IRREDUCIBLE SMOOTH ADMISSIBLE MOD p REPRESENTATIONS OF $\mathrm{GL}_n(F)$

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Contents

1. Introduction

In this talk we will present the main result of [Her11a] and sketch a proof. The theorem is a classification of the irreducible admissible representations of $\mathrm{GL}_n(F)$ for $F/\mathbb{Q}_p$ a p-adic field with ring of integers $\mathscr{O}$ and residue field k.

2. Preliminaries

We will now give a mod p analog of the classical Satake isomorphism in characteristic 0 and define a notion of supersingularity for representations of $\mathrm{GL}_n(F)$.

2.1. **Weights.** Let $G = \mathrm{GL}_n(F)$. Recall that G has a maximal compact $K = \mathrm{GL}_n(\mathscr{O})$ and a distinguished pro-p subgroup $K_1 = I_n + \mathrm{Mat}_n(\mathfrak{m}_F)$.

Lemma 2.1.1 (p-group Lemma)**.** *Let τ be a nonzero smooth $\overline{\mathbb{F}}_p$-representation of a pro-p group H. Then τ has an H-fixed vector.*

Proof. Since $\overline{\mathbb{F}}_p$ is an $\mathbb{F}_p$-vector space, view H as an $\mathbb{F}_p$-representation. Pick a nonzero $x \in \tau$. By definition of smoothness there is a an open normal subgroup $U \leq H$ fixing x. By compactness H/U is a p-group which acts on $\mathbb{F}_p[H/U] \cdot x$, which has some finite dimension d, so we get a map $H \to \mathrm{GL}_d(\mathbb{F}_p)$, whose image must live in a p-Sylow. But for some basis, every p-Sylow subgroup is

$$\begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix}$$

so the image fixes the first basis vector. □

So any nonzero smooth representation π of K has $\pi^{K_1} \neq 0$. But $K_1 \leq K$ is a normal subgroup so π^{K_1} is a subrepresentation of K. So if π is irreducible, $\pi = \pi^{K_1}$, so π is really a representation of

$$K/K_1 \cong \mathrm{GL}_n(\mathbb{F}_p)$$

So irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(k)$ are the same as irreducible smooth $\overline{\mathbb{F}}_p$-representations of K.

Definition 2.1.2. If π is a smooth representation of G, then V is a *weight* of π if $V \subseteq \pi|_K$.

Every nonzero smooth representation π of G contains a weight: to see this, note that the representation of $\mathrm{GL}_n(k)$ generated by any nonzero vector in π^{K_1} is finite dimensional, so must contain an irreducible representation.

2.2. **Hecke Algebras.** If π is unramified, so that $\pi^K \neq 0$, then the spherical Hecke algebra $\mathcal{C}_c(K\backslash G/K, \overline{\mathbb{F}}_p)$ acts on π^K. But if not, then we can still define an action of a certain Hecke algebra which depends on the weight of π. To see how this might work, note that a spherical representation is by definition one that has the trivial weight as one of its weights, and

$$\begin{aligned}\mathcal{C}_c(K\backslash G/K, \overline{\mathbb{F}}_p) &\cong \left\{\varphi : G \to \overline{\mathbb{F}}_p \text{ compactly supported mod } K \mid \varphi(kgk') = k\varphi(g)k' \text{ for all } k, k' \in K\right\} \\ &= \mathrm{Hom}_K(1, \text{c-Ind}_K^G(1)|_K) \\ &= \mathrm{End}_G(\text{c-Ind}_K^G(1))\end{aligned}$$

But then $\pi^K = \mathrm{Hom}_K(1, \pi|_K) = \mathrm{Hom}_G(\text{c-Ind}_K^G(1), \pi)$, and the Hecke action turns out to be the natural action by precomposing by endomorphisms of $\mathrm{End}_G(\text{c-Ind}_K^G(1))$.

This motivates the following definition:

Definition 2.2.1. Given a weight V, we define

$$\mathcal{H}_G(V) = \mathrm{End}_G(\text{c-Ind}_K^G(V))$$

Then $\mathcal{H}_G(V)$ naturally acts (via Frobenius reciprocity) on $\mathrm{Hom}_K(V, \pi|_K)$, and this is nontrivial if π has weight V.

Remark 2.2.2. In fact, this definition works for any split[1] connected reductive group G defined over $\mathscr{O}$ and any compact open $H \subseteq G$. In particular if $M \leq G$ is a standard Levi subgroup we can look at $\mathcal{H}_M(V)$ where V is a weight for the group M and this acts on $\mathrm{Hom}_{M(\mathscr{O})}(V, \pi|_{M(\mathscr{O})})$ for π a representation of M.

