p-ADIC MODULAR FORMS

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Notes taken by Ashwin Iyengar[1] *and have not been checked by the speaker. Any errors are due to me.*

Contents

1. Talk I

This course is about p-adic modular forms. We will talk about:

(1) Ordinary p-adic forms, and ordinary families of modular forms.

(2) Overconvergent p-adic modular forms, and the theory of Coleman and Mazur's eigencurve (but we explain using Emerton's completed cohomology)

(3) Applications to the infinite fern of Gouvea-Mazur.

1.1. Geometric Modular Forms.

First we define the modular curve over $\mathbf{Z}_p$. Throughout the course, p is a fixed prime, and $N \geq 5$ prime to p is an integer (the "base level" or "tame level"). Then the modular curve $Y_1(N)/\mathbf{Z}_p$ is a scheme representing the functor $\mathsf{Sch}_{\mathbf{Z}_p} \to \mathsf{Set}$ taking

$$S \mapsto \{(E/S \text{ elliptic curve}, \alpha : \mu_N \hookrightarrow E[N])\} /\text{isomorphism}$$

Here recall that:

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(1) an elliptic curve $E \to S$ is a proper and smooth morphism with geometrically connected fibers, along with a zero section $S \to E$, and

(2) $\mu_N = \ker(\mathbf{G}_m \xrightarrow{N} \mathbf{G}_m)$ and $E[N] = \ker(E \xrightarrow{N} E)$

This functor is representable by $Y_1(N)$, and is a smooth affine curve over $\mathbf{Z}_p$ with geometrically connected fibers.

Why do we call this a modular curve? If we fix an embedding $\mathbf{Z}_p \hookrightarrow \mathbf{C}$ and $\zeta_n \in \mu_N(\mathbf{C})$, then the usual complex uniformization of elliptic curves says that

$$Y_1(N)(\mathbf{C}) \cong \Gamma_1(N)\backslash\mathbf{H}$$

where as usual,

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

We have a finite flat map $Y_1(N) \to \mathbf{A}^1_{\mathbf{Z}_p}$, taking $(E, \alpha) \mapsto j(E)$ (the j-invariant), of degree $[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_1(N)]$. Furthermore, this map is étale outside the locus $j - 1728 = 0$, and thus $Y_1(N)$ is normal outside this locus. So to compactify the modular curve, we take the normalization of $\mathbf{P}^1_{\mathbf{Z}_p}$ in $Y_1(N)$ (see Section 8.6 of [KM85] for the details, or [Sta19, Tag 0BAK]), and get:

$$\begin{array}{ccc} Y_1(N) & \longrightarrow & X_1(N) \\ \downarrow & & \downarrow \\ \mathbf{A}^1_{\mathbf{Z}_p} & \longrightarrow & \mathbf{P}^1_{\mathbf{Z}_p} \end{array}$$

Then $X_1(N) \to \mathbf{Z}_p$ is a proper smooth curve. On the other hand, you can't immediately extend the universal family of elliptic curves $E \to Y_1(N)$ to $X_1(N)$. But for each $M \geq 1$, we have $E[M] = \ker(E \xrightarrow{M} E)$, which is a finite flat group scheme over $Y_1(N)$ (not necessarily étale because M is not necessarily prime to p). But there is a unique extension of $E[M]$ to a finite flat group scheme over $X_1(N)$ such that (letting $D = X_1(N)\backslash Y_1(N)$) on $\widehat{X_1(N)}_D$, we have an extension

$$0 \to \mu_M \to E[M] \to \underline{\mathbf{Z}/M\mathbf{Z}}_{X_1(N)} \to 0,$$

where $\underline{\mathbf{Z}/M\mathbf{Z}}_{X_1(N)}$ is the constant group scheme.

Now we want to define a coherent sheaf on $X_1(N)$, in order to define our forms.

Fact 1.1.1. *There exists a coherent sheaf $\omega = \omega_{E/Y_1(N)}$ on $X_1(N)$ which restricts to $\pi_*\Omega_{E/Y_1(N)}$ over $Y_1(N)$ and around the cusps you have to compare the restriction of the elliptic curve to the Tate curve $Tate(q^n)$ and construct some sheaf there (still need to work this out).*

Definition 1.1.1. Now if $k \in \mathbf{Z}$ and A is some $\mathbf{Z}_p$-algebra, then a **modular form** of weight k, level N and coefficients in A is an element of

$$M_k(N, A) := H^0(X_1(N)_A, \omega^{\otimes k})$$

Remark 1.1.1.

(1) If $A = \mathbf{C}$ (for some map $\mathbf{Z}_p \hookrightarrow \mathbf{C}$), this recovers the usual $\mathbf{C}$-vector space of modular forms.

(2) If $k \geq 2$, and $A \to B$, then we have base change:

$$M_k(N, B) \cong M_k(N, A) \otimes_A B.$$

1.2. **p-adic modular forms.** The usual construction, originally due to Serre, is to reduce q-expansions mod p^n and take an inverse limit, but we want to do something more geometric.

For $m \geq 1$, we let

$$X_1(N)_m = X_1(N) \times_{\operatorname{Spec} \mathbf{Z}_p} \operatorname{Spec} \mathbf{Z}/p^m\mathbf{Z}$$

be the reduction mod p^m of the modular curve, and we let

$$X_1(N)_m^\circ = X_1(N)_m \setminus \{\text{supersingular closed points}\},$$

where we say a closed point (E, α) is supersingular if $E \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ is a supersingular elliptic curve: this forms a finite set of points. Then $X_1(M)_m^\circ$ is an open and affine subscheme of $X_1(N)_m$.

Using these, we can construct the Igusa tower. Note $E[p^r]$ is a finite flat group scheme, and we have a Frobenius (defined over $X_1(N)_1$)

$$F : E[p^r] \to E[p^r]^{(p)}.$$

We can iterate this Frobenius:

$$F^r : E[p^r] \to E[p^r]^{(p)} \to \cdots \to E[p^r]^{(p^r)}.$$

This is a finite flat morphism, and we let $V^r = (F^r)^D$ denote the Verschiebung, where here D denotes the Cartier dual and F^r is the Frobenius on $E[p^r]^D$. If $E_0/\overline{\mathbf{F}_p}$ is an elliptic curve, then E_0 is ordinary (non-supersingular) if and only if $\ker V$ is étale, if and only if $\ker V^r$ is étale. So therefore, $\ker(V^r : E[p^r]^{(p^r)} \to E[p^r])$ is a finite étale group scheme of $X_1(N)_1^\circ \subseteq X_1(N)_1$, essentially because we removed the supersingular points.

Then $\ker(V^r)$ specialized at a geometric point of $X_1(N)_1^\circ$ is $\mathbf{Z}/p^r\mathbf{Z}$, so $\ker(V^r)$ is a finite étale group scheme of geometric fiber $\mathbf{Z}/p^r\mathbf{Z}$.

Definition 1.2.1. We define $X_1(Np^r)_1^\circ$ to be the finite étale covering of $X_1(N)_1^\circ$ parametrizing isomorphisms

$$\ker(V^r) \cong (\mathbf{Z}/p^r\mathbf{Z}).$$

In other words, for $S/\mathbf{F}_p$,

$$X_1(Np^r)_1^\circ(S) \cong \left\{ (S \to X_1(N)_1^\circ, \alpha) \mid \alpha : \ker(V^r) \times_{X_1(N)_1^\circ} S \cong \underline{\mathbf{Z}/p^r\mathbf{Z}}_S \right\}.$$

Note this is surjective because $\ker(V^r)$ is étale-locally trivial.

Using Cartier duality, an isomorphism $\ker(V^r) \cong \mathbf{Z}/p^r\mathbf{Z}$ is the same as

$$\mu_{p^n} \cong \ker(F^r)$$

which is the same as an injection $\mu_{p^n} \hookrightarrow E[p^r]$.

Now what if we vary $m \geq 1$? The reduction maps $\mathbf{Z}/p^m\mathbf{Z} \to \mathbf{F}_p$ induce closed immersions $X_1(N)_1^\circ \hookrightarrow X_1(N)_m^\circ$ which are nilpotent thickenings. Since a nilpotent thickening $X \to Y$ is an isomorphism of topology spaces, it induces an equivalence of categories (see [Sta19, Tag 039R]) $\mathrm{Et}(X) \to \mathrm{Et}(Y)$, so we can form a diagram

$$\begin{tikzcd} X_1(Np^n)_1^\circ \arrow[r, "\text{étale}"] \arrow[d, hook] & X_1(N)_1^\circ \arrow[d, hook] \\ X_1(Np^r)_m^\circ \arrow[r, "\text{étale}"] & X_1(N)_m^\circ \end{tikzcd}$$

In fact $X_1(Np^r)_m^\circ$ is an affine smooth $\mathbf{Z}/p^m$-scheme. So $V_{m,r} = \mathscr{O}(X_1(Np^r)_m^\circ)$ is a smooth $\mathbf{Z}/p^m$-algebra. We let

$$V_{m,\infty} = \varinjlim_r V_{m,r} = \mathscr{O}(\varprojlim_r X_1(Np^r)_m^\circ)$$

Then $V_{m,\infty}$ is a $\mathbf{Z}/p^m$-algebra with a smooth (why?) action of $\mathbf{Z}_p^\times = \varprojlim_r (\mathbf{Z}/p^r)^\times$. We can recover:

$$V_{m,r} = V_{m,\infty}^{\ker(\mathbf{Z}_p^\times \to \mathbf{Z}/p^r)}.$$

Furthermore we have the compatibility between thickenings:

$$V_{m+1,\infty} \otimes_{\mathbf{Z}/p^{m+1}} \mathbf{Z}/p^m \cong V_{m,\infty}.$$

Therefore we can take the limit again to obtain

$$V_{\mathbf{Z}_p}(N) := \varprojlim_m V_{m,\infty}$$

which is a complete (by construction), torsion-free $\mathbf{Z}_p$-module, with a continuous $\mathbf{Z}_p$-linear action of $\mathbf{Z}_p^\times$.

