is locally profinite. In particular, it is Hausdorff, locally compact and totally disconnected.

2 Basic structure of $\mathrm{GL}_2(\mathbb{Q}_p)$

In this section let $G = \mathrm{GL}_n(\mathbb{Q}_p)$.

(2.1) Definition (Important subgroups)**.**

- The ***Borel*** $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ upper triangular matrices in G.
- The ***torus*** $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ diagonal matrices in B.
- The ***unipotent radical*** $U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ upper triangular matrices with 1's on the diagonal in B.
- If $G = \mathrm{GL}_n$, $n = \sum_{i=1}^r n_i$, then $P \cong \begin{pmatrix} \mathrm{GL}_{n_1} & & * \\ & \ddots & \\ 0 & & \mathrm{GL}_{n_r} \end{pmatrix}$ denotes the ***standard parabolic*** subgroup of G with r blocks of size $n_1, \ldots, n_r$ (so the Borel is $B = P_{1,\ldots,1}$).
  Note $B \cong T \ltimes U$ (a case of Levi decomposition).
- Let $\overline{P}$ denote the opposite parabolic (just the transpose for GL_n) and let $\overline{N}$ be its unipotent radical.
- Let $\Lambda = \mathrm{diag}(p^{\mathbb{Z}}, \cdots, p^{\mathbb{Z}})$ and let $\Lambda^+ = \{\mathrm{diag}(p^{a_1}, \cdots, p^{a_n}) \mid a_1 \geq \cdots \geq a_n\}$. Then Λ runs over cocharacters of T (homomorphisms $\mathbb{G}_m \to T$).

(2.2) Proposition (Levi Decomposition)**.** If $P \cong \begin{pmatrix} \mathrm{GL}_{n_1} & & * \\ & \ddots & \\ 0 & & \mathrm{GL}_{n_r} \end{pmatrix}$ is the standard parabolic, then we can define the ***standard Levi*** as the corresponding Levi $M \cong \begin{pmatrix} \mathrm{GL}_{n_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{GL}_{n_r} \end{pmatrix} \cong$ $\prod_i \mathrm{GL}_{n_i}$ (block diagonal) and let U denote the unipotent radical $N \cong \begin{pmatrix} \mathrm{id}_{n_1} & & * \\ & \ddots & \\ 0 & & \mathrm{id}_{n_r} \end{pmatrix}$.

Then $P = MN = M \ltimes N$ ($N \triangleleft P$, $M \cong P/N$).

(2.3) Proposition (Iwasawa Decomposition)**.** For any P standard parabolic, $G = PK$.

(2.4) Proposition (Cartan Decomposition)**.** $G = K\Lambda^+K = \amalg_{\lambda \in \Lambda^+} K\lambda K$.
This is essentially stating that each element has a Smith normal form.

(2.5) Remark. This also implies that $\mathrm{GL}_n(\mathbb{Q}_p)/K$ is countable as $K\lambda K = \amalg_g g\lambda K$ where g are finitely many coset representatives for $K/K \cap \lambda K \lambda^{-1}$ (noting that $K \cap \lambda K \lambda^{-1}$ is compact open in K so $K/K \cap \lambda K \lambda^{-1}$ is finite) and a countable union of finite sets is countable.

(2.6) Proposition (Bruhat Decomposition)**.** $\mathrm{GL}_n(\mathbb{Q}_p) = BWB$ where $W = N_{\mathrm{GL}_n}(T)/T \cong S_n =$ permutation matrices $\subset \mathrm{GL}_n(\mathbb{Q}_p)$ is the Weyl group.

In the $n = 2$ case this gives $\mathrm{GL}_n(\mathbb{Q}_p) = B \amalg B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B$.

3 Augmented G-representations

From here on let G be a p-adic analytic group and let A be a complete local Noetherian $\mathbb{Z}_p$-algebra with finite residue field and maximal ideal $\mathfrak{m}$. Let $\mathrm{Mod}_G(A)$ denote the category of $A[G]$-modules.

We wish to define various categories of G-representations and their relationships to eachother.

