DEFORMATION THEORY FOR SUPERSINGULAR REPRESENTATIONS

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Contents

1. Introduction

So far we've explained a lot of the setup that one needs to build towards the main theorem. In this final lecture, we will discuss one of the key ingredients, which compares the deformation theory of an irreducible mod p representation (or its dual) and the deformation theory of the association mod p Galois representation, via Colmez's Montréal functor.

For simplicity, we will treat the supersingular case. This is by far the least technical case: the deformation theory on the automorphic side simplifies considerably, and one ends up just needing to compute the dimensions of certain Ext groups to conclude.

Breuil proved that every supersingular representation is of the form

$$\pi(r, 0, \chi) := (\chi \circ \det) \otimes \text{c-Ind}_{KZ}^{G} \operatorname{Sym}^r k^2/(T)$$

where $\chi : \mathbb{Q}_p^\times \to k^\times$ is a smooth character and $0 \leq r \leq p-1$: here T is the Hecke operator generating the Hecke algebra over k. The case $0 < r < p-1$ is called the *regular case*, and the case $r = 0$ (equivalently, $r = p-1$) is called the *Iwahori* case.

2. Montréal Functor and Deformation Theory

In Waqar's talk we discussed the (exact) Montréal functor

$$\mathbf{V} : \mathrm{Mod}^{\mathrm{fin}}_{G,\zeta}(\mathcal{O}) \to \mathrm{Mod}^{\mathrm{fin}}_{G_{\mathbb{Q}_p}}(\mathcal{O})$$

which extends to a functor $\check{\mathbf{V}} : \mathfrak{C}(\mathcal{O}) \to \mathrm{Rep}_{G_{\mathbb{Q}_p}}(\mathcal{O})$ via

$$M = \varprojlim M_i \mapsto \varprojlim V(M_i^\vee)^\vee(\epsilon\zeta)$$

where the limit is taken over the finite length quotients of M. The functor $\mathbf{V}$ satisfies

$$\mathbf{V}(\pi(r, 0, \chi)) = \chi \otimes \operatorname{Ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p} \omega_2$$

which is absolutely irreducible. In the above statement, ω_2 is the character of $G_{\mathbb{Q}_{p^2}}$ corresponding to $x \mapsto x|x| \bmod \varpi$ under local class field theory.

We now recall the formalism of deformation theory introduced in my last talk. We fix some continuous character $\zeta : Z \to \mathcal{O}^\times$ and then $\mathfrak{C}(\mathcal{O})$ and $\mathfrak{C}(k)$ are, as before, the duals of the categories $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})$ and $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(k)$. We let $S = \pi^\vee \in \mathbf{C}(k)$, and then in order to deform it we wanted an object $Q \in \mathfrak{C}(k)$ satisfying

- (H1) $\mathrm{Hom}_{\mathfrak{C}(k)}(Q, S') = 0$ for $S' \not\cong S$ irreducible in $\mathfrak{C}(k)$
- (H2) S occurs as a subquotient of Q with multiplicity 1
- (H3) $\mathrm{Ext}^1_{\mathfrak{C}(k)}(Q, S') = 0$ for $S' \not\cong S$ irreducible in $\mathfrak{C}(k)$
- (H4) $\dim_k \mathrm{Ext}_{\mathfrak{C}(k)}(Q, S) < \infty$
- (H5) $\mathrm{Ext}^2_{\mathfrak{C}(k)}(Q, \mathrm{rad}\, Q) = 0$

Why do we need this formalism? Recall that in general we're deforming entire blocks, not necessarily single mod p representation. For instance, for the block $\left\{\mathrm{Ind}_P^G \chi_1 \otimes \chi_2\omega^{-1}, \mathrm{Ind}_P^G \chi_2 \otimes \chi_1\omega^{-1}\right\}$ (with $\chi_1\chi_2^{-1} \neq 1, \omega^{\pm}$) we take $Q = \kappa^\vee$ where κ is the unique non-split extension of the two representations in the block, and it ends up satisfying the axioms (H1)-(H5).

But since π is supersingular (and is the only representation in its block), we can actually take $Q = S$. Then (H1), (H2) and (H5) are immediate.