Moreover, $V_{\mathbf{Z}_p}$ contains no p-divisible elements, so $V_{\mathbf{Z}_p} \hookrightarrow V_{\mathbf{Q}_p} := V_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. In fact, $V_{\mathbf{Z}_p}$ is the closed unit ball of the unique p-adic norm on $V_{\mathbf{Q}_p}$: note $V_{\mathbf{Q}_p}$ is complete for that norm (this is the Gauss norm: roughly, locally taking the colimit over r means that you're extracting some roots of the elements of $V_{m,1}$. Then taking the limit over m means that you're allowing yourself power series whose coefficients go to 0... I think? I don't understand the algebra structure completely).

The action of $\mathbf{Z}_p^\times$ preserves $V_{\mathbf{Z}_p}$, hence the norm as well. So $V_{\mathbf{Q}_p}$ is a unitary representation of the p-adic Lie group $\mathbf{Z}_p^\times$.

Elements of $V_{\mathbf{Q}_p}(N)$ are called **p-adic modular functions of tame level N.**

Definition 1.2.2. If $K/\mathbf{Q}_p$ is a finite extension, a **weight** of $\mathbf{Z}_p^\times$ with values in K is a continuous group homomorphism

$$\chi : \mathbf{Z}_p^\times \to K^\times.$$

Definition 1.2.3. If $\chi : \mathbf{Z}_p^\times \to K^\times$ is a weight, we can define

$$V_K[\chi] = \left\{ f \in V_K = V_{\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} K \mid \text{ for all } a \in \mathbf{Z}_p^\times, a \cdot f = \chi(a) f \right\}.$$

Then $V_K[\chi]$ are **p-adic modular forms** of weight χ (with coefficients in K).

If $k \in \mathbf{Z}$, then $\chi_k(a) = a^k$ is a weight (and we abuse notation thus).

1.3. **Modular Forms are p-adic modular forms.** The key observation is that under

$$X_1(Np^m)_m^\circ \to X_1(N)_m^\circ,$$

the pullback of the line bundle ω is trivial.

On $X_1(N)_1^\circ$, recall we have the connected-étale sequence

$$0 \to \ker(F^m) \to E[p^m] \to \ker(V^m) \to 0$$

and $\ker(F^m) \cong \ker(V^m)^D$. Then, again using the fact that the étale site behaves well with respect to nilpotent thickenings, $\ker(V^m)$ extends uniquely to some finite étale group scheme H_m^D over $X_1(N)_m^\circ$. On $X_1(N)_m^\circ$, we have

$$0 \to H_m \to E[p^m] \to H_m^D \to 0.$$

Here $H_m := (H_m^D)^D$ is the canonical subgroup.

Fact 1.3.1. $\omega_{E/X_1(N)_m^\circ} \xrightarrow{\sim} \omega_{E[p^m]/X_1(N)_m^\circ} \xrightarrow{\sim} \omega_{H_m/X_1(N)_m^\circ}$.

Sketch of Proof. The first isomorphism follows from the fact that $[a] : G \to G$ (where G is any group scheme) induces the multiplication by a map on tangent spaces, so the induced map of tangent bundles $\Omega_{E/X_1(N)_m^\circ} \to [p^m]_* \Omega_{E/X_1(N)_m^\circ}$ is zero since we're working mod p^m.

The second isomorphism follows from the fact that H_m^D is étale. □

There is a Hodge-Tate map

$$\mathrm{HT} : H_m^D \to \omega$$

of sheaves on the étale site. Note

$$H_m^D = \mathrm{Hom}(H_m, \mathbf{G}_m) = \mathrm{Hom}(H_m, \mu_{p^m})$$

so we define $\mathrm{HT}(f) = f^*(dx/x)$. Since dx/x is an invariant differential and HT is a group homomorphism, we get an isomorphism

$$H_m^D \otimes_{\underline{\mathbf{Z}/p^m\mathbf{Z}}_{X_1(N)_m^\circ}} \mathscr{O}_{X_1(N)_m^\circ} \xrightarrow{\sim} \omega_{H_m}.$$

2. Talk II

2.1. **From Yesterday.** For $N \geq 5$ and $p \nmid n$, we defined a tower of curves and their thickenings which fit into Cartesian diagrams:

$$\begin{tikzcd}
\cdots \arrow[r] & X_1(Np^r)_1^\circ \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & X_1(Np)_1^\circ \arrow[r] \arrow[d, hook] & X_1(N)_1^\circ \arrow[d, hook] \\
\cdots \arrow[r] & X_1(Np^r)_2^\circ \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & X_1(Np)_2^\circ \arrow[r] \arrow[d, hook] & X_1(N)_2^\circ \arrow[d, hook] \\
 & \vdots \arrow[d, hook] & & \vdots \arrow[d, hook] & \vdots \arrow[d, hook] \\
\cdots \arrow[r] & X_1(Np^r)_m^\circ \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & X_1(Np)_m^\circ \arrow[r] \arrow[d, hook] & X_1(N)_m^\circ \arrow[d, hook] \\
 & \vdots & & \vdots & \vdots
\end{tikzcd}$$

Recall each composition $X_1(Np^r)_m^\circ \to X_1(N)_m^\circ$ is étale with covering group $\mathbf{Z}/p^r\mathbf{Z}$.

From this, we constructed $V_{m,\infty} = \mathscr{O}(\varprojlim_r X_1(Np^r)_m^\circ)$, and taking the limit over the mod p^m coefficients, we took

$$V_{\mathbf{Z}_p}(N) = \varprojlim_m V_{m,\infty}$$

and rationally we have

$$V_{\mathbf{Q}_p} = V_{\mathbf{Q}_p}(N) = V_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

which is a p-adic Banach space and carries a continuous action of $\mathbf{Z}_p^\times$.

Given a weight $\chi : \mathbf{Z}_p^\times \to K^\times$ we can take the eigenspace

$$V_K[\chi] = \{f \in V_K \mid a \cdot f = \chi(a)f\}$$

Then given the universal elliptic curve $E \to Y_1(N)_m^\circ$ we have for all r an exact sequence

$$0 \to H_r \to E[p^r] \to H_r^D \to 0$$

and H_r^D is étale.

Some notation: if G/S is a group scheme, we write

$$\omega_G = e^*\Omega^1_{G/S} \cong \pi_*\Omega^1_{G/S}$$

where $e : S \to G$ is the identity section.

We showed there are isomorphisms

$$\omega_{E/X_1(N)_m^\circ} \xrightarrow{\sim} \omega_{E[p^r]} \xrightarrow{\sim} \omega_{H_r}$$

We also had existence of the Hodge-Tate map $HT : H_r^D \to \omega_{H_r}$. This is a map of sheaves on the étale site, and

$$H_r^D = \mathrm{Hom}(H_r, \mathbf{G}_m) \cong \mathrm{Hom}(H_r, \mu_{p^r})$$

Then $HT(f) = f^* \frac{dx}{x}$. We can extend it to a map

$$H_r^D \otimes_{\mathbf{Z}/p^r} \mathscr{O}_{X_1(N)_m^\circ} \xrightarrow{HT} \omega_{H_r}.$$

Proposition 2.1.1. *HT is an isomorphism.*

Proof. By étale descent for coherent sheaves we can check this after étale base change. So we base change to $X_1(Np^r)_m^\circ$, because over this étale cover H_r^D becomes isomorphic to $\underline{\mathbf{Z}/p^r\mathbf{Z}}_{X_1(Np^r)_m^\circ}$. Then you just need to check that HT is surjective, which comes from the fact that dx/x generates $\omega_{\mu_{p^r}} \cong \omega_{H_r}$. ∎

So on $X_1(Np^r)_m^\circ$ (for $r \geq m$) we get an isomorphism

$$\gamma_{m,r} : \omega \xrightarrow{\sim} \omega_{H^r} \cong H_r^D \otimes_{\underline{\mathbf{Z}/p^r\mathbf{Z}}} \mathscr{O}_{X_1(Np^r)_m^\circ} \cong \mathscr{O}_{X_1(Np^r)_m^\circ}.$$

By construction, the $\gamma_{m,r}$ are compatible in both m and r. This allows us to construct a map from modular forms to p-adic modular forms. Again recall that we defined

$$M_k(N, \mathbf{Z}/p^r\mathbf{Z}) = H^0(X_1(N)_m^\circ, \omega^{\otimes k}).$$

But we can restrict to some layer of the Igusa tower:

$$H^0(X_1(N)_m^\circ, \omega^{\otimes k}) \hookrightarrow H^0(X_1(Np^r)_m^\circ, \omega^{\otimes k}) \xrightarrow{\gamma_{m,r}} \mathscr{O}(X_1(Np^r)_m^\circ) = V_{m,r} \hookrightarrow V_{m,\infty}$$

This map doesn't depend on r and is compatible in m, i.e.

$$\begin{tikzcd} M_k(N, \mathbf{Z}/p^{m+1}\mathbf{Z}) \arrow[r, hook] \arrow[d] & V_{m+1,\infty} \arrow[d] \\ M_k(N, \mathbf{Z}/p^m\mathbf{Z}) \arrow[r, hook] & V_{m,\infty} \end{tikzcd}$$

so after taking the limit we get a map $M_k(N, \mathbf{Z}_p) \to V_{\mathbf{Z}_p}(N)$, and by tensoring with $\mathbf{Q}_p$ we get

$$\iota_k : M_k(N, \mathbf{Q}_p) \hookrightarrow V_{\mathbf{Q}_p}(N).$$

Fact 2.1.1. *Letting $\chi_k(a) = a^k$, we have*

$$\mathrm{im}(\iota_k) \subseteq V_{\mathbf{Q}_p}(N)[\chi_k]$$

In other words, "modular forms of weight k are p-adic modular forms of weight k".

Sketch of Proof. There is an action of $\mathbf{Z}_p^\times$ on each $Y_1(Np^r)_m^\circ$: if we take $a \in \mathbf{Z}_p^\times$, then

$$a \cdot (E/S, \alpha_N : \mu_N \hookrightarrow E[N], \alpha_{p^r} : \mu_{p^r} \hookrightarrow E[p^r]) = (E/S, \alpha_N, \alpha_{p^r} \circ \gamma_a)$$

where $\gamma_a(z) = z^a$ (this is an isomorphism since $a \in \mathbf{Z}_p^\times$).

