p-ADIC LOCAL LANGLANDS FOR $\mathrm{GL}_2(\mathbb{Q}_p)$ STUDY GROUP

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Contents

1. Introduction

The (classical) local Langlands correspondence for GL_n asserts that there is a bijection

$$\mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_n(\mathbb{Q}_p)) \longleftrightarrow \mathrm{WD}_n(\mathbb{C})$$

where

- $\mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_n(\mathbb{Q}_p))$ is the set of isomorphism classes of smooth irreducible $\mathbb{C}$-valued representations of $\mathrm{GL}_n(\mathbb{Q}_p)$
- $\mathrm{WD}_n(\mathbb{C})$ is the set of isomorphism classes of n-dimensional $\mathbb{C}$-valued Weil–Deligne representations, i.e. all pairs (ρ, N) where ρ is a semisimple representation $\rho\colon W_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathbb{C})$ with open kernel, and N is an endomorphism which satisfies
$$\rho(\sigma)N\rho(\sigma)^{-1} = |\,\mathrm{Art}_{\mathbb{Q}_p}^{-1}(\sigma)|_p N$$

which satisfies certain functorial properties (which determine the correspondence uniquely). We refer to the left-hand side of this correspondence as the *automorphic side* and the right-hand side as the *Galois side*. This conjecture now has many proofs, for example, by Harris–Taylor [HT01] (which has global input) and a purely local proof by Scholze [Sch13].

In fact, there is nothing special about having complex coefficients – the usual topology on $\mathbb{C}$ plays no role in either side of the correspondence – and we could essentially work with any characteristic 0 field, as long as it is equipped with the discrete topology. For example, let $L/\mathbb{Q}_p$ be a finite extension. Then we have the following lemma:

Lemma 1.0.1. *The category of Weil–Deligne representations over L is equivalent to the category of $(\varphi, N, G_{\mathbb{Q}_p})$-modules, i.e. the category of finite free $\mathbb{Q}_p^{\mathrm{ur}} \otimes L$-modules D with*

- *A $\mathbb{Q}_p^{\mathrm{ur}}$-semilinear Frobenius $\varphi\colon D \to D$*
- *A $\mathbb{Q}_p^{\mathrm{ur}} \otimes L$-linear endomorphism $N\colon D \to D$ satisfying*
$$N\varphi = p\varphi N$$
- *An action of $G_{\mathbb{Q}_p}$ that is $\mathbb{Q}_p^{\mathrm{ur}}$-semilinear and factors through a finite quotient.*

This almost looks like a common object in p-adic Hodge theory, except there is one thing missing – there is no filtration on D. Essentially this is because we have equipped L with the discrete topology. From now on we equip L with the *p-adic* topology.

To obtain a p-adic refinement of the classical local Langlands correspondence, we enrich the $(\varphi, N, G_{\mathbb{Q}_p})$-module with a filtration such that the module is *weakly admissible*. Then, by the work of Colmez, Fontaine, Berger and André–Kedlaya–Mebkhout, weakly admissible filtered $(\varphi, N, G_{\mathbb{Q}_p})$-modules are the same thing as de Rham representations of $G_{\mathbb{Q}_p}$. This is the candidate of the Galois side of the p-adic correspondence. But what should the candidate on the automorphic side be?

It turns out that the data of the Hodge–Tate weights of a (regular) filtration are in bijection with algebraic representations of GL_n. For example, for $n = 2$, the Hodge–Tate weights[1] $\{a, a+k\}$ correspond to the algebraic representation $\mathrm{Sym}^{k-1} \otimes \det^a$. Let D be the $(\varphi, N, G_{\mathbb{Q}_p})$-module associated with π via the classical local Langlands correspondence, and suppose that we can equip D with the filtration corresponding to an algebraic representation V such that D inherits the structure of a weakly admissible filtered $(\varphi, N, G_{\mathbb{Q}_p})$-module. Then the corresponding object on the automorphic side should be an appropriate p-adic completion of $\pi \otimes V$.

Example 1.0.2. Let $n = 1$ and $L = \mathbb{Q}_p^\times$. Then the classical local Langlands correspondence is just local class field theory. In particular, the norm character

$$|\cdot| \colon \mathbb{Q}_p^\times \to \mathbb{Q}^\times \subset \mathbb{Q}_p^\times$$

corresponds to the (φ, N)-module $D = \mathbb{Q}_p \cdot e$ where $\varphi(e) = p^{-1}e$ and $N = 0$ (the Galois action is trivial so we have suppressed it from the notation). We can equip D with the filtration satisfying $F^{-1}D = D$ and $F^0 D = \{0\}$, which turns D into a weakly admissible filtered φ-module. This corresponds to the crystalline representation:

$$\chi_{\mathrm{cyc}} \colon G_{\mathbb{Q}_p^\times} \to \mathbb{Q}_p^\times$$

given by the cyclotomic character.