Lemma 2.0.1. *(H3) and (H4) are satisfied for S.*

Proof. (H3) was proven by Paškūnas in [Paš10], and was covered in Andy's talk. (H4) was also proven in *op. cit*[1] by Paškūnas, who showed that $\mathrm{Ext}^1_{\mathfrak{C}(k)}(S, S) = 3$. □

There is a condition (H0) which holds vacuously in the supersingular case, which ensures that (H1)-(H5) actually hold with $\mathfrak{C}(\mathcal{O})$ in place of $\mathfrak{C}(k)$ (in the other cases you have to check something).

Then we let $\mathfrak{A}$ denote the category of possibly non-commutative local Artinian $\mathcal{O}$-algebras $(A, \mathfrak{m}_A)$ with $\sigma : \mathcal{O} \to A$ factoring through $Z(A)$ and inducing an isomorphism $k \to A/\mathfrak{m}_A$, with the evident morphisms. We let $\widehat{\mathfrak{A}}$ denote the category of local $\mathcal{O}$-algebras R such that $R/\mathfrak{m}_R^n \in \mathfrak{A}$ for $n > 0$, and $R \xrightarrow{\sim} \varprojlim_n R/\mathfrak{m}_R^n$.

Definition 2.0.2.

1. A deformation of S to $A \in \mathfrak{A}$ is a pair (M, α) where $M \in \mathfrak{C}(\mathcal{O})$ comes with a map of $\mathcal{O}$-algebras $A \to \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M)$ making M A-flat, and $\alpha : k\widehat{\otimes}_A M \xrightarrow{\sim} Q$ is an isomorphism in $\mathfrak{C}(k)$.
2. The deformation functor of S is
$$\begin{aligned} D_S : \mathfrak{A} &\to \mathrm{Set} \\ A &\mapsto \{\text{deformations of } S \text{ to } A\} / \simeq \end{aligned}$$
taken up to the evident notion of isomorphism of deformations. We let D_S^{ab} be the restriction of this functor to the full subcategory $\mathfrak{A}^{\mathrm{ab}}$ of commutative rings in $\mathfrak{A}$.

Paškūnas shows that if $\widetilde{P} \twoheadrightarrow S$ is a projective envelope in $\mathfrak{C}(\mathcal{O})$, then D_S is represented by $\widetilde{E} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P})$ with universal object $(\widetilde{P}, \alpha_{\widetilde{P}})$ (one has to do some work to show that in fact $\widetilde{P}$ is a deformation of S: also, you may object that $\widetilde{E}$ is not isomorphic to any object in $\mathfrak{A}$, but in fact it lives in the bigger category $\widehat{\mathfrak{A}}$, so one usually says that D_S is pro-represented by $\widetilde{E}$).

More precisely the set $D_S(A)$ is in natural bijection with the $A^\times$-conjugacy classes of $\mathrm{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}, A)$ for every $A \in \mathfrak{A}$.

[1] did I use this correctly? I don't speak Latin.

2.1. **Interlude on Galois deformation theory.** Let's recall what a Galois deformation ring is, for everyone's convenience.

In both cases, we start with an object defined over k and we deform it to local Artinian $\mathcal{O}$-algebras with residue field k. This means that we look for objects living over such algebras which recover our object defined over k once we base change back down to k.

Geometrically, one can think of this as follows: the object defined over k is a k-point of a moduli scheme or stack, and the deformation theory tells you algebraic information about the infinitesimal neighborhoods of that point. For example, looking at lifts to $k[x]/x^2$ gives you the tangent space to our k-point inside the moduli space.

So to deform a Galois representation we work as follows. Fix a continuous representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}(V)$ for some finite dimensional k-vector space V: here we've given k and $\mathrm{GL}(V)$ the discrete topology. Then define a functor

$$D_{\overline{\rho}} : \mathfrak{A}^{\mathrm{ab}} \to \mathsf{Set}$$

which takes $A \in \mathfrak{A}^{\mathrm{ab}}$ to the set of isomorphism classes of pairs (M, φ_M) where M is a finite free A-module equipped with a continuous A-linear action of $G_{\mathbb{Q}_p}$ and $\varphi_M : M \otimes_A k \xrightarrow{\sim} V$ is an isomorphism of $G_{\mathbb{Q}_p}$-representations.