2.3. **Satake.** What does this Hecke algebra look like? Is it commutative?

Theorem 2.3.1 ([Her11b]). *For any standard parabolic $P = MN$, there are injective maps*

$$\mathcal{H}_G(V) \hookrightarrow \mathcal{H}_M(V_{N(k)}) \hookrightarrow \mathcal{H}_T(V_{U(k)}) = \overline{\mathbb{F}}_p[X_*(T)]$$

with image $\overline{\mathbb{F}}_p[X_*(T)_+]$, where N is the unipotent radical of the standard Borel in $\mathrm{GL}_n(\mathbb{F}_p)$, and $X_*(T)_+ = \{(a_1 \geq \cdots \geq a_n) \in \mathbb{Z}^n\}$ is the set of dominant coweights. In fact, the map

$$\mathcal{H}_G(V) \to \mathcal{H}_M(V_{N(k)})$$

is the localization at a non-invertible $\lambda \in X_*(T)_+$ corresponding to the shape of M.

[1] For a general connected reductive group, the whole classification is done in [AHHV17], but the methods are more involved, and in particular one looks at a special maximal parahoric K instead of the maximal compact K.

So for instance, if $G = \mathrm{GL}_3$ and we pick the Levi $M = \mathrm{GL}_2 \times \mathrm{GL}_1$, then we invert $(a_1 = a_2 > a_3)$.

Note admissibility of π implies that $\mathrm{Hom}_K(V, \pi_K) < \infty$. By the Satake isomorphism, $\mathcal{H}_G(V)$ is commutative and so there is a decomposition into generalized eigenspaces

$$\mathrm{Hom}_K(V, \pi|_K) = \bigoplus_{\chi : \mathcal{H}_G(V) \to \overline{\mathbb{F}}_p} \mathrm{Hom}_K(V, \pi|_K)_\chi$$

A system $\chi : \mathcal{H}_G(V) \to \overline{\mathbb{F}}_p$ appears in $\mathrm{Hom}_K(V, \pi|_K)$ if and only if there exists a nonzero map

$$\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V),\chi} \overline{\mathbb{F}}_p \to \pi,$$

which follows easily from Frobenius reciprocity.

2.4. **Supersingularity.** Supersingular representations are the building blocks of the representations we care about, along with Steinberg representations.

If V is a weight for π, then any system of eigenvalues $\varphi : \mathcal{H}_G(V) \to \overline{\mathbb{F}}_p$ appearing in $\mathrm{Hom}_K(V, \pi|_K)$ induces a monoid homomorphism $\varphi' : X_*(T)_+ \to \overline{\mathbb{F}}_p$ via Satake.

Definition 2.4.1. A representation π is *supersingular* if for every weight V and system of $\mathcal{H}_G(V)$-eigenvalues χ appearing in $\mathrm{Hom}_K(V, \pi|_K)$, the corresponding map φ' takes every non-invertible antidominant coweight to 0.

Why does this definition make sense? Supersingular representations should be the analog of supercuspidal representations in the $\mathbb{C}$-valued world. But supercuspidal representations are the ones which are not subquotients of parabolic inductions.

Since $\mathcal{H}_G(V) \to \mathcal{H}_M(V_{N(k)})$ is, as we saw, a localization of $\mathbb{F}_p[X_*(T)_+]$ at certain non-invertible dominant coweights which depend on the shape of M, we get the following.

Lemma 2.4.2. *A representation π is supersingular if and only if any system $\chi : \mathcal{H}_G(V) \to \overline{\mathbb{F}}_p$ appearing in $\mathrm{Hom}_K(V, \pi|_K)$ for any weight V does not factor through*

$$\mathcal{H}_G(V) \to \mathcal{H}_M(V_{N(k)})$$

for any proper parabolic $P = MN < G$.

3. Main Theorem

We will state the main theorem, but first we need to define generalized Steinberg representations.

3.1. **Steinberg.** Let P be a standard parabolic. Then

$$\mathrm{Sp}_P = \frac{\mathrm{Ind}_P^G 1}{\sum_{P \subsetneq Q} \mathrm{Ind}_Q^G 1}$$

Theorem 3.1.1 ([GK14]). *The generalized Steinberg representations are irreducible and admissible, and pairwise non-isomorphic.*

3.2. **Statement.** With these ingredients in place, we can state the main theorem.