Then letting $\lambda_{\mathrm{can}} := \mathrm{HT}(1) \in H^0(X_1(Np^r)_m^\circ, \omega)$, we check that

$$a^* \lambda_{\mathrm{can}} = a^{-1} \lambda_{\mathrm{can}}.$$

∎

Remark 2.1.1.

(1) We also have $M_k(Np^r, \mathbf{Q}_p) \hookrightarrow V_{\mathbf{Q}_p}(N)$. To see this, let $Y_1(Np^r)'_{\mathbf{Z}_p}$ be the scheme representing the functor

$$S \mapsto (E/S, \mu_{Np^r} \hookrightarrow E).$$

To have such an embedding, we must have E ordinary, so $Y_1(Np^r)' \otimes \mathbf{Z}/p^m\mathbf{Z} \cong Y_1(Np^r)^\circ_m$. Consider the normalization $X_1(Np^r)_{\mathbf{Z}_p}$ of $\mathbf{P}^1_{\mathbf{Z}_p}$ in $Y_1(Np^r)'_{\mathbf{Z}_p}$. This is proper but not smooth because we have level at p, but we can define

$$M_k(Np^r, \mathbf{Q}_p) = H^0(X_1(Np^r)_{\mathbf{Q}_p}, \omega^{\otimes k}).$$

The target receives a map from

$$H^0(X_1(Np^r)_{\mathbf{Z}_p}, \omega^{\otimes k})$$

Reduce mod m to get a map $H^0(X_1(Np^r)_m, \omega^{\otimes k}) \to H^0(X_1(Np^r)^\circ_m, \omega^{\otimes k}) \hookrightarrow V_{m,\infty}$. Finally, we get a map

$$H^0(X_1(Np^r)_{\mathbf{Z}_p}, \omega^{\otimes k}) \to V_{\mathbf{Z}_p}(N)$$

and

$$H^0(X_1(Np^r)_{\mathbf{Q}_p}, \omega^{\otimes k}) \to V_{\mathbf{Q}_p}(N)$$

But careful, because $M_k(Np^r, \mathbf{Q}_p) \not\subset V_{\mathbf{Q}_p}(N)[\chi_k]$. If we fix a nebentypus $\epsilon : (\mathbf{Z}/p^r\mathbf{Z})^\times \to K^\times$, then we get an injection

$$M_k(Np^r, \epsilon, K) \hookrightarrow V_{\mathbf{Q}_p}(N)[\chi_k \epsilon]$$

So "modular forms of weight k and level Np^r are p-adic modular forms of tame level N".

(2) Other properties:

(a)

$$\bigoplus_{k \geq 0} M_k(N, \mathbf{Q}_p) \subseteq V_{\mathbf{Q}_p}(N)$$

is a dense subspace.

(b) Igusa proved that each $X_1(Np^r)_1$ is connected. So the map

$$\pi_1(X_1(N)^\circ_1) \to (\mathbf{Z}/p^r\mathbf{Z})^\times$$

is surjective. Therefore, $V_{1,\infty}^{1+p\mathbf{Z}_p} = \sum_{k\geq 0} \mathrm{im}(\iota_k)$ where $\iota_k : M_k(N, \mathbf{F}_p) \to V_{1,\infty}$. Also $\omega^{\otimes(p-1)}$ is trivialized by the Hasse invariant on $X_1(N)^\circ_1$. Multiplication by the Hasse invariant induces

$$\iota_k(M_k(N, \mathbf{F}_p)) \subseteq \iota_{k+(p-1)}(M_{k+p-1}(N, \mathbf{F}_p)).$$

so we have

$$\bigcup_{n \geq 0} M_{a+n(p-1)}(N, \mathbf{F}_p) \subseteq V_{1,\infty}$$

and we call this $M(n, a, \mathbf{F}_p)$ for $0 \leq a \leq p-2$, and

$$V_{1,\infty}^{1+p\mathbf{Z}_p} = \bigoplus_{a=0}^{p-2} M(N, a, \mathbf{F}_p).$$

2.2. **Hecke Operators.** We will do this via correspondences. Fix $\ell \neq p$. Define $Y_1(Np^r;\ell)^\circ_m$ to be the scheme representing the functor

$$S \mapsto \{(E/S, \alpha : \mu_{Np^r} \hookrightarrow E, H \subseteq E[\ell] \text{ finite flat gp. scheme of order } \ell, H \cap \operatorname{im}(\alpha) = 0)\}$$

There is a correspondence

$$Y_1(Np^r)^\circ_m \leftarrow Y_1(Np^r, \ell) \rightarrow Y_1(Np^r)^\circ_m$$

The left map takes $(E, \alpha, H) \mapsto (E, \alpha)$ and the right map takes $(E, \alpha, H) \mapsto (E/H, \mu_{Np^r} \xrightarrow{\alpha} E \to E/H)$. This correspondence defines $T(\ell) \in \operatorname{End}(V_{m,r})$ as follows:

$$T(\ell) = \frac{1}{\ell} \operatorname{tr}(\pi_1) \circ \pi_2^*.$$

These are compatible in m, r, so we get $T(\ell) \in \operatorname{End}(V_{\mathbf{Q}_p}(N))$ and $|T(\ell)| \leq 1$. This is compatible with the Hecke operator $T_k(\ell) \in \operatorname{End}(M_k(N, \mathbf{Q}_p))$ under the embedding $M_k(N, \mathbf{Q}_p) \hookrightarrow V_{\mathbf{Q}_p}(N)$.

The tricky part is when $\ell = p$. The Frobenius $Y_1(N)^\circ_1 \xrightarrow{\text{Frob}} Y_1(N)^\circ_1$ takes $(E, \alpha) \mapsto (E/H_1, \pi_{H_1} \circ \alpha)$. There exists a lift of Frob to the thickenings $Y_1(N)^\circ_m$. In fact, we can define

$$Y_1(N)^\circ_m \to Y_1(N)^\circ_m$$

taking $(E, \alpha) \mapsto (E/H_1, \pi_{H_1} \circ \alpha)$. On the Igusa tower we can lower the level:

$$Y_1(Np^r)^\circ_m \to Y_1(Np^{r-1})^\circ_m$$

taking $(E, \alpha, \mu_{p^r} \hookrightarrow E[p^r]) \mapsto (E/H_1, \alpha, \pi_{H_1} \circ \alpha_{p^r})$. These maps induces some algebra endomorphism

$$F : V_{\mathbf{Z}_p}(N) \to V_{\mathbf{Z}_p}(N)$$

which is finite flat of degree p.

Define $U = 1/p \operatorname{tr}(F)$. Since F is finite flat of degree p,

$$\operatorname{tr}(F)(V_{\mathbf{Z}_p}) \subseteq pV_{\mathbf{Z}_p}$$

This is compatible with the classical Hecke operator, in the sense that if $r \geq 1$,

$$\begin{array}{ccc} M_k(Np^r, \mathbf{Q}_p) & \xrightarrow{T_k(p)=U_p} & M_k(Np^r, \mathbf{Q}_p) \\ \big\downarrow & & \big\downarrow \\ V_{\mathbf{Q}_p}(N) & \xrightarrow{U} & V_{\mathbf{Q}_p}(N). \end{array}$$

If $r = 0$, then $U \equiv T_k(p) \bmod p$ on $M_k(N, \mathbf{Z}_p)$.

2.3. **Ordinary p-adic modular forms.** First, take $\Gamma = 1 + p\mathbf{Z}_p$. If $p = 2$, take $1 + 4\mathbf{Z}_2$. Recall the Iwasawa algebra $\Lambda = \mathbf{Z}_p[[\Gamma]]$ the completed algebra of Γ, i.e.

$$\Lambda = \varprojlim_n \mathbf{Z}_p[\Gamma/\Gamma^{p^m}]$$

Then $\Lambda \cong \mathbf{Z}_p[[x]]$ by sending $1 + p \mapsto 1 + x$. Since $\mathbf{Z}_p^\times$ acts on $V_{\mathbf{Q}_p}(N)$ continuously, $V_{\mathbf{Q}_p}(N)$ turns into a Λ-module.

Let $\mathcal{V}_{\mathbf{Q}_p} = \operatorname{Hom}_{\text{cts}}(V_{\mathbf{Q}_p}, \mathbf{Q}_p)$. We put the weak topology on $\mathcal{V}_{\mathbf{Q}_p}$, i.e. the coarsest topology for which all of the evaluation maps are continuous. More concretely, we have a lattice $\mathcal{V}_{\mathbf{Z}_p} = \operatorname{Hom}(V_{\mathbf{Z}_p}, \mathbf{Z}_p) = \varprojlim_m \operatorname{Hom}(V_{\mathbf{Z}_p} \otimes \mathbf{Z}/p^m, \mathbf{Z}/p^m)$, which we claim is compact. The action Γ on $V \otimes \mathbf{Z}/p^m$ is smooth, and for the weak topology, the spaces $\operatorname{Hom}(V_{\mathbf{Z}_p} \otimes \mathbf{Z}/p^m, \mathbf{Z}/p^m)$ are compact $\mathbf{Z}_p$-modules. Why? $V_{\mathbf{Z}_p} \otimes \mathbf{Z}/p^m = \varinjlim_{W \text{ finite submodules}} W$, and

$$\operatorname{Hom}(V_{\mathbf{Z}_p} \otimes \mathbf{Z}/p^m, \mathbf{Z}/p^m) = \varprojlim_W \operatorname{Hom}(W, \mathbf{Z}/p^m),$$

which is then compact, so $\mathcal{V}_{\mathbf{Z}_p}$ is a compact $\mathbf{Z}_p$-module, and so is a compact Λ-module.

Why are we doing this? The problem is that $\mathcal{V}_{\mathbf{Z}_p}$ is not typically a finite type Λ-module. When $\mathcal{V}_{\mathbf{Z}_p}$ is a finite type Λ-module, we say that $V_{\mathbf{Q}_p}$ is an **admissible representation** of $\mathbf{Z}_p^\times$.

3. Talk III

Yesterday, we had this space $V_{\mathbf{Q}_p}(N)$ of p-adic modular forms, with a continuous action of $\mathbf{Z}_p^\times$.