(3.1) Definition (Completed group ring)**.** Let H be a compact open subgroup of G. We can define the ***completed group ring*** of H over A

$$A[[H]] := \varprojlim_{H' \leq H \text{ open}} A[H/H'].$$

We can endow the rings $A[H/H']$ with the $\mathfrak{m}$-adic topology (that is, we have a fundamental system of open neighbourhoods of 0 given by the submodules $\mathfrak{m}^n(A[H/H'])$). Then we equip $A[[H]]$ with the projective limit topology (the coarsest topology such that each $A[[H]] \xrightarrow{f_{H'}} A[H/H']$ is continuous (generated by $f_{H'}^{-1}(U)$ for U open)).

Explicitly, $A[[H]] = \{\mathbf{a} = (a_{H'})_{H'} \in \prod_{H' \leq H \text{ open}} A[H/H'] \mid a_{H''} = a_{H'} \text{ in } A[H/H'] \text{ for all } H'' \leq H'\}$ so in particular $A[H] \xhookrightarrow{a \mapsto (a \mod H')_{H'}} A[[H]]$.

(3.2) Remark. The rings $A[H/H']$ are profinite as each H/H' is finite (by compactness of H) and hence $A[[H]]$ is profinite (the inverse limit of an inverse system of profinite groups with continuous transition maps is profinite). In particular, $A[[H]]$ is a compact, topological ring.

(3.3) Fact. $A[[H]]$ is Noetherian.

(3.4) Proposition. Any finitely generated $A[[H]]$-module admits a unique profinite topology with respect to which the $A[[H]]$-action becomes continuous.

Further, any $A[[H]]$-linear morphism of finitely generated $A[[H]]$-modules is continuous with respect to the profinite topologies defined above.

Proof. Since $A[[H]]$ is Noetherian, we can find a presentation

$$\begin{array}{ccccccc} A[[H]]^s & \longrightarrow & & \longrightarrow & A[[H]]^r & \xrightarrow{\pi} & M \longrightarrow 0 \\ & \searrow & & \nearrow & & & \\ & & \ker(\pi) & & & & \end{array}$$

(induction on number of generators shows that any submodule of a finitely generated module is finite, so $\ker(\pi)$ is finitely generated).

As $A[[H]]$ is profinite, $A[[H]]^s \to A[[H]]^r$ has closed image N and hence $M = A[[H]]^r/N$ is the quotient of the profinite module $A[[H]]^r$ by a closed $A[[H]]$-submodule and if we equip M with the induced quotient topology then it becomes a profinite module with the desired properties. □

(3.5) Definition (Canonial topology)**.** If M is a finitely generated $A[[H]]$-module, we refer to the topology given before as the ***canonical topology*** on M.

(3.6) Definition (Augmented representation)**.** An ***augmented representation of*** G ***over*** A is $M \in \mathrm{Mod}_G(A)$ equipped with an $A[[H]]$-module structure for some (equiv. any) compact open $H \leq G$ such that the two induced $A[H]$-actions induced by the inclusions $A[H] \subset A[[H]]$ and $A[H] \subset A[G]$ coincide.

We denote by $\mathrm{Mod}_G^{\mathrm{aug}}(A)$ the abelian ***category of augmented*** G***-representations over*** A with morphism maps that are simultaneously G-equivariant and $A[[H]]$-linear for some (equiv. any H) compact $H \leq G$.

(3.7) Definition (Profinite augmented representation)**.** If $M \in \mathrm{Mod}_G^{\mathrm{aug}}(A)$ is equipped with a profinite topology such that the $A[[H]]$-action on M is jointly continuous for some (equiv. any) compact open $H \leq G$ over A, then we say that M is a ***profinite augmented representation of*** G ***over*** A.

(Equivalently, the profinite topology on M admits a neighbourhood basis at the origin consisting of $A[[H]]$-submodules.)

We denote by $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ the abelian ***category of profinite augmented*** G***-representations over*** A with morphisms the continuous A-linear G-equivariant maps ($A[H]$ is dense in $A[[H]]$ so any such map is automatically $A[[H]]$-linear for any compact open $H \leq G$).