Proposition 2.1.1. *If $\mathrm{End}_{k[G_{\mathbb{Q}_p}]}(V) = k$, then $D_{\overline{\rho}}$ is pro-represented by a complete local Noetherian ring $R_{\overline{\rho}}$ with residue field k. In other words, there is a natural isomorphism*

$$\mathrm{Hom}_{\widehat{\mathfrak{A}^{\mathrm{ab}}}}(R_{\overline{\rho}}, A) \xrightarrow{\sim} D_{\overline{\rho}}(A)$$

In particular, there is a universal deformation $\rho^{\mathrm{univ}} : G_{\mathbb{Q}_p} \to \mathrm{GL}(M^{\mathrm{univ}}))$ where M^{univ} is a finite free $R_{\overline{\rho}}$-module (of rank $\dim_k V$), such that a map $f : R_{\overline{\rho}} \to A$ corresponds to the deformation $\mathrm{GL}_n(f) \circ \rho^{\mathrm{univ}}$.

If $\mathrm{End}_{k[G_{\mathbb{Q}_p}]}(V)$ contains extra endomorphisms, there is a way to rigidify this functor and make it representable by "framing" it, but since in the supersingular case the mod p Galois representation is already absolutely irreducible, we are in the situation of the above Proposition with $\overline{\rho} = \mathbf{V}(\pi)$ so we don't need to do this.

Note also that the fact that $R_{\overline{\rho}}$ is Noetherian is a consequence of the fact that $G_{\mathbb{Q}_p}$ satisfies what's called a "p-finiteness condition", which for instance is not satisfied for $G_{\mathbb{Q}}$ (but is satisfied for $G_{\mathbb{Q},S}$, the Galois group of the maximal extension unramified outside of a finite set of finite places S of $\mathbb{Q}$).

You can also modify $D_{\overline{\rho}}$ slightly to only consider deformations with a fixed determinant. If $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ is a continuous character, then we let

$$D^{\psi}_{\overline{\rho}}(A) = \{(M, \varphi_M) \in D_{\overline{\rho}}(A) \mid \det(g|M) = \psi(g) \text{ for all } g \in G\}$$

Abusively, we've written $\psi(g)$ to mean its image under the map $\mathcal{O} \to A$. If $D_{\overline{\rho}}$ is representable, then $D^{\psi}_{\overline{\rho}}$ is as well, and the inclusion $D^{\psi}_{\overline{\rho}} \hookrightarrow D_{\overline{\rho}}$ gives rise to a quotient map

$$R_{\overline{\rho}} \twoheadrightarrow R^{\psi}_{\overline{\rho}}$$

(one can show that an injection of representable functors on $\mathfrak{A}$ gives rise to a surjection on the corresponding map of rings)

Lastly, one can ask about ring theoretic properties of $R_{\overline{\rho}}$ or $R^{\psi}_{\overline{\rho}}$. These are summarized in the following Proposition. Let $\mathrm{ad}\,\overline{\rho}$ denote the adjoint representation: i.e., 2 by 2 matrices with the conjugation action $G_{\mathbb{Q}_p}$ via ρ and let $\mathrm{ad}^0\,\overline{\rho}$ denote the subrepresentation consisting of trace zero matrices.

Proposition 2.1.2. *Let $h := \dim_k H^1(G_{\mathbb{Q}_p}, \mathrm{ad}\,\overline{\rho})$. Then there is a surjection*

$$\mathcal{O}[[x_1, \ldots, x_h]] \twoheadrightarrow R_{\overline{\rho}}$$

such that the minimal number of generators of the kernel is bounded above by $\dim_k H^2(G_{\mathbb{Q}_p}, \mathrm{ad}\,\overline{\rho})$. In particular, if $H^2(G_{\mathbb{Q}_p}, \mathrm{ad}\,\overline{\rho})$ then the above surjection is an isomorphism, and $R_{\overline{\rho}}$ is formally smooth.