More generally, if V is a p-adic Banach space with a continuous action of $\mathbf{Z}_p^\times$, then V gives rise to a Λ-module, where $\Lambda := \mathbf{Z}_p[[\Gamma]]$ is the Iwasawa algebra for $\Gamma = 1 + p\mathbf{Z}_p$. We consider the dual $V' = \mathrm{Hom}_{\mathrm{cts}}(V, \mathbf{Q}_p)$ with its weak topology, which induces on the $\mathbf{Z}_p$-lattice $V'_{\mathbf{Z}_p} = \mathrm{Hom}(V_{\mathbf{Z}_p}, \mathbf{Z}_p)$ the inverse limit topology coming from

$$V'_{\mathbf{Z}_p} = \varprojlim_m \varprojlim_{W \subseteq V_{\mathbf{Z}_p} \otimes \mathbf{Z}/p^m \text{ finite}} \mathrm{Hom}(W, \mathbf{Z}/p^m).$$

and obtain a compact Λ-module.

Now if we let $\Lambda_{\mathbf{F}_p} = \Lambda \otimes \mathbf{F}_p = \mathbf{F}_p[[x]]$, then

$$(V_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p)^\Gamma = (V_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p)[x] \ (x\text{-torsion})$$

and

$$V_{\mathbf{Z}_p} = \mathrm{Hom}_{\mathrm{cts}}(V'_{\mathbf{Z}_p}, \mathbf{Z}_p)$$

so

$$(V_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p)^\Gamma = \mathrm{Hom}_{\mathrm{cts}}(V'_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p, \mathbf{F}_p)$$

and

$$(V_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p[x])' = V'_{\mathbf{Z}_p} \otimes_\Lambda \mathbf{F}_p.$$

So by using a topological version of Nakayama's lemma, we see that $V'_{\mathbf{Z}_p}$ is a module of finite type if and only if $\dim_{\mathbf{F}_p}(V_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p)^\Gamma < \infty$. When these conditions are satisfied, we say that V is an **admissible** $\mathbf{Z}_p^\times$**-module**.

If we apply this to $V_{\mathbf{Q}_p}(N) \supseteq V_{\mathbf{Z}_p(N)}$, we get

$$(V_{\mathbf{Z}_p}(N) \otimes_{\mathbf{Z}_p} \mathbf{F}_p)^\Gamma = V_{1,1} = \mathscr{O}(X_1(Np)_1^\circ)$$

But this is the space of functions on an affine curve, so it is not finite dimensional. There is a construction of Hida that lets us get around this.

3.1. **Ordinary Projector.**

This is due to Hida.

$$\bigoplus_k M_k(N, \mathbf{Q}_p) \longhookrightarrow \bigoplus_k M_k(Np, \mathbf{Q}_p) \xhookrightarrow{\text{dense}} V_{\mathbf{Q}_p}(N)$$

These have actions of U. Then $V_{\mathbf{Z}_p}(N) \otimes \mathbf{Z}/p^m$ is an increasing union of finite submodules stable under U. Which?

$$\left(\bigoplus_{k=0}^{j} M_k(Np, \mathbf{Q}_p) \cap V_{\mathbf{Z}_p}\right) \otimes \mathbf{Z}/p^m$$

Fact 3.1.1. *If $v \in V_{\mathbf{Z}_p}(N)$, then $(U^{n!}v)_{n\geq 0}$ converges to some vector $e_{\mathrm{ord}}v$. Furthermore e_{ord} is a continuous projector in* $\mathrm{End}(V_{\mathbf{Z}_p}(N))$.

We call the **ordinary projector**.

If W is an object "coming from $V_{\mathbf{Z}_p}(N)$" (for example, could be $V_{\mathbf{Z}_p} \otimes \mathbf{Z}/p^m$ or $V_{\mathbf{Q}_p}$ or $\mathcal{V}_{\mathbf{Z}_p}$) then e_{ord} acts on W and we let

$$W^{\mathrm{ord}} = e_{\mathrm{ord}}W$$

Theorem 3.1.1 (Hida)**.** *Take $p > 2$. Then $\mathcal{V}_{\mathbf{Z}_p}^{\mathrm{ord}}$ is a finite free Λ-module. This implies that $V_{\mathbf{Q}_p}^{\mathrm{ord}}(N)$ is an admissible $\mathbf{Z}_p^\times$-representation.*

If $k \geq 3$, then we have an inclusion

$$V_{\mathbf{Q}_p}(N)[\chi_k] \supseteq M_k(N, \mathbf{Q}_p)$$

and in fact

$$V_{\mathbf{Q}_p}^{\mathrm{ord}}(N)[\chi_k] \supseteq M_k(N, \mathbf{Q}_p)^{\mathrm{ord}}.$$

Let $V_{\mathbf{Q}_p,\mathrm{cusp}}(N) = \left\{ f \in V_{\mathbf{Q}_p}(N) \mid f \text{ vanishes along } X_1(Np^r)^\circ \backslash Y_1(Np^r)^\circ \right\}$. Similarly, we have (still $k \geq 3$)

$$V_{\mathbf{Q}_p,\mathrm{cusp}}^{\mathrm{ord}}(N)[\chi_k] \supseteq S_k(N, \mathbf{Q}_p)^{\mathrm{ord}}.$$

and $\mathcal{V}_{\mathbf{Z}_p,\mathrm{cusp}}^{\mathrm{ord}}$ is finite free over Λ.

Theorem 3.1.2 (Hida)**.**

(1) *For $k \geq 3$, $\dim_{\mathbf{Q}_p} M_k(N, \mathbf{Q}_p)^{\mathrm{ord}}$ depends only on the class of k mod $p - 1$. (same result for cuspidal spaces).*

(2) *If $\epsilon : (\mathbf{Z}/p^r\mathbf{Z})^\times \to K^\times$ is a Dirichlet character of conductor p^r $(r \geq 1)$ and $k \geq 2$, then*

$$\dim_{\mathbf{Q}_p} S_k(Np^r, \epsilon, K)$$

depends only on the class of k mod $p - 1$.

Proof of Theorem 3.1.1. We show how Theorem 3.1.2 implies Theorem 3.1.1. First we need to show that $\mathcal{V}_{\mathbf{Z}_p}^{\mathrm{ord}}$ is finitely generated as a Λ-module. By the above discussion, this is true iff $\dim_{\mathbf{F}_p}((V_{\mathbf{Z}_p} \otimes \mathbf{F}_p)^\Gamma)^{\mathrm{ord}} < \infty$. But

$$((V_{\mathbf{Z}_p} \otimes \mathbf{F}_p)^\Gamma)^{\mathrm{ord}} = V_{1,1}^{\mathrm{ord}} = \bigoplus_{a \in \mathbf{Z}/p} M(N, a, \mathbf{F}_p)^{\mathrm{ord}}$$

But remember that $M(N, a, \mathbf{F}_p)^{\mathrm{ord}} = \bigcup_{n \geq 0} M_{a+n(p-1)}(N, \mathbf{F}_p)^{\mathrm{ord}}$, and that

$$\dim_{\mathbf{F}_p} M_k(N, \mathbf{F}_p)^{\mathrm{ord}} = \dim_{\mathbf{Q}_p} M_k(N, \mathbf{Q}_p)^{\mathrm{ord}},$$

so Theorem 3.1.2 tells you that this infinite union stabilizes and thus

$$\bigoplus_{a \in \mathbf{Z}/p} M(N, a, \mathbf{F}_p)^{\mathrm{ord}} = \bigoplus_{a \in \mathbf{Z}/p} M_{j(a)}(N, \mathbf{F}_p)^{\mathrm{ord}}$$

for large enough $j(a) \equiv a$ for each a.

So $\mathcal{V}_{\mathbf{Z}_p}^{\mathrm{ord}}$ is finite over Λ, and we need to show that it's actually free. Note $\mathbf{Z}_p^\times \cong \Delta \times \Gamma$: if $a \in \mathbf{Z}/p - 1$, let $\chi_a : \Delta \to \mathbf{Z}_p^\times$ be the character defined by raising to the power a. Then we get a decomposition into eigenspaces

$$V_{\mathbf{Z}_p} = \bigoplus_{a \in \mathbf{Z}/p-1} V_{\mathbf{Z}_p,a}$$

where $V_{\mathbf{Z}_p,a} = V_{\mathbf{Z}_p}[\chi_a]$. Letting $\mathcal{V}_{\mathbf{Z}_p,a} = \mathrm{Hom}_{\mathrm{cts}}(V_{\mathbf{Z}_p,a}, \mathbf{Z}_p)$, we have

$$\mathcal{V}_{\mathbf{Z}_p,a}^{\mathrm{ord}} \otimes_\Lambda \mathbf{F}_p \cong \mathbf{F}_p^{r(a)}$$

where $r(a) = \dim_{\mathbf{Q}_p} M_{j(a)}(N, \mathbf{Q}_p)^{\mathrm{ord}}$ (now we let $j(a) \equiv a \bmod p - 1$). Pick some surjection $\Lambda^{r(a)} \twoheadrightarrow \mathcal{V}_{\mathbf{Z}_p,a}^{\mathrm{ord}}$ and let $P_{j(a)} = ((1 + x) - (1 + p)^{j(a)}) = \ker(\Lambda \xrightarrow{\chi_{j(a)}} \mathbf{Z}_p)$. Using the co-torsion-free inclusion

$$M_{j(a)}(N, \mathbf{Z}_p)^{\mathrm{ord}} \subseteq V_{\mathbf{Z}_p}^{\mathrm{ord}}(N)[\chi_{j(a)}]$$

we extend the surjection to

$$\Lambda^{r(a)} \twoheadrightarrow \mathcal{V}_{\mathbf{Z}_p,a}^{\mathrm{ord}} \twoheadrightarrow \mathrm{Hom}(M_{j(a)}(N, \mathbf{Z}_p), \mathbf{Z}_p)$$

These maps factor through the quotient by the ideal $P_{j(a)}$:

$$\Lambda^{r(a)}/P_{j(a)} \twoheadrightarrow \mathcal{V}^{\mathrm{ord}}_{\mathbf{Z}_p,a}/P_{j(a)} \twoheadrightarrow \mathrm{Hom}(M_{j(a)}(N,\mathbf{Z}_p),\mathbf{Z}_p)$$

