PAŠKŪNAS'S DEFORMATION THEORY

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1. Introduction

Let us remember what we're trying to prove. Assume $p \geq 5$ is a prime number and let $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Fix a finite extension $L/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. Let $Z = \mathbb{Q}_p^\times$ be the center of G, and fix a central character $\zeta : Z \to \mathcal{O}^\times$. Let $\epsilon : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ denote the p-adic cyclotomic character.

Theorem 1.0.1. *Colmez's Montréal functor $\mathbf{V}$ induces a bijection between:*

(1) *absolutely irreducible admissible unitary non-ordinary L-Banach space representations of G with central character ζ, and*

(2) *absolutely irreducible 2-dimensional L-representations of $G_{\mathbb{Q}_p}$ with determinant $\zeta\epsilon$.*

I want to talk about the the strategy of the proof on the Banach space representation side in this introduction, and I want to do this by drawing a parallel with the Galois representation side.

If we start with a semisimple mod p representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$, then we can form the generic fiber of its universal deformation space, which is $X_{\overline{\rho}} = \mathrm{MaxSpec}(R_{\overline{\rho}}[1/p])$: the points in this space give rise to p-adic Galois representations, valued in some finite extension of L. Furthermore, the formal spectrum of the local ring of $\mathrm{Spec}(R_{\overline{\rho}}[1/p])$ at a closed point corresponding to an absolutely irreducible representation recovers the universal deformation space $X_\rho = \mathrm{MaxSpec}(R_\rho[1/p])$. On the other hand you could start with $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(L)$, pick a lattice and reduce and semisimplify to get $\overline{\rho}$, and then find ρ again in $X_{\overline{\rho}}$.

Remark 1.0.2. This is done properly in [EG19], where Emerton and Gee construct a moduli space of Galois representations with $\mathbb{Z}_p$-coefficients, which puts all of the p-adic deformation spaces attached to every mod p representation together into a global object called the *Emerton-Gee stack*.

Paškūnas's idea, and the central idea in the paper, is to try to reconstruct this setup on the Banach space side. The idea is that via Breuil's semisimple mod p Langlands correspondence, each semisimple mod p Galois representation with fixed determinant $\zeta\epsilon$ corresponds to a block in $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})$. So morally what you want to do is deform this block! So you set up a kind of non-commutative deformation theory in this categorical framework, and you get deformation rings, etc.

But then it turns out that Gabriel's theory can also be used to compute these deformation rings! Suppose for a moment that $\mathfrak{B}$ is a supersingular block so it has one absolutely irreducible representation. If we let $\mathsf{Mod}^{\mathrm{lfin}}_{G,\zeta}(k)$ denote the category of ϖ-torsion objects, and take the duals $\mathfrak{C}(\mathcal{O})$ and $\mathfrak{C}(k)$, and take $S_{\mathfrak{B}} = \pi_{\mathfrak{B}}^{\vee}$ and $P_{\mathfrak{B}} \twoheadrightarrow S_{\mathfrak{B}}$ a projective envelope, then $E_{\mathfrak{B}} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}})$ is exactly the universal deformation ring of $S_{\mathfrak{B}}$!

Unfortunately $E_{\mathfrak{B}}$, which is an $\mathcal{O}$-algebra, is not necessarily commutative anymore. But if we invert p we should still expect this to say something about the p-adic representations which have a lattice reducing to $\pi_{\mathfrak{B}}$.

Note we're still in the supersingular case here, for simplicity.

Theorem 1.0.3. *Assuming $Z(E_{\mathfrak{B}})$ is Noetherian and $E_{\mathfrak{B}}$ is a finitely generated module over its center, then there is a bijection between isomorphism classes of*

(1) *irreducible right $E_{\mathfrak{B}}[1/p]$-modules which are finite dimensional over L, and*

(2) *irreducible $\Pi \in \mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)$ such that $\pi_{\mathfrak{B}}$ occurs as a subquotient of $\Theta/\varpi\Theta$ for some open bounded G-lattice $\Theta \subseteq \Pi$.*

In fact it doesn't matter which lattice you take in part (2) of the theorem, since they're all commensurable, and furthermore Paškūnas shows [Paš13, Corollary 5.37] that all irreducible subquotients are actually contained in $\mathfrak{B}$, so in this case are isomorphic to $\pi_{\mathfrak{B}}$. So another way to state this is that

Corollary 1.0.4. $\mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}}$ *is anti-equivalent to the category of irreducible right $E_{\mathfrak{B}}[1/p]$-modules which are finite dimensional over L.*

If you've forgotten what this means, recall that we have two decompositions (which follows from last week's talk)

$$\mathsf{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O}) \cong \prod_{\mathfrak{B}} \mathsf{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})^{\mathfrak{B}}$$

and Pol's talk + Ashvni's talk gives

$$\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L) \cong \prod_{\mathfrak{B}} \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathfrak{B}}$$

The superscript $\mathfrak{B}$ means to take the full subcategory of objects whose irreducible subquotients all live in $\mathfrak{B}$.