On the automorphic side, the corresponding representation is the "p-adic completion" of $|\cdot| \otimes V$, where V is the standard representation of GL_1. Or to put it another way, under the p-adic local langlands correspondence, we have

$$\begin{pmatrix} \mathbb{Q}_p^\times \to L^\times \\ x \mapsto x|x| \end{pmatrix} \longleftrightarrow \text{ cyclotomic character}$$

Unfortunately, the only precise statement of the p-adic local Langlands correspondence is for $n \leq 2$. For the case $n = 2$, the correspondence is now a theorem due to Colmez–Dospinescu–Paškūnas:

Theorem 1.0.3 ([CDP14]). *There is a bijection between the isomorphism classes of:*

- *absolutely irreducible, non-ordinary, admissible, unitary L-Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$*
- *2-dimensional absolutely irreducible continuous L-representations of $G_{\mathbb{Q}_p}$*

which is compatible with local class field theory.

Remark 1.0.4. All the remaining cases not appearing in Theorem 1.0.3 essentially follow from local class field theory. This is in analogy with the proof of the classical local Langlands correspondence for GL_n; you first show it for supersingular representations and then use induction. Furthermore, this correspondence restricts to one between certain completions of $\pi \otimes V$ and de Rham representations, as described above.

The goal of this study group is to understand the proof of Theorem 1.0.3.

[1] We normalise so that the cyclotomic character has Hodge–Tate weight 1.

1.1. **Outline of the study group.** The starting point of the proof of Theorem 1.0.3 is the Montreal functor constructed by Colmez, which is an exact, covariant functor from well-behaved Banach representations to continuous Galois representations. In [Paš13], Paškūnas investigates the image of this functor and, for $p \geq 5$, shows that this functor has all the required properties needed to be called a p-adic Langlands correspondence. It is this paper that we will follow for the study group. The remaining cases ($p = 2, 3$) are considered in [CDP14].

Since we are essentially trying to take a "p-adic completion" of the classical local Langlands correspondence, it is natural to first study the Montreal functor on mod p representations, then consider "deformations" on both sides of the mod p correspondence and show the corresponding "deformation rings" are isomorphic. On the Galois side, we consider the deformation ring parameterising pseudo-deformations of mod p Galois representations. On the automorphic side, we consider the injective envelope of the mod p representation and study the centre of its endomorphism ring.

The latter can (in an appropriate sense) be thought of as a localised Hecke algebra, and then the goal is to prove a (local version of a) R=T theorem. This analogy is pushed further in the work of Caraiani–Emerton–Gee–Geraghty–Paškūnas–Shin [CEG+13] where a conjectural description of a p-adic local Langlands correspondence for GL_n is given, via patching. If time permits, we could possibly look at this paper too.

2. Schedule of Talks

(1) *Abi*. **Mod p and integral p-adic representations of $\mathrm{GL}_n(\mathbb{Q}_p)$**: This talk will discuss the general theory of representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ with coefficients in complete local Noetherian $\mathbb{Z}_p$-algebras with finite residue field, following [Hera], [Herb], [Paš13], and [Eme10a]. In particular, the main goal is to understand in as much detail as possible the diagram on page 7 of [Eme10a], so try to cover all of the material until that diagram.

(2) *Ashwin*. **Classification of irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$** The goal of this talk is to state and prove (as much as possible) the classification given by Herzig in [Her11] following work of Barthel-Livné in [BL94] and [BL95] and [Bre03]. The work of Vignéras in [Vig04] might be helpful as well.

(3) *Andy*. **Ext groups between irreducible representations**: We now wish to understand the decomposition of $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})$ into blocks, where ζ is a fixed central character and "lfin" means locally finite. By definition we need to understand the Ext^1 groups between irreducible representations. This is done in [Paš10], [BP12], [Col10], [Eme10b]. For this talk, state the vanishing results in all cases, and sketch as many of the proofs as you can. Once we've gone through the second talk we will add here precise locations of the statements and proofs in the various cases.

(4) *Ashvni*. L-**Banach space representations**: For this talk we wish to cover the representation theory when p is inverted. For this we follow [Paš13, Section 4].

(5) *Pol*. **Gabriel's Theory**: Gabriel has a general theory of block decomposition of locally finite categories. It would be useful to go over this, and then show that our situation fits into his framework.

(6) *Ashwin*. **Deformation Theory**: The goal of this talk is to cover [Paš13, Section 2 and 3] where Paskunas introduces his noncommutative deformation theory and studies the "Bernstein centers" of the categories of mod p and p-adic representations. If time permits finish the end of [Paš13, Section 4].

(7) *Sam*. **Galois Representations** This will cover mod p and p-adic Galois representations, as well as the formalism of (φ, Γ)-modules over the Robba ring.

(8) *Waqar*. **Montréal Functor** The goal of this talk is to cover the Montréal functor. We'll give more details in due course.

(9) ???

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