So we have

$$(\Lambda/P_{j(a)})^{r(a)} \twoheadrightarrow \mathcal{V}^{\mathrm{ord}}_{\mathbf{Z}_p,a}/P_{j(a)} \twoheadrightarrow \mathrm{Hom}(M_{j(a)}(N,\mathbf{Z}_p),\mathbf{Z}_p)$$

But the source and target are $\mathbf{Z}_p^{r(a)}$, so this is an isomorphism, and

$$\ker(\Lambda^{r(a)} \twoheadrightarrow \mathcal{V}^{\mathrm{ord}}_{\mathbf{Z}_p,a}) \subseteq P_{j(a)}(\Lambda)^{r(a)}$$

so $\ker = 0$. □

3.2. **Hida Families.** We have $\bigoplus_k M_k(Np,\mathbf{Q}_p) \subseteq V_{\mathbf{Q}_p}(N)$. There is a big endomorphism $\bigoplus_k T_k(\ell)$ acting on the first sum. For $\ell \neq p$ there is a $T(\ell)$ acting on $V_{\mathbf{Q}_p}(N)$ compatibly. But we also have $\bigoplus_k T_k(\ell,\ell) = \bigoplus_k \ell^{k-2}\langle \ell\rangle_{Np}$ acting, which corresponds to $\ell^{-1}\langle\ell\rangle_{Np}$: this operator $\langle\ell\rangle_{Np}$ acts on each $V_{m,r}$ by

$$\langle\ell\rangle_{N_p}(E/S, \alpha_{Np^r}\cdot\mu_{Np^r} \hookrightarrow E) = (E/S, \alpha_{Np^r\circ}\gamma_\ell)$$

We let

$$\mathcal{H}(N) = \text{ weak closure of } \mathbf{Z}_p[T(\ell), \ell^{-2}\langle\ell\rangle_{Np}, U, \ell\neq p] \text{ in } \mathrm{End}(V_{\mathbf{Z}_p}(N))$$

For $j \geq 0$ let $\mathcal{H}^j(N)$ be the $\mathbf{Z}_p$-algebra $\mathrm{End}(\bigoplus_{k=0}^j M_k)$ and we have

$$\mathcal{H}(N) = \varprojlim_j \mathcal{H}^j(N).$$

By construction, for all k, we have

$$\mathcal{H}(N) \twoheadrightarrow \mathcal{H}_k(Np)$$

With this weak topology, $\mathcal{H}(N)$ is a compact $\mathbf{Z}_p$-algebra, which call the "big Hecke algebra".

We have $\mathcal{H}(N) \subseteq \mathrm{End}(V_{\mathbf{Z}_p}(N))$ and $\Lambda \subseteq \mathrm{End}(V_{\mathbf{Z}_p}(N))$. In fact, the action of Λ factors through $\mathcal{H}(N)$. Why? Take some $\ell \equiv 1 \bmod pN$. If $\delta_\ell \in \mathbf{Z}[\Gamma] \subseteq \Lambda$, then $\ell^{-2}\delta_\ell$ acts like $\bigoplus T_k(\ell,\ell)$ on $\bigoplus_k M_k(Np,\mathbf{Z}_p)$. This implies that $\delta_\ell \in \Lambda$ are in $\mathcal{H}(N)$. By density of such ℓ, $\Lambda \subseteq \mathcal{H}(N)$.

There are other variants if $\mathcal{H}(N)$. We know that $e_{\mathrm{ord}} \in \mathcal{H}(N)$, so we can define the ordinary part $\mathcal{H}^{\mathrm{ord}} = e_{\mathrm{ord}}\mathcal{H}(N)$. There is also a cuspidal version $h(N) = \mathrm{im}(\mathcal{H}(N) \to \mathrm{End}(V_{\mathrm{cusp}}(N))$. Can do $h(N)^{\mathrm{ord}}$...

Theorem 3.2.1 (Hida)**.** *Fix $\epsilon : (\mathbf{Z}/p^r\mathbf{Z})^\times \to K^\times$ a Dirichlet character. Then for $k \geq 2$, let $P_{k,\epsilon} = \ker(\Lambda \xrightarrow{\chi_k\epsilon} \mathbf{Z}_p)$. Then $\mathcal{H}(N)^{\mathrm{ord}}$ and $h(N)^{\mathrm{ord}}$ are finite free Λ-modules.*

If we fix $a \in \mathbf{Z}/p-1$ let

$$\mathcal{H}(N,a) = \mathrm{im}(\mathcal{H}(N) \to \mathrm{End}(V_{\mathbf{Z}_p,a})).$$
$$h(N,a) = \mathrm{im}(h(N) \to \mathrm{End}(V_{\mathbf{Z}_p,\mathrm{cusp},a})).$$

Then $\mathcal{H}(N) = \bigoplus_{a\in\mathbf{Z}/p-1}\mathcal{H}(N,a)$. For $k \geq 3$,

$$\mathcal{H}(N,a)^{\mathrm{ord}}/P_{k,1} \xrightarrow{\sim} \mathcal{H}_k(Np,\omega^{a-k})^{\mathrm{ord}}$$

Now for $k \geq 2$,

$$h(N,a)^{\mathrm{ord}}/P_{k,\epsilon} \xrightarrow{\sim} h_k(Np^r,\omega^{a-k})^{\mathrm{ord}}$$

Sketch of Proof. Using q-expansions, prove first that $\mathcal{V}_{\mathbf{Z}_p,\mathrm{cusp}} \cong h(N)$ as an $h(N)$-module. Then

$$\dim \mathcal{H}(Np^r,\epsilon)_{\mathbf{Q}_p} = \dim_{\mathbf{Q}_p} M_k(Np^r,\epsilon).$$

and prove the same thing in the cuspidal case (do this using some perfect duality).

You know that $\mathcal{V}_{\mathbf{Z}_p,\mathrm{cusp}}$ is finite free over Λ is $h(N)$ is as well. Then use Theorem 2 and the above dimension equality to conclude. □

By the Theorem $h(N)^{\mathrm{ord}}$ is a finite free Λ-algebra, and thus is a semilocal algebra, i.e. has finitely many maximal ideals, so we can decompose it as

$$h(N)^{\mathrm{ord}} = \bigoplus_{\mathfrak{m}\subseteq h(N)} h(N)^{\mathrm{ord}}_{\mathfrak{m}}$$

Then the rigid analytic spectrum of this Hecke algebra maps to weight space: $\mathcal{E}_m = (\operatorname{Spf} h(N)^{\mathrm{ord}}_m)^{\mathrm{rig}}) \xrightarrow{\pi} (\operatorname{Spf}\Lambda)^{\mathrm{rig}} =: \mathcal{W}$. A closed point of this family is called a **Hida family**, and corresponds to a morphism $h(N,a)^{\mathrm{ord}} \to K(/\mathbf{Q}_p)$ which is a system of eigenvalues of $h(N)$ acting on $V^{\mathrm{ord}}_{\mathbf{Q}_p(N)}$. Theorem 3 implies that if we consider some $y = \chi_k\epsilon$ (for $k \geq 2$) in the weight space $\mathcal{W}$, then $\pi^{-1}(y)$ contains only classical points.

4. Talk IV

Recall: we fix $N \geq 5$ and $p \nmid 2N$. We had the following theorem:

Theorem 4.0.1. *Fix $r \geq 1$ and $\chi : \mathbf{Z}_p^\times \to K^\times$ with open kernel $1 + p^n\mathbf{Z}_p$ for some $K/\mathbf{Q}_p$ finite and $k \geq 2$. Then*

$$\dim_K S_k(\Gamma_1(Np^r), \chi, K)^{\mathrm{ord}}$$

depends only on the the class of k mod $p-1$ and $\chi|_\Delta$, where

$$\mathbf{Z}_p^\times \cong \Delta \times \Gamma.$$

To prove this theorem we want to give a proof using completed cohomology.

4.1. Cohomological Methods.

Let $Y_1(Np^r) = \Gamma_1(Np^r)\backslash\mathbf{H}$, and for $k \geq 2$ we denote $V_k = \operatorname{Sym}^{k-2}(\mathbf{Z}^2)$ with its usual action of $\mathrm{SL}_2(\mathbf{Z})$. This gives rise to a local system on $Y_1(Np^r)$.

Then we define the parabolic cohomology

$$H^1_{\mathrm{par}}(Y_1(Np^r), \mathcal{V}_k) := \operatorname{im}(H^1_c(Y_1(Np^r), \mathcal{V}_k) \to H^1(Y_1(Np^r), \mathcal{V}_k)).$$

Then the Eichler-Shimura isomorphism is a Hecke-equivariant isomorphism

$$H^1_{\mathrm{par}}(Y_1(Np^r), \mathcal{V}_k) \otimes_{\mathbf{Z}} \mathbf{C} \cong S_k(\Gamma_1(Np^r)) \oplus \overline{S_k(\Gamma_1(Np^r))}.$$

So if we consider the image of the abstract Hecke algebra $h_k(\Gamma_1(Np^r))/ \subseteq \operatorname{End}(S_k(\Gamma_1(Np^r)))$, this matches the image in $\operatorname{End}(H^1_{\mathrm{par}}(Y_1(Np^r), \mathcal{V}_k))$.

Then the space $\operatorname{Hom}_{\mathbf{Q}_p}(H^1_{\mathrm{par}}(Y_1(Np^r), \mathcal{V}_k \otimes_{\mathbf{Z}} \mathbf{Q}_p), \mathbf{Q}_p)$ is free of rank 2 over $h_k(\Gamma_1(Np^r))_{\mathbf{Q}_p}$. We have to prove that

$$\dim_K H^1_{\mathrm{par}}(Y_1(Np^r), \chi, \mathcal{V}_k \otimes K)^{\mathrm{ord}}$$

depends only on k mod $p-1$ and $\chi|_\Delta$.