(3.8) Remark. The equivalence of the "some" and "any"'s come from the fact that if $H_1, H_2 \leq G$ are open, compact, then $H_1 \cap H_2$ has finite index in each of H_1 and H_2.

(3.9) Remark. Forgetting the topology gives a functor

$$\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A) \xrightarrow{\text{forget topology}} \mathrm{Mod}_G^{\mathrm{aug}}(A)$$

(3.10) Definition (Finitely generated augmented representations)**.** We let $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{aug}}(A)$ consisting of augmented G-modules that are finitely generated over $A[[H]]$ for some (equiv. any) compact open $H \leq G$.

(3.11) Remark. (3.4) shows that equipping each object of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ with its canonical topology means that we can lift $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A) \hookrightarrow \mathrm{Mod}_G^{\mathrm{aug}}(A)$ to an inclusion

$$\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A) \xhookrightarrow{\text{canonical topology}} \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A).$$

(3.12) Definition (Serre subcategory)**.** A full subcategory $\mathcal{D}$ of an abelian category $\mathcal{C}$ is a ***Serre subcategory*** if for any SES in $\mathcal{C}$

$$\begin{gathered} 0 \to M \to M' \to M'' \to 0, \\ M' \in \mathcal{D} \quad \Leftrightarrow \quad M, M'' \in \mathcal{D}. \end{gathered}$$

In our case this just means that it is closed under taking subobjects, quotients and extensions.

(3.13) Proposition. The category $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ forms a Serre subcategory of the abelian category $\mathrm{Mod}_G^{\mathrm{aug}}(A)$. In particular, it itself forms an abelian category.

Proof. Closure under the formation of quotients and extensions is clear, and closure under the formation of subobjects follows from the fact that $A[[H]]$ is Noetherian. $\square$

(3.14) Notation. Let $A_{\mathbb{Q}_p} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$ and $A[[H]]_{\mathbb{Q}_p} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A[[H]]$.

(3.15) Remark. The fact that $A[[H]]$ is Noetherian implies that $A[[H]]_{\mathbb{Q}_p}$ is Noetherian as well.

(3.16) Definition. An ***augmented representation of** G **over** $A_{\mathbb{Q}_p}$* is an $A_{\mathbb{Q}_p}[G]$-module M equipped with an $A[[H]]_{\mathbb{Q}_p}$-module structure for some (equiv. any) compact open $H \leq G$ such that the two $A_{\mathbb{Q}_p}[H]$ actions induced by $A_{\mathbb{Q}_p}[H] \subset A[[H]]_{\mathbb{Q}_p}$ and $A_{\mathbb{Q}_p}[H] \subset A_{\mathbb{Q}_p}[G]$.

(3.17) Definition. We let $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A_{\mathbb{Q}_p})$ denote the ***category of augmented** G**-modules over** $A_{\mathbb{Q}_p}$ **that are finitely generated over*** $A[[H]]_{\mathbb{Q}_p}$ for some (equiv. any) compact open $H \leq G$ and whose morphisms are $A_{\mathbb{Q}_p}[G]$ and $A[[H]]_{\mathbb{Q}_p}$-linear.

(3.18) Lemma. The functor $M \mapsto \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$ induces an equivalence of categories

$$\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)_{\mathbb{Q}_p} \xrightarrow{\sim} \mathrm{Mod}_G^{\mathrm{fg\,aug}}(A_{\mathbb{Q}_p})$$

4 Smooth G-representations

(4.1) Definition (Smooth representation)**.** Let $V \in \mathrm{Mod}_G(A)$. We say that a vector $v \in V$ is ***smooth*** if

1. v is fixed by some open subgroup of G,
2. v is annihilated by some power $\mathfrak{m}^i$ of the maximal ideal of A.

Equivalently, $v \in V$ is smooth iff v is annihilated by an open ideal in $A[[H]]$ for some (equiv. any) open $H \leq G$.

We let $V_{\mathrm{sm}} = \{v \in V \mid v \text{ is smooth}\}$ and we say that $V \in \mathrm{Mod}_G(A)$ is a ***smooth representation*** if $V = V_{\mathrm{sm}}$.