Theorem 3.2.1 ([Her11a, Theorem 1.1]). *All irreducible admissible smooth representations of $\mathrm{GL}_n(F)$ are uniquely in the form $\mathrm{Ind}_{P(F)}^{G(F)}(\sigma_1 \otimes \cdots \otimes \sigma_r)$ where*

1. *P is a standard parabolic with Levi $\prod_{i=1}^r \mathrm{GL}_{n_i}$,*
2. *σ_i is an irreducible admissible $\mathrm{GL}_{n_i}(F)$-representation which is either supersingular (in this case $n_i > 1$) or isomorphic to $\mathrm{Sp}_{Q_i} \otimes (\eta_i \circ \det)$ for some smooth character $\eta_i : F^\times \to \overline{\mathbb{F}}_p^\times$ and some standard parabolic Q_i in GL_{n_i}, and*

(3) *$\eta_i \neq \eta_{i+1}$ if σ_i and σ_{i+1} are both twists of generalized Steinberg as in part (2).*

3.3. **Proof.** First note that Ind_P^G preserves admissibility and smoothness. Then the proof splits into two parts: irreducibility of the parabolic inductions, and the classification.

3.3.1. *Irreducibility.* Let $\pi = \mathrm{Ind}_{P(F)}^{G(F)} \sigma$ as in Theorem 3.2.1, where $\sigma := \sigma_1 \otimes \cdots \otimes \sigma_n$.

First of all, it suffices to show that for any weight $f : V \hookrightarrow \pi|_K$, $f(V)$ generates π as a G-representation: if $\pi' \subseteq \pi$ is a nonzero proper subrepresentation then take a weight $f : V \hookrightarrow \pi'|_K$, and note that $\pi = \langle f(V) \rangle \subseteq \pi'$, so $\pi = \pi'$. In fact it suffices to show this for $f : V \hookrightarrow \pi|_K$ which are $\mathcal{H}_G(V)$-eigenvectors.

Now pick such a weight $f \in \mathrm{Hom}_K(V, \pi|_K)_\chi$. But by Frobenius reciprocity

$$\mathrm{Hom}_K(V, \mathrm{Ind}_{P(F)}^{G(F)} \sigma|_K) = \mathrm{Hom}_K(V, \mathrm{Ind}_{P(\mathscr{O})}^{K} \sigma) \cong \mathrm{Hom}_{M(\mathscr{O})}(V_{N(k)}, \sigma)$$

But $\mathcal{H}_G(V)$ acts on the right hand side via the localization $\mathcal{H}_G(V) \hookrightarrow \mathcal{H}_M(V_{N(k)})$, so χ factors through $\mathcal{H}_M(V_{N(k)})$.

Thus by the above, we get a map

$$\text{c-Ind}_{M(\mathscr{O})}^{M} V_{N(k)} \otimes_{\mathcal{H}_M(V_{N(k)}),\chi} \overline{\mathbb{F}}_p \twoheadrightarrow \sigma$$

which is surjective since σ is irreducible since each σ_i is irreducible. Now here comes the main ingredient:

Theorem 3.3.2 ("Parabolic and compact induction are compatible", [Her11a, Theorem 3.1])**.** *Assuming V is M-regular*[2]*, we have a natural isomorphism*

$$\mathrm{Ind}_P^G(\text{c-Ind}_{M(\mathscr{O})}^{M} V_{N(k)} \otimes_{\mathcal{H}_M(V_{N(k)}),\chi} \overline{\mathbb{F}}_p) \xrightarrow{\sim} \text{c-Ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} \overline{\mathbb{F}}_p$$

So why is this useful? Well, exactness of parabolic induction combined with the theorem yields

$$\text{c-Ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} \overline{\mathbb{F}}_p \twoheadrightarrow \mathrm{Ind}_P^G \sigma = \pi$$

But tracing through the definitions, one sees that this map is induced by the original Hecke eigenvector $V \hookrightarrow \pi$. But V generates the left hand side, so $f(V)$ generates π.

Then doing some group theory, an inductive argument shows that π always contains an M-regular weight: this is the part that uses the assumption that $\eta_i \neq \eta_{i+1}$ if σ_i and σ_{i+1} are twists of generalized Steinberg.

3.3.3. *Classification.* We now prove the classification theorem.

Remark 3.3.4 (Ordinary Parts)**.** We'll need the following tool. Emerton defines a functor

$$\mathrm{Ord}_P : \{\text{smooth representations of } G\} \to \{\text{smooth representations of } M\}$$

which is left exact and preserves admissibility. In particular this functor satisfies

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G \sigma, \tau) = \mathrm{Hom}_M(\sigma, \mathrm{Ord}_P \tau)$$

for σ and τ admissible representations: i.e. Ord_P is right adjoint to Ind_P^G on the full subcategory of admissible representations of G.

Now we give a rough-n-sketchy of the classification proof.