Even better, we get the following:

Theorem 1.0.5. *The category $\mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}}$ decomposes as:*

$$\mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}} \cong \bigoplus_{\mathfrak{m} \in \mathrm{MaxSpec}(Z(E_{\mathfrak{B}})[1/p])} \mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}}_{\mathfrak{m}}$$

where $\mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}}_{\mathfrak{m}}$ is the dual of the category of right modules of finite length over the $\mathfrak{m}$-adic completion of $E_{\mathfrak{B}}[1/p]$, or equivalently, the full subcategory of $\mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}}$ consisting of objects which are killed by a power of $\mathfrak{m}$.

Paškūnas actually also shows that $\mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}}_{\mathfrak{m}}$ contains a unique irreducible object $\Pi_{\mathfrak{m}}$.

So right now we have a table of analogies.

Galois "rings"	Galois "spaces"	Automorphic "rings"	Automorphic "spaces"
$\overline{\rho}$	$\bullet$	each $\pi \in \mathfrak{B}$	$\mathfrak{B}$
$R_{\overline{\rho}}$	$\operatorname{Spf} R_{\overline{\rho}}$	$E_{\mathfrak{B}}$	$\mathsf{Mod}^{\mathrm{ladm}}_{G,\zeta}(\mathscr{O})^{\mathfrak{B}}$
$R_{\overline{\rho}}[1/p]$	$X_{\overline{\rho}} = \operatorname{Spec} R_{\overline{\rho}}[1/p]$	$E_{\mathfrak{B}}[1/p]$	$\mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}}$
maximal $\mathfrak{m}_\rho \subseteq R_{\overline{\rho}}[1/p]$	$x \in X_{\overline{\rho}}$	maximal $\mathfrak{m} \subseteq Z(E_{\mathfrak{B}}[1/p])$	unique irred. $\Pi_{\mathfrak{m}} \in \mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}}_{\mathfrak{m}}$
$R_\rho = R_{\overline{\rho}}[1/p]^\wedge_{\mathfrak{m}_\rho}$	$\operatorname{Spf} R_\rho$	$E_{\mathfrak{B}}[1/p]^\wedge_{\mathfrak{m}}$	$\mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}}_{\mathfrak{m}}$

The goal, then, is to turn this analogy into an actual correspondence. To do this, we use Colmez's functor $\mathbf{V}$, which shows in particular the following result:

Theorem 1.0.6. *The functor $\mathbf{V}$ induces an isomorphism*

$$Z(E_{\mathfrak{B}}) \xrightarrow{\sim} R^{\zeta\epsilon}_{\mathrm{tr}\,\overline{\rho}}$$

where $R^{\zeta\epsilon}_{\mathrm{tr}\,\overline{\rho}}$ is the universal pseudodeformation ring of the trace of $\overline{\rho}$ with determinant $\zeta\epsilon$.

In fact in the supersingular case, the corresponding Galois representation is absolutely irreducible, so this pseudodeformation ring is actually just the deformation ring.

So really, the Banach space representations are parametrized by the p-adic Galois representations! And this is what we wanted.

To prove the theorem, we go via deformation theory: if we have deformation theory set up on both sides then we can just use Colmez's functor to show that they match up with each other in an exact way.

2. Deformation Theory

2.1. **Generalities.** First, take any full abelian subcategory $\mathfrak{C} \subseteq \mathsf{Mod}^{\mathrm{pro-aug}}_G(\mathscr{O})$ closed under direct products and subquotients. Assume $\mathfrak{C}^\vee \subseteq \mathsf{Mod}^{\mathrm{lfin}}_G(\mathcal{O})$, which for $\mathrm{GL}_2(\mathbb{Q}_p)$ is $\mathsf{Mod}^{\mathrm{ladm}}_G(\mathcal{O})$.

Now take $S \in \mathfrak{C}$ an irreducible object with $\mathrm{End}_{\mathfrak{C}}(S) = k$. Write $P \twoheadrightarrow S$ for its projective envelope and $E = \mathrm{End}_{\mathfrak{C}}(P)$.