We denote by $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ the full subcategory of $\mathrm{Mod}_G(A)$ consisting of smooth G-representations.

(We can equivalently let $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ denote the full subcategory of $\mathrm{Mod}_G(A)$ with objects $V = \bigcup_{H \text{ open}, i \geq 1} V^H[\mathfrak{m}^i]$.)

(4.2) Remark. If A is Artinian, then $\mathfrak{m}^i = 0$ for sufficiently large i, so automatically $\mathfrak{m}^i v = 0$ for any $v \in V$ for any V an A-module. Then we can omit condition (2) and have the usual definition.

(4.3) Example. Take a character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. Then the representation $\mathrm{GL}_n(\mathbb{Q}_p) \xrightarrow{\det} \mathbb{Q}_p^\times \xrightarrow{\chi} \mathbb{C}^\times$ is smooth iff χ is locally constant (trivial on some open subgroup of $\mathbb{Q}_p^\times$ (i.e. $\chi|_{1+p^r\mathbb{Z}_p}$ is trivial for some r)).

(4.4) Lemma. V_{sm} is an $A[G]$-submodule of V.

Proof. If $v_j \in V_{\mathrm{sm}}$ are fixed by open $H_j \leq G$ and annihilated by $\mathfrak{m}^{i_j}$ for $j = 1, 2$, then any A-linear combination of v_1 and v_2 is fixed by $H_1 \cap H_2$ and annihilated by $\mathfrak{m}^{\max(i_1, i_2)}$ and hence also lies in V_{sm} so V_{sm} is an A-submodule.

Further, gv_1 is fixed by $gH_1g^{-1} \leq G$ for any $g \in G$ and is annihilated by $\mathfrak{m}^{i_1}$, so V_{sm} is closed under the action of G. □

(4.5) Remark. $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ is a Serre subcategory of $\mathrm{Mod}_G(A)$ and so in particular is an abelian category.

(4.6) Remark. There is a left-exact functor

$$\mathrm{Mod}_G(A) \xrightarrow{V \mapsto V_{\mathrm{sm}}} \mathrm{Mod}_G^{\mathrm{sm}}(A)$$

which is right adjoint to the inclusion

$$\mathrm{Mod}_G^{\mathrm{sm}}(A) \hookrightarrow \mathrm{Mod}_G(A)$$

$i \dashv (-)_{\mathrm{sm}}$ ($\mathrm{Hom}_G(V, W) \cong \mathrm{Hom}_G(V, W_{\mathrm{sm}})$ for all V, W with W smooth).

The following is easy to prove and useful.

(4.7) Lemma. Let $V \in \mathrm{Mod}_G(A)$ by torsion as an $\mathbb{Z}_p$-module (holds if A is Artinian) and let $V^* = \mathrm{Hom}_{\mathbb{Z}_p}^{\mathrm{cont}}(V, \mathbb{Q}_p/\mathbb{Z}_p)$ (where $\mathbb{Q}_p/\mathbb{Z}_p$ has the discrete topology and V^* has the compact open topology). The following are equivalent

1. V is smooth.
2. The A-action and the G-action on V are both jointly continuous (i.e. $A \times V \to V$ and $G \times V \to V$ are both continuous where $A \times V$ and $G \times V$ have the product topology and A as the $\mathfrak{m}$-adic topology) when V is given the discrete topology.
3. The A-action and the G-action on V^* are both jointly continuous when V^* is given its natural profinite topology and A is given the $\mathfrak{m}$-adic topology.
4. For some (equiv. any) compact open $H \leq G$, the $A[H]$-action on V extends to an $A[[H]]$-action continuous when V has the discrete topology.
5. For some (equiv. any) compact open $H \leq G$, the H-action on V^* extends to a continuous action of $A[[H]]$ on V^*.

(4.8) Remark. (4.7) implies that passing to Pontrjagin duals gives an anti-equivalence

$$\begin{aligned} \mathrm{Mod}_G^{\mathrm{sm}}(A) &\xrightarrow{\mathrm{anti}\sim} \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A) \\ V &\mapsto V^*. \end{aligned} \tag{1}$$

(Note $V^{**} = V$.)