We'll proceed by induction. For $n = 1$ there's nothing to prove. So now let $n > 1$. Suppose π is an irreducible admissible smooth representation of G. Pick a weight V for π and a system of Hecke eigenvalues χ corresponding to V. Then π is a quotient of $\text{c-Ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} \overline{\mathbb{F}}_p$. Then the monoid homomorphism $\chi' : X_*(T)_+ \to \overline{\mathbb{F}}_p$

[2]this is a term that I haven't defined: it's a group theoretic condition, but since I'm bad at group theory I'll leave this out.

determines a parabolic subgroup $P = MN$ by looking which dominant coweights χ' takes to 0. If V is M-regular, then Theorem 3.3.2 tells us that

$$\mathrm{Ind}_P^G(\text{c-Ind}_{M(\mathscr{O})}^M(V_{N(k)}) \otimes_{\mathcal{H}_M(V_{N(k)}),\chi} \overline{\mathbb{F}}_p) \twoheadrightarrow \pi$$

Call the thing in parentheses $\pi_{M,\chi}$. Then Emerton shows that the natural map is injective:

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G \pi_{M,\chi}, \pi) \hookrightarrow \mathrm{Hom}_M(\pi_{M,\chi}, \mathrm{Ord}_P \pi)$$

So in particular $\mathrm{Ord}_P \pi \neq 0$. But since $\mathrm{Ord}_P \pi$ is admissible smooth, we can find an irreducible admissible smooth $\tau \hookrightarrow \mathrm{Ord}_P \pi$, which by Frobenius reciprocity gives us a surjection

$$\mathrm{Ind}_P^G \tau \twoheadrightarrow \pi$$

We can decompose $\tau = \tau_1 \otimes \cdots \otimes \tau_r$ over the Levi blocks, and then by induction we can decompose $\tau_i = \mathrm{Ind}_{Q_i}^{M_i} \sigma_i$ into the desired form, so we get $\mathrm{Ind}_P^G \tau = \mathrm{Ind}_{P'}^G \sigma$ for some smaller parabolic P' in the desired form, except that the $\eta_i \neq \eta_{i+1}$ condition is not necessarily satisfied: but in this case, $\mathrm{Ind}_P^G \sigma$ breaks into a bunch of irreducible constituents which are all of the desired form, so one of them is isomorphic to π.

On the other hand if V is not M-regular, then we have our second main ingredient.

Theorem 3.3.5 ("Change of Weight", [Her11a, Theorem 6.10]). *Under a group theoretic condition on χ we have*

$$\text{c-Ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} \overline{\mathbb{F}}_p \cong \text{c-Ind}_K^G V' \otimes_{\mathcal{H}_G(V'),\chi} \overline{\mathbb{F}}_p$$

with V' an M-regular weight.

If the group theoretic condition on χ is satisfied, then we can change the weight and finish the proof.

If the group theoretic condition is not satisfied, then it turns out that V is the trivial weight and in fact we can show that $\pi = 1$, or there exists a proper parabolic P such that $\mathrm{Ord}_P \pi \neq 0$. But note $\mathrm{Ord}_P \pi$ is admissible, so we can find a nonzero irreducible admissible smooth subrepresentation $\sigma \hookrightarrow \mathrm{Ord}_P \pi$, and by Frobenius reciprocity this gives us a surjective map $\mathrm{Ind}_P^G \sigma \twoheadrightarrow \pi$ as before.

4. Supersingular Representations for $\mathrm{GL}_2(\mathbb{Q}_p)$

For $\mathrm{GL}_2(\mathbb{Q}_p)$ the result translates down to:

Theorem 4.0.1 (Barthel-Livné). *The irreducible admissible smooth representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ are*

1. $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ *with* $\chi_1 \neq \chi_2$*,*
2. $\mathrm{Sp}_B \otimes (\eta \otimes \det)$ *for η a smooth character,*
3. $\eta \otimes \det$ *for η a smooth character*
4. *Supersingular representations.*

The first three are easy enough to describe, because smooth characters of $F^\times$ are easy to understand, and we can write down inductions and Steinberg representations.

What about supersingular representations? These were first classified by Breuil in [Bre03].

Theorem 4.0.2 (Breuil). *The supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ are classified by*

$$\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V),\chi} \overline{\mathbb{F}}_p$$

with $\chi' : \mathbb{F}[X_*(T)_+] \to \overline{\mathbb{F}}_p$ taking all strictly dominant weights to 0.

Note that we already showed that if π is irreducible admissible, then it must be a quotient of the above. So it suffices to show that the $\text{c-Ind}_K^G(V) \otimes_{\mathcal{H}_G(V),\chi}$ are themselves irreducible and admissible. This is what Breuil proves. Note that this classification does not extend to GL_n for any $n > 2$.

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