Now assume there exists an object $Q \in \mathfrak{C}$ of finite length such that the following conditions hold:

1. $\mathrm{Hom}_{\mathfrak{C}}(Q, S') = 0$ for $S' \in \mathfrak{C}$ irreducible and $S' \not\cong S$.
2. S occurs as a subquotient of Q with multiplicity 1. Equivalently, $\dim_k \mathrm{Hom}_{\mathfrak{C}}(P, Q) = 1$.
3. $\mathrm{Ext}^1_{\mathfrak{C}}(Q, S') = 0$ for $S' \in \mathfrak{C}$ irreducible and $S' \not\cong S$.
4. $\dim_k \mathrm{Ext}^1_{\mathfrak{C}}(Q, S) < \infty$.
5. $\mathrm{Ext}^2_{\mathfrak{C}}(Q, \mathrm{rad}\, Q) = 0$, where $\mathrm{rad}\, Q$ is the maximal proper subobject of Q.

Remark 2.1.1.

1. Assume $Q = S$. Then (H1) and (H2) are immediate, as well as (H5). So the only nontrivial thing is (H3) and (H4). If these hold, then we want $S = Q$. If not, then we need a bigger Q, and I'll discuss this in the principal series case in a bit. For the supersingular case, this is fine.
2. (H1) (H2) and (H3) actually characterize Q uniquely, up to nonunique isomorphism, so then you just need to check that (H4) and (H5) hold.

We say that Q satisfies $(H, \mathfrak{C})$ if these axioms are satisfied.

A consequence of H1 and H2 is that $\mathrm{Hom}_{\mathfrak{C}}(Q, S)$ is 1-dimensional, and is spanned by a surjection $f : Q \twoheadrightarrow S$. But since P is a projective envelope, there exists a surjection $\theta : P \twoheadrightarrow Q$ such that $f \circ \theta$ is $P \twoheadrightarrow S$.

Assuming these axioms, one can prove, amongst other things:

Proposition 2.1.2. *If M is a right pseudo-compact E-module, then the functor $-\widehat{\otimes}_E P$ is exact, and preserves torsion-freeness.*

2.2. **Deformation Theory.** Now let $\mathfrak{C}(\mathcal{O})$ and $\mathfrak{C}(k)$ denote the duals of $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})$ and the full subcategory of ϖ-torsion modules. Let S be an irreducible object in $\mathfrak{C}(k)$, and suppose Q is an object in $\mathfrak{C}(k)$ which satisfies $(H, \mathfrak{C}(k))$. Let $P \twoheadrightarrow S$ be a projective envelope in $\mathfrak{C}(k)$ as before, and let $E = \mathrm{End}_{\mathfrak{C}(k)}(P)$, which is a local ring with maximal ideal $\mathfrak{m}$.

Now we let $\widetilde{P} \twoheadrightarrow S$ denote the projective envelope in $\mathfrak{C}(\mathcal{O})$ and let $\widetilde{E} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P})$ which has a two-sided ideal $\widetilde{m}$. In fact note that if $M \in \mathfrak{C}(k)$ then

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, M) = \mathrm{Hom}_{\mathfrak{C}(k)}(\widetilde{P}/\varpi\widetilde{P}), M$$

so $\widetilde{P}/\varpi\widetilde{P}$ is projective in $\mathfrak{C}(k)$ and the map $\widetilde{P}/\varpi\widetilde{P} \to S$ is essential, so $P \cong \widetilde{P}/\varpi\widetilde{P}$.

Lemma 2.2.1. *The snake lemma implies that for $A, B \in \mathfrak{C}(k)$, we have an exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathfrak{C}(k)}(A, B) \to \mathrm{Ext}^1_{\mathfrak{C}(\mathcal{O})}(A, B) \to \mathrm{Hom}_{\mathfrak{C}(k)}(A, B)$$

Now what about the deformation theory axioms? Since $\mathfrak{C}(\mathcal{O})$ and $\mathfrak{C}(k)$ have the same irreducible objects, H1 and H2 for $\mathfrak{C}(k)$ immediately imply H1 and H2 for $\mathfrak{C}(\mathcal{O})$. One can use the lemma above along with H1 to show that H3 and H4 for $\mathfrak{C}(k)$ imply H3 and H4 for $\mathfrak{C}(\mathcal{O})$. So what about H5?