Let $h := \dim_k H^1(G_{\mathbb{Q}_p}, \mathrm{ad}^0 \overline{\rho})$*. Then there is a surjection*

$$\mathcal{O}[[x_1, \dots, x_h]] \twoheadrightarrow R^{\psi}_{\overline{\rho}}$$

such that the minimal number of generators of the kernel is bounded above by $\dim_k H^2(G_{\mathbb{Q}_p}, \mathrm{ad}^0 \overline{\rho})$*. In particular, if* $H^2(G_{\mathbb{Q}_p}, \mathrm{ad}^0 \overline{\rho})$ *then the above surjection is an isomorphism, and* $R^{\psi}_{\overline{\rho}}$ *is formally smooth.*

Note that $H^1(G_{\mathbb{Q}_p}, \mathrm{ad}\,\overline{\rho})$ is actually the tangent space to D_S, i.e. is k-linearly isomorphic to $D_S(k[x]/x^2)$ (there is a natural k-vector space structure you can put on this set). So the dimension of the tangent space puts an upper bound on the number of generators of $R_{\overline{\rho}}$ over $\mathcal{O}$.

2.2. **End of interlude.** Our goal is to show that $R^{\epsilon\zeta}_{\mathbf{V}(\pi)} \cong \widetilde{E}$. Here $R^{\epsilon\zeta}_{\mathbf{V}(\pi)}$ is the universal Galois deformation ring of $\mathbf{V}(\pi) \cong \check{\mathbf{V}}(S)$ with fixed determinant $\epsilon\zeta$, as defined above.

First we can show that there is a map of deformation functors.

Proposition 2.2.1. *The functor* $\mathbf{V}$ *induces a morphism of functors*

$$D^{\mathrm{ab}}_S \to D^{\mathrm{ab}}_{\check{\mathbf{V}}(S)}$$

Proof sketch. One first shows that $\check{\mathbf{V}}(m \otimes_A M) \cong m \otimes_A \check{\mathbf{V}}(M)$ for $M \in \mathfrak{C}(\mathcal{O})$ and m a finitely generated A-module. This implies that for any deformation S_A of S to $A \in \mathfrak{A}$, $\check{\mathbf{V}}(S_A)$ is A-flat (recall S_A is A-flat) hence free over A since A is local Artinian. Furthermore we have the isomorphism

$$k \otimes_A \check{\mathbf{V}}(S_A) \cong \check{\mathbf{V}}(k \otimes_A S_A) \cong \check{\mathbf{V}}(S)$$

so by Nakayama's lemma, $\check{\mathbf{V}}(S_A)$ has rank $\dim_k \check{\mathbf{V}}(S)$. Thus it's a well-defined deformation of $\check{\mathbf{V}}(S) \cong \mathbf{V}(\pi)$. □

This gives us a map of rings $R_{\check{\mathbf{V}}(S)} \to \widetilde{E}^{\mathrm{ab}}$ in the other direction.

3. $R = E$ Theorem

A standard fact in deformation theory: if a map of deformation problems induces an injection on the corresponding tangent spaces, then the corresponding map of rings is a surjection. In other words, if

$$\mathrm{Ext}^1_{\mathfrak{C}(k)}(S, S) \to \mathrm{Ext}^1_{k[G_{\mathbf{Q}_p}]}(\check{\mathbf{V}}(S), \check{\mathbf{V}}(S))$$

is injective then $R_{\check{\mathbf{V}}(S)} \to \widetilde{E}^{\mathrm{ab}}$ is surjective. A theorem of Colmez says that this is indeed the case.

In fact this factors as

$$R_{\check{\mathbf{V}}(S)} \twoheadrightarrow R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)} \twoheadrightarrow \widetilde{E}^{\mathrm{ab}}$$

Or in other words, if you take a deformation of S and apply $\check{\mathbf{V}}$, the resulting deformation of $\check{\mathbf{V}}(S)$ has determinant $\epsilon\zeta$. This is because deformations of S have, by definition, central character ζ, and the Montréal functor $\check{\mathbf{V}}$ takes central characters to determinants, twisted by the cyclotomic character.