4.2. Completed Cohomology.

If $K \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$ is a compact open subgroup, then let

$$Y_K = \mathrm{GL}_2(\mathbf{Q})\backslash(\mathcal{H}^\pm \times \mathrm{GL}_2(\mathbf{A}^\infty)/K).$$

We work at level $K = K^p \times K_p$ where

$$K^p = \left\{ g \in \mathrm{GL}_2(\widehat{Z}^p) \mid g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

Then write $Y_{K_p} := Y_{K^pK_p}$. Note

$$Y_{\mathrm{GL}_2(\mathbf{Z}_p)} = Y_1(N) = Y(\Gamma_1(N)).$$

Increasing level at p, we have

$$Y_1(Np^r) = Y_{K_1(p^r)}$$

where $K_1(p^r) = \left\{ \begin{pmatrix} \mathbf{Z}_p^\times & \mathbf{Z}_p \\ p^r\mathbf{Z}_p & 1 + p^r\mathbf{Z}_p \end{pmatrix} \right\}$.

We first take a limit:

$$H^1_{\mathbf{Z}_p} = \varinjlim_{K_p \subseteq \mathrm{GL}_2(\mathbf{Q}_p)} H^1(Y_{K_p}, \mathbf{Z}_p)$$

This becomes a $\mathbf{Z}_p$-module. Then define the **completed cohomology** to be $\widehat{H}^1_{\mathbf{Z}_p}$, the p-adic completion, which is now a p-adic Banach space. Finally we let

$$\widehat{H}^1(N) = \widehat{H}^1 = \widehat{H}^1_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

This naturally has an action of $\mathrm{GL}_2(\mathbf{Q}_p)$, which is unitary and continuous representation of $\mathrm{GL}_2(\mathbf{Q}_p)$.

There is alternative characterization given by

$$\widehat{H}^1_{\mathbf{Z}_p} \cong \widetilde{H}^1_{\mathbf{Z}_p} := \varprojlim_n (\varinjlim_{K_p \subseteq \mathrm{GL}_2(\mathbf{Q}_p)} H^1(Y_{K_p}, \mathbf{Z}/p^n\mathbf{Z})).$$

which is the same because we're working with curves. In general these won't be the same, but in general this is the better definition of completed cohomology.

Theorem 4.2.1 (Emerton)**.** *$\widehat{H}^1$ is an admissible continuous representation of $\mathrm{GL}_2(\mathbf{Q}_p)$.*

One should compare this to $V_{\mathbf{Q}_p}$ with its $\mathbf{Z}_p^\times$-action: but this is far from being admissible.

For $k \geq 2$, consider the map

$$\begin{tikzcd} H^1(Y_{K_p}, \mathcal{V}_k \otimes_{\mathbf{Z}} \mathbf{Z}_p) \arrow[r] & \varprojlim_n H^1(Y_{K_p}, \mathcal{V}_k \otimes_{\mathbf{Z}} \mathbf{Z}/p^n) \arrow[d] \\ & \varprojlim_n \varinjlim_{K_p' \subseteq K_p} H^1(Y_{K_p'}, \mathcal{V}_k \otimes_{\mathbf{Z}} \mathbf{Z}/p^n) \end{tikzcd}$$

If K_p' is small enough, then $\mathcal{V}_k \otimes_{\mathbf{Z}_p} \mathbf{Z}/p^n$ is trivial on $Y_{K_p'}$. In other words, for small enough K_p',

$$H^1(Y_{K_p'}, \mathcal{V}_k \otimes_{\mathbf{Z}} \mathbf{Z}/p^n) \cong H_1(Y_{K_p'}, \mathbf{Z}/p^n) \otimes_{\mathbf{Z}} V_k.$$

This gives us a map

$$H^1(Y_{K_p}, \mathcal{V}_k \otimes_{\mathbf{Z}} \mathbf{Z}_p) \to ((V_k \otimes_{\mathbf{Z}} \mathbf{Z}_p) \otimes \widehat{H}^1(N))^{K_p}.$$

So here $V_k \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is an algebraic representation of $\mathrm{GL}_2(\mathbf{Q}_p)$, and we tensor it with the huge unitary representation $\widehat{H}^1(N)$.

Theorem 4.2.2 (Emerton)**.** *The map*

$$H^1(Y_{K_p}, \mathcal{V}_k \otimes_{\mathbf{Z}} \mathbf{Q}_p) \to \mathrm{Hom}_{K_p}((V_{k,\mathbf{Q}_p})', \widehat{H}^1)$$

is a Hecke-equivariant isomorphism.

For more general Shimura varieties, you have a spectral sequence instead of an isomorphism.

Remark 4.2.1. If we do this for each k and put them together,

$$\bigoplus_k H^1(Y_1(N), \mathcal{V}_k) \otimes V'_{k,\mathbf{Q}_p} \hookrightarrow \widehat{H}^1$$

with dense image. Therefore, if you construct a "big" Hecke algebra using completed cohomology, you don't get anything new. If we let $\mathbf{T}^{\mathrm{sph}}$ be the weak completion of $\mathbf{Z}_p[T(\ell), T(\ell,\ell) \mid \ell \nmid Np] \subseteq \mathrm{End}(\widehat{H}^1)$, then actually $\mathbf{T}^{\mathrm{sph}} \subseteq h(N) \subseteq \mathrm{End}(V_{\mathbf{Q}_p,\mathrm{cusp}})$ (note this is not an equality because we left out the U_p-operator).

Fact 4.2.1. *The spherical Hecke algebra $\mathbf{T}^{\mathrm{sph}}$ is a semi-local $\mathbf{Z}_p$-algebra, so decomposes as a finite product*

$$\mathbf{T}^{\mathrm{sph}} = \prod_{\mathfrak{m}} \mathbf{T}_{\mathfrak{m}}.$$

As a consequence, we get a decomposition $\widehat{H}^1 \cong \bigoplus_{\mathfrak{m}} \widehat{H}^1_{\mathfrak{m}}$.

Now fix some $\mathfrak{m} \subseteq \mathbf{T}$ which is non-Eisenstein. This means that the associated Galois representation $\overline{\rho}_{\mathfrak{m}}$: $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is irreducible. This is useful because

$$H^1(Y_1(N), \mathcal{V}_k)_{\mathfrak{m}} = H^1_{\mathrm{par}}(Y_1(N), \mathcal{V}_k)_{\mathfrak{m}}.$$

Theorem 4.2.3 (Emerton)**.** *The* $\mathrm{GL}_2(\mathbf{Z}_p)$*-representation* $\widehat{H}^1_{\mathbf{Z}_p,\mathfrak{m}}$ *is isomorphic to a direct factor of* $\mathcal{C}(\mathrm{GL}_2(\mathbf{Z}_p), \mathbf{Z}_p)^{\oplus s}$*, for some* $s \geq 0$.

Now we will give a proof of Hida's vertical control theorem, as stated at the beginning of the talk.

Proof. We have an isomorphism

$$H^1(Y_1(Np^r), \mathcal{V}_{k,\mathbf{Q}_p})^{\mathrm{ord}}_{\mathfrak{m}} \xrightarrow{\sim} \mathrm{Hom}_{K_1(p^r)}(V'_{k,\mathbf{Q}_p}, \widehat{H}^1_{\mathfrak{m}}).$$

We have a U_p-operator acting on the left, and on the right, we do as well. This is defined as follows: if π is any representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ and $H \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$ is a compact subgroup, then we define

$$U_p : \pi^H \xrightarrow{\mathrm{diag}(p,1)} \pi^{H \cap \mathrm{diag}(p,1) H \,\mathrm{diag}(p,1)^{-1}} \xrightarrow{\mathrm{tr}} \pi^H$$

In fact the map above is then equivariant for these actions.

Now we have an injection

$$\mathrm{Hom}_{K_1(p^r)}(V'_{k,\mathbf{Q}_p}, \widehat{H}^1_{\mathfrak{m}}) \hookrightarrow \mathrm{Hom}_{\begin{pmatrix} \mathbf{Z}_p^\times & \mathbf{Z}_p \\ 0 & 1+p^r\mathbf{Z}_p \end{pmatrix}}(V'_{k,\mathbf{Q}_p}, \widehat{H}^1_{\mathfrak{m}}),$$

But

$$V'_{k,\mathbf{Q}_p}|_{\begin{pmatrix} \mathbf{Q}_p^\times & \mathbf{Q}_p \\ 0 & \mathbf{Q}_p^\times \end{pmatrix}}$$

contains a weight $1 \otimes \chi_{k-i}^{-1}$, where

$$1 \otimes \chi_k^{-1} : (a,b,0,d) \mapsto d^{-(k-i)}$$

and

$$\mathrm{Hom}_{\mathbf{Z}_p^\times \times (1+p^r\mathbf{Z}_p)}(1 \otimes \chi_{k-2}^{-1}, (\widehat{H}^1_{\mathfrak{m}})^{N(\mathbf{Z}_p),\mathrm{ord}})$$

But

$$N(\mathbf{Z}_p) \subseteq K_1(p^r)$$

But then putting ord everywhere we get isomorphisms. $N(\mathbf{Z}_p)/(N(\mathbf{Z}_p) \cap \mathrm{diag}(p,1) N(\mathbf{Z}_p) \,\mathrm{diag}(p,1)^{-1}) \xrightarrow{\sim} K_1(p^r)/K_1(p^r) \cap \mathrm{diag}(p,1) K_1(p^r) \,\mathrm{diag}(p,1)^{-1}$.

Definition 4.2.1. Take π some continuous unitary admissible representation $\mathrm{GL}_2(\mathbf{Q}_p)$ and define $\mathrm{ord}(\pi) = e_{\mathrm{ord}} \pi^{N(\mathbf{Z}_p)}$. Here we use the fact that $(U_p^{n!}) \to e_{\mathrm{ord}}$. This is a particular case of Emerton's ordinary functor

In summary, $H^1(Y_1(Np^r), \mathcal{V}_{k,\mathbf{Q}_p})^{\mathrm{ord}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}_p^\times \times (1+p^r\mathbf{Z}_p)}(1 \otimes_{k-1}^{-1}, \mathrm{ord}(\widehat{H}^1_{\mathfrak{m}}))$.