Proposition 2.2.2. *If $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}[\varpi], \mathrm{rad}\, Q) = 0$ then H5 for $\mathfrak{C}(k)$ implies H5 for $\mathfrak{C}(\mathcal{O})$.*

For $\mathrm{GL}_2(\mathbb{Q}_p)$, this will always hold.

Definition 2.2.3.

- Let $\mathfrak{A}$ denote the category of finite local Artinian $\mathcal{O}$-algebras $(A, \mathfrak{m}_A)$ such that the image of $\mathcal{O}$ under the structure map $\mathcal{O} \to A$ lies in $Z(A)$, and the structure map induces
$$\mathcal{O}/\varpi\mathcal{O} \xrightarrow{\sim} A/\mathfrak{m}_A$$
- Let $\widehat{\mathfrak{A}}$ denote the category of local $\mathcal{O}$-algebras $(R, \mathfrak{m}_R)$ such that $R/\mathfrak{m}_R^n R \in \mathfrak{A}$ and $R \cong \lim_n R/\mathfrak{m}_R R$. Morphisms are given by
$$\mathrm{Hom}_{\widehat{\mathfrak{A}}}(R, S) = \varprojlim_n \mathrm{Hom}_{\mathfrak{A}}(R/\mathfrak{m}_R^n, S/\mathfrak{m}_S^n)$$

If you're used to commutative deformation theory, this looks pretty similar to the usual definition.

Definition 2.2.4. Let $(A, \mathfrak{m}_A) \in \mathfrak{A}$. A *deformation* of Q to A is a pair (M, η, α) where $M \in \mathfrak{C}(\mathcal{O})$ along with a map $\eta : A \to \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M)$ which makes M into a flat A-module, and $\alpha : k \otimes_A M \cong Q$. We then define a functor $D_Q : \mathfrak{A} \to \mathrm{Set}$ taking

$$(A, \mathfrak{m}_A) \mapsto \{\text{deformations of } Q\} / \cong$$

There is a similar definition for $\widehat{\mathfrak{A}}$, and one particular deformation is given by $\widetilde{P}$. Recall that there is a surjection $\theta : \widetilde{P} \twoheadrightarrow Q$, and in fact this induces

$$\alpha^{univ} : k\widehat{\otimes}_{\widetilde{E}}\widetilde{P} \cong Q.$$

Also note that $\widetilde{E} = \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P)$, so this defines a deformation.

We want to show that this functor is pro-represented by $\widetilde{E}$. Given a $\varphi \in \mathrm{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}, A)$ we can construct a deformation

$$(A \widehat{\otimes}_{\widetilde{E},\varphi} \widetilde{P}, \alpha_\varphi)$$

where $\alpha_\varphi = \alpha_\varphi \circ (A \widehat{\otimes}_{\widetilde{E},\varphi} \widetilde{P} \to k \widehat{\otimes}_{\widetilde{E},\varphi} \widetilde{P})$. Flatness of this deformation as an A-algebra follows from the H axioms as well. This gives a map

$$\mathrm{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}, -) \xrightarrow{\sim} D_Q$$

Theorem 2.2.5. *The above map induces a natural isomorphism*

$$\mathrm{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}, A)/A^\times\text{-}\mathit{conjugacy} \xrightarrow{\sim} D_Q(A)$$

This result once again uses "flatness" of $\widetilde{P}$ over $\widetilde{E}$.

Remark 2.2.6. If we restrict this functor to abelian objects in $\mathfrak{A}$, the resulting functor is isomorphic to $\mathrm{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}^{ab}, -)$.

2.3. **Final Remark.** Lastly, I just want to say that in the supersingular case, $S = Q$. In the generic principal series case, i.e. the block $\left\{\pi_1 = \mathrm{Ind}_B^G \chi_1 \otimes \chi_2\omega^{-1}, \pi_2 = \mathrm{Ind}_B^G \chi_2 \otimes \chi_1\omega^{-1}\right\}$ with $\chi_1\chi_2^{-1} \neq 1, \omega^{\pm 1}$, we end up using $S = \pi_1$ and $Q = \kappa^\vee$ where

$$0 \to \pi_1 \to \kappa \to \pi_2 \to 0$$

is the unique nonsplit extension.

References

[EG19] Matthew Emerton and Toby Gee. Moduli stacks of étale (phi,Gamma)-modules and the existence of crystalline lifts. *arXiv e-prints*, page arXiv:1908.07185, August 2019.

[Paš13] Vytautas Paškūnas. The image of Colmez's Montreal functor. *Publ. Math. Inst. Hautes Études Sci.*, 118:1–191, 2013.