5 Induced representations

(5.1) Definition (Induced representation)**.** Let $H \leq G$ be closed and let $V \in \mathrm{Mod}_H^{\mathrm{sm}}(A)$. Then we can define the ***induced representation***

$$\begin{aligned} \mathrm{Ind}_H^G(V) :&= \left\{ f : G \to V \mid \begin{array}{c} f(hg) = hf(g)\, \forall h \in H, g \in G \text{ and} \\ \exists U \text{ compact open with } f(gu) = f(g) \forall g \end{array} \right\} \\ &= \{ f : G \to V \mid f(hg) = hf(g)\, \forall h \in H, g \in G \}_{\mathrm{sm}} \end{aligned}$$

with G-action $(\gamma \cdot f)(g) = f(g\gamma)$.

We can also define the ***compact induction***

$$\mathrm{c}-\mathrm{Ind}_H^G(V) = \left\{ f \in \mathrm{Ind}_H^G(V) \mid \mathrm{supp}(f) \text{ compact} \right\},$$

a subrepresentation of $\mathrm{Ind}_H^G(V)$.

In the case that H is a parabolic subgroup of GL_n, and more generally if $H \setminus G$ is compact, $\mathrm{c} - \mathrm{Ind}_H^G = \mathrm{Ind}_H^G$.

(5.2) Remark (Frobenius reciprocity)**.** Say $V \in \mathrm{Mod}_G^{\mathrm{sm}}(A)$, $W \in \mathrm{Mod}_H^{\mathrm{sm}}(A)$. Then

1. $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G V) \cong \mathrm{Hom}_H(V|_H, W)$ naturally.
2. If $U \leq G$ is open, then $\mathrm{Hom}_G(\mathrm{c} - \mathrm{Ind}_U^G W, V) \cong \mathrm{Hom}_U(W, V|_U)$. Moreover, $\mathrm{c} - \mathrm{Ind}_U^G$ is an exact functor.
3. If $G = \mathrm{GL}_n(\mathbb{Q}_p)$ and P is a standard parabolic, Ind_P^G is an exact functor.

(5.3) Remark. If $K \subset H \subset G$, then $\mathrm{Ind}_H^G \mathrm{Ind}_K^H \cong \mathrm{Ind}_K^G$, and similarly for $\mathrm{c} - \mathrm{Ind}$.

(5.4) Definition (Parabolic Induction)**.** If P is a parabolic subgroup, we can define ***parabolic induction*** Ind_P:

$$\begin{array}{ccc} \mathrm{Mod}_M^{\mathrm{sm}}(A) & \xrightarrow{\mathrm{Ind}_P} & \mathrm{Mod}_G^{\mathrm{sm}}(A) \\ {\scriptstyle \mathrm{inflate}}\big\downarrow & \nearrow_{\mathrm{Ind}_P^G} & \\ \mathrm{Mod}_P^{\mathrm{sm}}(A) & & \end{array}$$

where the inflation is given by taking the action as the identity on N and noting $M \cong P/N$.

(5.5) Example. Take $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be smooth so we get $\chi_1 \otimes \chi_2 : T \xrightarrow{\mathrm{diag}(a,b) \mapsto \chi_1(a)\chi_2(b)} \mathbb{C}^\times$. Take $B \twoheadrightarrow T$ with kernel N. Then we can extend $\chi_1 \otimes \chi_2$ to B in the obvious way and take $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$, the ***principal series representation***.

6 Admissible representations

(6.1) Definition (Admissible representations)**.** We say that $V \in \mathrm{Mod}_G^{\mathrm{sm}}(A)$ is ***admissible*** if $V^H[\mathfrak{m}^i]$ (the elements of the subspace of fixed vectors V^H of V killed off by $\mathfrak{m}^i$) is finitely generated over A for every open $H \leq G$ and every $i \geq 0$.

We let $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ consisting of admissible representations.