Lemma 3.0.1. *If* $\mathfrak{m} \subseteq R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}[1/p]$ *is a maximal ideal, then the corresponding* p*-adic representation*

$$\rho_{\mathfrak{m}} : G_{\mathbf{Q}_p} \xrightarrow{\rho^{\mathrm{univ}}} \mathrm{GL}_2(R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}) \to \mathrm{GL}_2(\kappa(\mathfrak{m}))$$

is absolutely irreducible, where $\kappa(\mathfrak{m})$ *is the* p*-adic field* $R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}[1/p]/\mathfrak{m}$*.*

Proof. If not, then there exists some finite extension $\kappa/\kappa(\mathfrak{m})$ over which $\rho_{\mathfrak{m}}$ becomes reducible, and then picking an integral lattice and reducing mod ϖ one gets a reducible mod ϖ representation defined over some finite extension of k. But this is just the extension of scalars of $\mathbf{V}(\pi)$ to a bigger field, which is still irreducible, so we get a contradiction. □

Lemma 3.0.2. *If* $\mathfrak{m} \subseteq R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}[1/p]$ *is a maximal ideal, then there exists a map* $x : \widetilde{E} \to \mathcal{O}_{\kappa(\mathfrak{m})}$ *such that*

$$\kappa(\mathfrak{m}) \otimes_{\widetilde{E}} \check{\mathbf{V}}(\widetilde{P}) \cong \kappa(\mathfrak{m}) \otimes_{R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}} \rho^{\mathrm{univ},\epsilon\zeta}$$

Proof. This is essentially due to Kisin: see [Paš13, Proposition 5.56] and [Kis10, Corollary 2.3.8]. □

Then since $\check{\mathbf{V}}(S)$ has only scalar endomorphisms (it's absolutely irreducible) there is a unique $G_{\mathbf{Q}_p}$-invariant $\mathcal{O}_{\kappa(\mathfrak{m})}$-lattice inside $\kappa(\mathfrak{m}) \otimes_{\widetilde{E}} \check{\mathbf{V}}(S)$ which reduces to $\check{\mathbf{V}}(S)$ mod $\varpi_{\kappa(\mathfrak{m})}$. This means that $\mathcal{O}_{\kappa(\mathfrak{m})} \otimes_{R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}} \rho^{\mathrm{univ},\epsilon\zeta}$ and $\mathcal{O}_{\kappa(\mathfrak{m})} \otimes_{\widetilde{E}} \check{\mathbf{V}}(\widetilde{P})$ are the same deformation. Therefore, the map $R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)} \to \kappa(\mathfrak{m})$ factors as

$$R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)} \twoheadrightarrow \widetilde{E}^{\mathrm{ab}} \xrightarrow{x} \kappa(\mathfrak{m})$$

Then by the Chinese remainder theorem and the fact that $R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}[1/p]$ is Jacobson, we get maps

$$R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)} \twoheadrightarrow \widetilde{E}^{\mathrm{ab}} \to \mathrm{im}(R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)} \to R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}[1/p]/\sqrt{0})$$

But now comes the nice part: one can compute actually that $H^2(G_{\mathbb{Q}_p}, \mathrm{ad}^0 \check{\mathbf{V}}(S)) = 0$ so that actually $R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}$ is formally smooth and thus really we have that the composition

$$R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)} \twoheadrightarrow \widetilde{E}^{\mathrm{ab}} \to R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}$$

is the identity, so since the first map is surjective, it's actually an isomorphism. Actually a tangent space computation shows that $\dim_k H^1(G_{\mathbb{Q}_p}, \mathrm{ad}^0 \overline{\rho}) = 3$, and so $R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)} \cong \mathcal{O}[[x_1, x_2, x_3]]$.

It remains to show that actually $\widetilde{E}$ itself is commutative, which is done using some commutative algebra and more of Paškūnas's results on Ext groups for irreducible representations. See [Paš13, Proposition 6.3] for the details.

4. A consequence for p-adic Langlands

Recall that the category $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})^{\mathfrak{B}}$ is the full subcategory of $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})$ consisting of objects with every irreducible subquotient isomorphic to π. Note that $\mathfrak{B} = \{\pi\}$ since we're in the supersingular case.

Then the category $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})^{\mathfrak{B}}$ is anti-equivalent to the category of compact $R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}$-modules and a corollary is the decomposition

$$\mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}} \cong \bigoplus_{\mathfrak{m} \subseteq \mathrm{MaxSpec}\, R^{\epsilon\zeta}_{\check{\mathbf{V}}(S)}[1/p]} \mathrm{Ban}^{\mathrm{adm,fl}}_{G,\zeta}(L)^{\mathfrak{B}}_{\mathfrak{m}}$$

References

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