Given a character $\chi : \mathbf{Z}_p^\times \to K^\times$ as above, we want to understand

$$H^1(Y_1(Np^r), \mathcal{V}_k, \epsilon)^{\mathrm{ord}} \xrightarrow{\sim} \mathrm{Hom}_{T(\mathbf{Z}_p)}(1 \otimes \chi_{k-2}^{-1} \epsilon^{-1}, \mathrm{ord}(\widehat{H}^1_{\mathfrak{m}}))$$

In summary, we started with $\widehat{H}^1_{\mathfrak{m}}$, constructed $(\widehat{H}^1_{\mathfrak{m}})^{N(\mathbf{Z}_p)}$ with a U_p-operator, and took $e_{\mathrm{ord}}(\widehat{H}^1_{\mathfrak{m}})^{N(\mathbf{Z}_p)}$, which is a representation of $T(\mathbf{Z}_p)$. Now we want to control $\mathrm{ord}(\widehat{H}^1_{\mathfrak{m}})[1 \otimes \chi_{k-2}\epsilon]$.

Theorem 4.2.4 (Hida). *Assume π is a unitary admissible continuous representation of $\mathrm{GL}_2(\mathbf{Q}_p)$. Then $\mathrm{ord}(\pi)$ is an admissible representation of $T(\mathbf{Z}_p)$. If $\pi|_{\mathrm{GL}_2(\mathbf{Z}_p)}$ is a direct summand of $\mathcal{C}(\mathrm{GL}_2(\mathbf{Z}_p), \mathbf{Z}_p)^{\oplus s}$. Then*

$$\mathrm{ord}(\pi)$$

is a direct summand of

$$\mathcal{C}(T(\mathbf{Z}_p), \mathbf{Z}_p)^{\oplus s}$$

Theorem 4.2.5 (Hecke). *If $\pi = \widehat{H}^1_{\mathfrak{m}}$, then*

$$\mathrm{ord}(\widehat{H}^1_{\mathfrak{m}}) = \bigoplus_{\psi : T(\mathbf{Z}_p) \to \overline{\mathbf{F}}_p^\times} \mathrm{ord}(\widehat{H}^1_{\mathfrak{m}})_\psi$$

where

$$\mathrm{ord}(\widehat{H}^1_{\mathfrak{m}})_\psi \cong \mathcal{C}(\Gamma, \mathbf{Z}_p)^{\oplus t}$$

where $\Gamma \cong (1 + p\mathbf{Z}_p)^2$.

Then if $\chi : T(\mathbf{Z}_p) \to \mathscr{O}_K^\times$, then

$$\dim \mathrm{ord}(\widehat{H}^1_{\mathfrak{m}})[\chi] = t_\psi$$

where $\chi \equiv \psi \bmod \varpi_K$. We used:

Fact 4.2.2. *If π is a representation of $T(\mathbf{Z}_p)$, then $\pi \cong \bigoplus_\psi \pi[\psi]$.*

□

5. Talk V

Today we'll talk about finite slope stuff.

5.1. **p-adic Jacquet functor.** Start with a locally analytic representation π of $\mathrm{GL}_2(\mathbf{Q}_p)$. Let $N_0 = \begin{pmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$, and take π^{N_0}, which has an action of

$$T_+ = \{(\alpha, \beta) \in T(\mathbf{Q}_p) \mid v_p(\alpha) \geq v_p(\beta)\}.$$

via

$$t \cdot v = \sum_{x \in N_0 / N_0 \cap t N_0 t^{-1}} xtv.$$

If we choose $(p, 1) \in T(\mathbf{Q}_p)$, this will act as

$$U_p = \sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}$$

However, π^{N_0} will not in general be an admissible representation of $T(\mathbf{Q}_p)$.

Remark 5.1.1. The distribution algebra $D(T(\mathbf{Z}_p), \mathbf{Q}_p)$ is a Fréchet-Stein algebra, isomorphic to

$$\mathscr{O}(\widehat{T(\mathbf{Z}_p)}),$$

i.e. the rigid analytic space parametrizing characters of $T(\mathbf{Z}_p)$, i.e. the weight space, i.e.

$$\widehat{T(\mathbf{Z}_p)} \cong (\widehat{\mathbf{Z}_p^\times})^2.$$

(hats here mean associated rigid space)

Now take $\widehat{T(\mathbf{Q}_p)} \cong \widehat{T(\mathbf{Z}_p)} \times (\mathbf{G}_m^{\mathrm{rig}})^2$, which is the space of "locally analytic characters of $T(\mathbf{Q}_p)$", and this is a Fréchet-Stein space. So $\mathscr{O}(\widehat{T(\mathbf{Q}_p)})$ is Fréchet-Stein. In fact

$$D(T(\mathbf{Q}_p), \mathbf{Q}_p) \hookrightarrow \mathscr{O}(\widehat{T(\mathbf{Q}_p)})$$

is a dense subspace. Let $\mathcal{T} = \widehat{T(\mathbf{Q}_p)}$.

Definition 5.1.1. We define the **Jacquet module of** π to be

$$J_B(\pi) = \mathcal{L}^b_{T^+}(\mathscr{O}(\mathcal{T}), \pi^{N_0}).$$

These are linear functions for the strong topology, which are T_+-equivariant. Or,

$$J_B(\pi)' = \mathscr{O}(\mathcal{T}) \widehat{\otimes}_{\mathbf{Q}_p[T_+]} (\pi^{N_0})'$$

Theorem 5.1.1 (Emerton)**.** *If π is a locally analytic representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ which is admissible, then $J_B(\pi)$ is a locally analytic representation of $T(\mathbf{Q}_p)$ such that $J_B(\pi)'$ is coadmissible as an $\mathscr{O}(\mathcal{T})$-module.*

A consequence of this theorem is that

$$\{\text{coadmissible } \mathscr{O}(\mathcal{T})\text{-modules}\} \cong \{\text{coherent sheaves over } \mathcal{T}\}$$

where we send $\mathcal{M} \mapsto \mathcal{M}(\mathcal{T})$. Thus we obtain $\mathcal{M}_\pi$ a coherent sheaf on $\mathcal{T} = \widehat{T(\mathbf{Q}_p)}$, which is the dual of the Jacquet module.

So let's try to understand this sheaf. Take a closed point $x \in \mathcal{T}$, i.e. a map $\chi_x : T(\mathbf{Q}_p) \to \overline{\mathbf{Q}_p}^\times$. Then

$$\mathcal{M}_\pi \otimes_{\mathscr{O}(\mathcal{T})} \kappa(x) \cong \mathrm{Hom}_{T_+}(\chi_x, \pi^{N_0}) \cong (\pi^{N_0}[\chi_x])'.$$

So in some sense, this coherent sheaf is the family of eigenspaces, for each character.

Now we try to apply this construction to

$$\pi = \widehat{H}^1(N)^{\mathrm{an}}$$

which is an admissible representation of $\mathrm{GL}_2(\mathbf{Q}_p)$. As before, fix a non-Eisenstein maximal ideal $\mathfrak{m}$ of the Hecke algebra, which corresponds to some irreducible Galois representation

$$\overline{\rho} : G_{\overline{\mathbf{Q}}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p).$$

Then $J_B(\widehat{H}^1_{\mathfrak{m}}(N)^{\mathrm{an}})' \cong \Gamma(\mathcal{T}, \mathcal{M}_{\overline{\rho}})$ for some $\mathcal{M}_{\overline{\rho}}$ coherent on $\mathcal{T}$.

Theorem 5.1.2. *If π is an admissible locally analytic representation of $\mathrm{GL}_2(\mathbf{Q}_p)$, then $\pi|_{\mathrm{GL}_2(\mathbf{Z}_p)}$ is a direct factor of*

$$\mathcal{C}^{\mathrm{an}}(\mathrm{GL}_2(\mathbf{Z}_p), \mathbf{Q}_p)^{\oplus s}$$

Then the map

$$\kappa : \operatorname{supp} \mathcal{M}_\pi \subseteq \mathcal{T} \to \mathcal{W} := \widehat{T(\mathbf{Z}_p)} \cong (\widehat{\mathbf{Z}_p})^2$$

has discrete fibers. Furthermore, locally on $\operatorname{supp} \mathcal{M}$, $\mathcal{M}$ is a finite free $\mathscr{O}_{\mathcal{W}}$-module.

Now $\mathbf{T}^{\mathrm{sph}}_{\overline{\rho}} \subseteq \mathrm{End}(\widehat{H}^1_{\overline{\rho}})$, and by functoriality of "an", we have

$$\psi : \mathbf{T}^{\mathrm{sph}}_{\overline{\rho}} \to \mathrm{End}(\widehat{H}^{1,\mathrm{an}}_{\overline{\rho}}) \to \mathrm{End}(\mathcal{M}_{\overline{\rho}}).$$

So now we can define $\mathcal{A}_{\overline{\rho}}$, the $\mathscr{O}_{\mathcal{W}}$-subalgebra of $\mathcal{E}\backslash\lceil(M_{\overline{\rho}})$ generated by $\mathrm{im}\,\psi$.

Then the eigenvariety $\mathcal{E}_{\overline{\rho}} = \mathrm{Sp}\,\mathcal{A}_{\overline{\rho}}$, i.e. the relative rigid analytic spectrum over $\operatorname{supp} \mathcal{M}_{\overline{\rho}}$. Thus, we get a rigid analytic variety $\mathcal{E}_{\overline{\rho}} \xrightarrow{\kappa} \mathcal{W}$ quasi-finite with discrete fibers. This is locally finite on $\mathcal{E}_{\overline{\rho}}$, and it's finite surjective on every irreducible component.

So what are the points of $\mathcal{E}_{\overline{\rho}}$? If we fix a closed point $x \in \mathcal{E}_{\overline{\rho}}$, we obtain a character

$$\lambda : \mathbf{T}^{\mathrm{sph}}_{\overline{\rho}} \to \overline{\mathbf{Q}}_p$$

by definition, and the image in the weight space gives a character $\delta : T(\mathbf{Q}_p) \to \overline{\mathbf{Q}}_p$.

Then points of $\mathcal{E}_{\overline{\rho}}(\overline{\mathbf{Q}}_p)$ are pairs $(\lambda, \delta) \in \mathrm{Spm}(\mathbf{T}^{\mathrm{sph}}[1/p]) \times \widehat{T(\mathbf{Q}_p)}$ such tha

$$\mathrm{Hom}_{T(\mathbf{Q}_p)}(\delta, J_B(\widehat{H}^{1,\mathrm{an}}[\lambda])) \neq 0.$$

Then $\widehat{H}^1_{\overline{\rho}}[\lambda] \subseteq \widehat{H}^1_{\overline{\rho}}$ is a closed subspace.