(6.2) Remark. If A is Artinian, then $\mathfrak{m}^i = 0$ for large i so $V^H[\mathfrak{m}^i] = V^H$ so this agrees with the usual definition (that V^H is finite dimensional for all $H \leq G$ open compact).

(6.3) Fact. $V \in \mathrm{Mod}_G^{\mathrm{sm}}(A)$ is admissible iff V^* is finitely generated as an $A[[H]]$-module for some (equiv. any) compact open $H \leq G$.

(6.4) Remark. (6.3) shows that the anti-equivalence (1) restricts to an anti-equivalence

$$\begin{aligned} \mathrm{Mod}_G^{\mathrm{adm}}(A) &\xrightarrow{\text{anti}\sim} \mathrm{Mod}_G^{\mathrm{fg\,aug}}(A) \\ V &\mapsto V^*. \end{aligned} \tag{2}$$

(6.5) Proposition. $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ forms a Serre subcategory of the abelian category $\mathrm{Mod}_G(A)$ and in particular is itself an abelian category.

Proof. This follows from the anti-equivalence (2) and (3.13). □

(6.6) Remark. Let $V \in \mathrm{Mod}_G(A)$ and $i \geq 0$. Then $V[\mathfrak{m}^i]$ is a successive extension of finitely many copies of $V[\mathfrak{m}]$. Therefore by (6.5), if $V \in \mathrm{Mod}_G^{\mathrm{sm}}(A)$ is such that $V[\mathfrak{m}] \in \mathrm{Mod}_G^{\mathrm{adm}}(A/\mathfrak{m})$ (equivalently if the definition for admissibility is assumed just in the case $i = 1$), then $V \in \mathrm{Mod}_G^{\mathrm{adm}}(A)$.

(6.7) Fact. If $H \setminus G$ is compact, then Ind_H^G preserves admissibililty.

(6.8) Definition (Locally admissible representations)**.** Let $V \in \mathrm{Mod}_G(A)$. We say that $v \in V$ is ***locally admissible*** if v is smooth and if the smooth G-subrepresentation of V generated by v is admissible.

Let $V_{\mathrm{l.adm}} = \{v \in V \mid v \text{ locally admissible}\}$.

We say that $V \in \mathrm{Mod}_G(A)$ ***is locally admissible*** if $V = V_{\mathrm{l.adm}}$ and we let $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ consisting of locally admissible representations.

(6.9) Lemma. $V_{\mathrm{l.adm}}$ is an $A[G]$-submodule of V.

(6.10) Proposition. $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ is closed under passing to subrepresentations, quotients, and inductive limits in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. In particular, it is an abelian category.

Proof. $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ is closed in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ under passing to subrepresentations and quotients so the same is true for $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$. Closure under inductive limits is also clear since local admissibility is checked elementwise. □

(6.11) Remark. Any finitely generated locally admissible representation is admissible smooth.

Since any representation is the inductive limit of finitely generated ones, a representation is locally admissible iff it is an inductive limit of admissible representations.

(6.12) Remark. There is a left-exact functor

$$\mathrm{Mod}_G(A) \xrightarrow{V \mapsto V_{\mathrm{l.adm}}} \mathrm{Mod}_G^{\mathrm{l.adm}}(A)$$

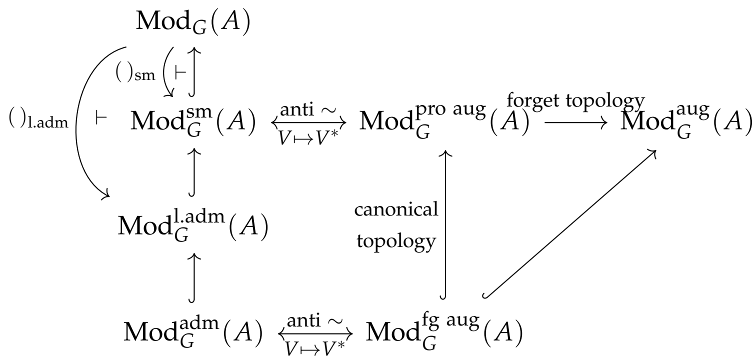
which is right adjoint to the inclusion

$$\mathrm{Mod}_G^{\mathrm{l.adm}}(A) \hookrightarrow \mathrm{Mod}_G(A).$$

$i \dashv (-)_{\mathrm{l.adm}}$.