Definition 5.1.2. A point $x = (\lambda, \delta)$ is called **classical** if λ is classical, i.e. there exists W some algebraic representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ and some open subgroup $K_p \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$ such that

$$H^1(Y_{K_p}, \mathcal{V}_{W^\vee})[\lambda] \cong \mathrm{Hom}_{K_p}(W, \widehat{H}^1[\lambda]) \neq 0.$$

For example, the λ associated to $f \in S_k(\Gamma_1(Np^r))$ (for $k \geq 2$) is classical by letting $W = (\mathrm{Sym}^{k-2}\,\mathbf{Q}_p^2)'$.

Now we want to show that classical points are dense.

5.2. **Classicality Result.** Start with $\delta \in \widehat{T(\mathbf{Q}_p)} = \mathrm{Hom}(T(\mathbf{Q}_p), \overline{\mathbf{Q}_p}^\times)$ and assume δ is locally algebraic, which means that

$$\delta = \delta^{\mathrm{alg}} \cdot \delta^{\mathrm{sm}},$$

where $\delta^{\mathrm{alg}}(a,d) = a^{k_1} d^{k_2}$ for some $k_1, k_2 \in \mathbf{Z}$, and δ^{sm} is a smooth character, i.e. locally constant.

Let B, T be the Borel and torus as usual, and write $\overline{B}$ for the opposite Borel. Write $\mathfrak{b} = \mathrm{Lie}(B)$. We let

$$M(\delta^{\mathrm{alg}}) = U(gl_2) \otimes_{U(\overline{\mathfrak{b}})} \delta^{\mathrm{alg}} \in \mathcal{O}^{\overline{\mathfrak{b}}}_{\mathrm{alg}}.$$

But given $M \in \mathcal{O}^{\overline{\mathfrak{b}}}_{\mathrm{alg}}$, we can construct the Orlik-Strauch $\mathrm{GL}_2(\mathbf{Q}_p)$-representation

$$\mathcal{F}^{\mathrm{GL}_2}_{\overline{B}}(M, \delta^{\mathrm{sm}})$$

Theorem 5.2.1 (Emerton, Breuil)**.** *If π is a unitary p-adic Banach representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ and $\delta = \delta^{\mathrm{alg}}\delta^{\mathrm{sm}} \in \widehat{T(\mathbf{Q}_p)}$, then*

$$\mathrm{Hom}_{T(\mathbf{Q}_p)}(\delta, J_B(\pi^{\mathrm{an}})) \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}(\mathcal{F}^{\mathrm{GL}_2(\mathbf{Q}_p)}_{\overline{B}(\mathbf{Q}_p)}((M(\delta^{\mathrm{alg}})^{-1})^\vee, \delta^{\mathrm{sm}}), \pi^{\mathrm{an}})$$

What is $\mathcal{F}(\delta)$? If $k_1 < k_2$

$$M((\delta^{\mathrm{alg}})^{-1})^\vee$$

is simple, so it's isomorphic to its dual, then

$$\mathcal{F}(\delta) = \left(\mathrm{Ind}^{\mathrm{GL}_2(\mathbf{Q}_p)}_{\overline{B(\mathbf{Q}_p)}}\,\delta\right)^{\mathrm{an}}$$

and this is topologically irreducible. If $k_1 \geq k_2$, then

$$0 \to M((\delta')^{-1}) \to M((\delta^{\mathrm{alg}})^{-1}) \to L((\delta^{\mathrm{alg}})^{-1}) \to 0$$

The quotient is finite dimensional, and $\delta' = \delta_{k_2-1,k_1+1}$. Now apply duality, we have

$$0 \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbf{Q}_p)}_{\overline{B}}(\delta'\delta^{\mathrm{sm}}) \to \mathcal{F}(\delta) \to L(\delta^{\mathrm{alg}}) \otimes (\mathrm{Ind}^{\mathrm{GL}_2}_{\overline{B}}\,\delta^{\mathrm{sm}})^{\mathrm{an}} \to 0$$

and we've essentially changed the order of the Jordan-Hölder filtration. Note $L(\delta^{\mathrm{alg}}) \cong \mathrm{Sym}^{k-2}\,\mathbf{Q}_p^2 \otimes \det^{k-2}$.

Theorem 5.2.2 (Coleman, Emerton)**.** *Assume $k_1 \geq k_2$, so we're in the second case. Take $x = (\lambda, \delta) \in \mathcal{E}_{\overline{\rho}}(\overline{\mathbf{Q}_p})$ and take $\delta = \delta^{\mathrm{alg}}\delta^{\mathrm{sm}} = \delta_{k_1,k_2}\delta^{\mathrm{sm}}$. Then if $v_p(\delta^{\mathrm{sm}}(p,1)) < -k_2 + 1$, then λ is classical.*

Proof. If $x = (\lambda, \delta) \in \mathcal{E}_{\overline{\rho}}(\overline{\mathbf{Q}_p})$, then

$$\mathrm{Hom}(\delta, J_B(\widehat{H}^{1,\mathrm{an}}_{\overline{\rho}}[\lambda])) \neq 0$$

By the reciprocity, there is a nonzero map $\mathcal{F}(\delta) \to \widehat{H}^1_{\overline{\rho}}[\lambda]$. But there is a quotient $L(\delta^{\mathrm{alg}}) \otimes (\mathrm{Ind}\, \delta^{\mathrm{sm}})^{\mathrm{sm}}$, and we want the first map to factor through the second.

Now it's sufficient to to prove that there is no map

$$(\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \delta' \delta^{\mathrm{sm}})^{\mathrm{an}} \to \widehat{H}^1_{\overline{\rho}}$$

but the target is a unitary representation of $\mathrm{GL}_2(\mathbf{Q}_p)$, so this map is injective (something about invariant norms?).

So it suffices to show that there is no $\mathrm{GL}_2(\mathbf{Q}_p)$-invariant norm on $(\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \delta_{k_2-1,k_1+1}\delta^{\mathrm{sm}})^{\mathrm{an}} = \pi$. If so then the U_p-action on π^{N_0} has norm $||U_p|| \leq 1$ for some invariant norm. So we define $f \in \pi$ by: if $b \in \overline{B}(\mathbf{Q}_p)$ and $\mu \in I$ (iwahori) then define

$$f(b(0,1,1,0)u) = 0$$

and

$$f(b(1,x,0,1)) = \widetilde{\delta}(b)$$

for $x \in \mathbf{Z}_p$ via $\mathrm{GL}_2(\mathbf{Q}_p) = \overline{B}I \sqcup \overline{B}(0,1,1,0)I$. Then one checks that f is N_0-invariant, and

$$U_p(f) = \widetilde{\delta}(p,0,0,1)f,$$

so $||\widetilde{\delta}(p,0,0,1)|| \leq 1$, so $v_p(\delta^{\mathrm{sm}}(p,0,0,1)) \geq -k_2+1$. $\square$

Example 5.2.1. If $f \in S_k(N)$ is an eigenform and $\overline{\rho}_f \cong \overline{\rho}$. If α is a root of the Hecke polynomial of f at p. Then a local-global compatibility result says that $(\lambda_f, \delta) \in \mathcal{E}_{\overline{\rho}}$ where

$$\delta = \delta_{0,2-k}(\omega_\alpha, \omega_{\alpha^{-1}}|\cdot|^{2-k})$$

Conversely, if $(\lambda, \delta) \in \mathcal{E}_{\overline{\rho}}$ and we assume that $\delta = \delta_{0,2-k}\delta^{\mathrm{sm}}$, where

$$\delta^{\mathrm{sm}} = \omega_\alpha \otimes \omega_{\alpha^{-1}}|\cdot|^{2-k}.$$

So if we assume that $v_p(\alpha) < k-1$ then λ is classical and comes from a modular eigenform f.

Corollary 5.2.1. *The classical points are Zariski dense in $\mathcal{E}_{\overline{\rho}}$.*

So $\mathcal{E}_{\overline{\rho}} \hookrightarrow (\mathrm{Spf}\, \mathbf{T}^{\mathrm{sph}}_{\overline{\rho}})^{\mathrm{rig}} \times \widehat{T(\mathbf{Q}_p)}$ is the Zariski closure of the pairs (λ, δ) with λ classical (unramified at p) and δ locally algebraic dominant, with unramified smooth part. So $\mathcal{E}_{\overline{\rho}}$ is a rigid analytic space whose closed points are systems of Hecke eigenvalues interpolating the classical systems.

5.3. **Galois representations.** Say $x = (\lambda, \delta) \in \mathcal{E}_{\overline{\rho}}$ is a classical point, then there exists a unique $\rho_x : G_{\overline{\mathbf{Q}}} \to \mathrm{GL}_2(\overline{\mathbf{Q}_p})$ unramified outside Np such that for all $\ell \nmid Np$, $\mathrm{tr}(\rho_x(\mathrm{Frob}_\ell)) = \lambda(T(\ell))$. From the density result plus some techniques of the theory of pseudo-representations, we can construct for all $x \in \mathcal{E}_{\overline{\rho}}$ some ρ_x as in the classical case satisfying the same properties.

Now fix $x = (\lambda, \delta) \in \mathcal{E}_{\overline{\rho}}$. Then ρ_x is determined by λ. Can we read δ on ρ_x?

Example 5.3.1. Look at (λ, δ) with λ classical and assume that $\delta = \delta_{k_1,k_2}\delta^{\mathrm{sm}}$ with δ^{sm} unramified. By local-global compatibility, ρ_x is semistable, and trianguline, i.e. $D_{\mathrm{rig}}(\rho_x|_{G_{\mathbf{Q}_p}})$ is an extension of rank one (φ, Γ)-modules, which are associated to two characters δ_1, δ_2, and these are the two components of δ (possibly need to twist δ_2).

Theorem 5.3.1 (Kisin)**.** *Take $x = (\lambda, \delta) \in \mathcal{E}_{\overline{\rho}}$ then $\rho_x|_{G_{\mathbf{Q}_p}}$ is trianguline, and there exists a nonzero morphism $\mathcal{R}(\delta_1) \hookrightarrow D_{\mathrm{rig}}(\rho_x)$.*

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