7 Relations

Now we can summarise the relations in the following diagram:



8 Some finiteness conditions

Let Z denote the centre of G.

(8.1) Definition (Z-finite representations)**.** Let $V \in \mathrm{Mod}_G(A)$.

1. We say that V is Z***-finite*** if $A[Z]/\operatorname{Ann}_{A[Z]} V$ is a finite A-algebra.
2. We say that $v \in V$ is ***locally*** Z***-finite*** if the $A[Z]$-submodule of V generated by v is Z-finite (equivalently it is finitely generated as an A-module). Let $V_{Z\text{-fin}}$ denote the subset of locally Z-finite elements of V.
3. We say that V is locally Z-finite over Z if $V = V_{Z\text{-fin}}$.

(8.2) Remark. $V_{Z\text{-fin}}$ is an $A[G]$-submodule of V and if V_1, V_2 are (locally) Z-finite then $V_1 \oplus V_2$ is (locally) Z-finite. Further, any $A[G]$-invariant submodule or quotient of a (locally) Z-finite representation is again (locally) Z-finite.

(8.3) Remark. If $V \in \mathrm{Mod}_G(A)$ is finitely generated over $A[G]$, then V is Z-finite iff V is locally Z-finite.

(If V is generated by finitely many locally Z-finite vectors, then it is a quotient of a direct sum of fintiely many Z-finite representations so is Z-finite.)

(8.4) Lemma. If $V \in \mathrm{Mod}_G(A)$, then any locally admissible vector in V is locally Z-finite ($V_{\mathrm{l.adm}} \subset V_{Z\text{-fin}}$).

Proof. This is just working through the definitions.

Let $v \in V$ be smooth locally admissible. Then, by smoothness, there is $H \leq G$ compact open fixing v and there is $i \geq 0$ with $\mathfrak{m}^i v = 0$.

Let W be the $A[G]$-submodule generated by v. Then W is admissible smooth and $\mathfrak{m}^i = 0$ so $W^H = W^H[\mathfrak{m}^i]$ is finitely generated over A. Also, W^H is Z-invariant ($h \cdot z \cdot w = z \cdot h \cdot w = z \cdot w$) so $W^H/\operatorname{Ann}_{A[Z]} W^H$ is finitely generated over A and we are done. $\square$

(8.5) Remark. The previous remark and lemma imply that if $V \in \mathrm{Mod}_G^{\mathrm{adm}}(A)$ is finitely generated over $A[G]$, then it is Z-finite.

(8.6) Lemma. Let $V \in \mathrm{Mod}_G^{\mathrm{sm}}(A)$. Consider the following conditions:

1. V is of finite length (as an object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, or equivalently of $\mathrm{Mod}_G(A)$) (has finite composition series) and is admissible.

2. V is finitely generated as an $A[G]$-module and is admissible.

3. V is of finite length (as an object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$) and is Z-finite.

Then $(1) \Rightarrow (2)$ (immediate) and $(1) \Rightarrow (3)$ (previous remark).

(8.7) Conjecture. If G is a reductive p-adic group, then the three conditions in (8.6) are equivalent.

(8.8) Theorem. The previous conjecture holds in the following two cases:

- G is a torus.
- $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

(8.9) Definition. Let Z be the centre of G and $\zeta : Z \to A^\times$ be a continuous character. We denote by $\mathrm{Mod}_{G,\zeta}^*(A)$ the full subcategory of $\mathrm{Mod}_G^*(A)$ consisting of objects on which Z acts by ζ, that is, $zv = \zeta(z)v$ for all $z \in Z$.

(8.10) Fact. If $G = \mathrm{GL}_2(\mathbb{Q}_p)$ or G is a torus, then

$$\mathrm{Mod}_{G,\zeta}^{\mathrm{l.fin}}(A) = \mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A).$$