p-ADIC HODGE THEORY AND DEFORMATIONS OF GALOIS REPRESENTATIONS

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Notes taken by Ashwin Iyengar[1] *and have not been checked by the speaker. Any errors are due to me.*

Contents

1. Talk 1

1.1. **Introduction.** We want to study $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and for all cases of interest for us, K will be a finite extension of $\mathbf{Q}_p$ or $K/\mathbf{F}_p((t))$. In particular, we want to study representations on finitely generated:

(1) $\mathbf{F}_p$-vector spaces,

(2) $\mathbf{Z}_p$-modules, and

(3) $\mathbf{Q}_p$-vector spaces.

We will appeal to a general strategy of Fontaine, which is to study representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ using very large period rings that are typically labeled by "B" with varying decorations.

Date: March 20, 2019.

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More precisely, let G be a group and L a field. Then we want to study representations of G on finite dimensional L-vector spaces. Fontaine's idea is to find an L-algebra B (say an integral domain) equipped with a G-action that is trivial on L: these are called "period rings". Then G acts on the finite free B-module $V \otimes_L B$ so we can look at

$$D_B(V) := (V \otimes_L B)^G,$$

the B^G-module of G-invariants. We have a canonical G-equivariant map

$$D_B(V) \otimes_{B^G} B \to V \otimes_L B \tag{1}$$

In good situations, this will be an isomorphism, and we can hopefully recover V.

Definition 1.1.1. An L-algebra B is called **G-regular** if

(1) B is an integral domain
(2) $B^G = \mathrm{Frac}(B)^G$
(3) For any nonzero $b \in B$ such that $L \cdot b$ is G-stable, then $b \in B^\times$.

In particular, they implies that $E := B^G$ is a field, and one can prove that $\dim_E D_B(V) \leq \dim_L V$ and that the map (1) is injective. If the dimensions agree, then (1) is an isomorphism. In this case, we say that V is **B-admissible**.

To see that the map (1) is injective, call $C = \mathrm{Frac}(B)$ and let $E := B^G$ and note we have a diagram

$$\begin{array}{ccc} D_B(V) \otimes_E B & \longrightarrow & V \otimes_L B \\ \big\downarrow & & \big\downarrow \\ D_C(B) \otimes_E C & \longrightarrow & V \otimes_L C \end{array}$$

So it suffices to show that the bottom is injective. In other words, it suffices to check that an E-linearly independent set of vectors in $D_C(V)$ is C-linearly independent. So suppose $X \subseteq D_C(V)$ is an E-linearly independent set of vectors, and suppose there is a nontrivial C-linear dependence $x_m = \sum_{i<m} c_i x_i$ of minimal length, where each $x_i \in X$. Then for any $\sigma \in G$,

$$\sum_{i<m} c_i v_i = x_m = \sigma(x_m) = \sum_{i<m} \sigma(c_i) v_i.$$

By minimality of m, we must have $c_i = \sigma(c_i)$, or in other words, $c_i \in C^G = E$. But then we have a nontrivial E-linear dependence between $x_1, \ldots, x_m$, which is a contradiction.

Remark 1.1.2.

(1) Given $\rho : G \to \mathrm{GL}(V)$ for V an L-vector space, if we choose a basis of V then ρ is a cocycle (for the trivial action on $\mathrm{GL}_n(L)$), i.e. $\rho \in H^1(G, \mathrm{GL}_n(L))$. Then (ρ, V) is B-admissible if and only if its image under the map of pointed cohomology sets
$$H^1(G, \mathrm{GL}_n(L)) \to H^1(G, \mathrm{GL}_n(B))$$
(induced by the G-equivariant map $L \to B$) is trivial.
(2) If G is a topological group and B is a topological L-algebra with continuous G-action and we consider only continuous ρ, then the above statement is still true if we replace H^1 with H^1_{cts}, i.e. continuous Galois cohomology.

Note that in practice, B comes with additional structures. For example, this could be an endomorphism, or a filtration, depending on its purpose: in any case, this extra structure will be "compatible with G", which means different things depending on the context. Given V this extra structure induces additional structure on $D_B(V)$, and the hope is that we can recover a B-admissible representation from $D_B(V)$.

1.2. **φ-modules and (φ, Γ)-modules.** Let's give perhaps the simplest case of the above phenomenon. Let F be a local field of characteristic p (i.e. $\mathbf{F}_q((t))$). Let $G = G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ and now let $L = \mathbf{F}_p$, so we work with mod p coefficients.

Let $B = F^{\mathrm{sep}}$. This is clearly G_F-regular, and by definition of G_F we get $B^{G_F} = F$.

Lemma 1.2.1. *Every continuous G_F-representation on a finite dimensional $\mathbf{F}_p$-vector space is F^{sep}-admissible.*

Proof. By the remark, $\rho : G_F \xrightarrow{\text{cts}} \mathrm{GL}_n(F)$ defines a class in $H^1(G_F, \mathrm{GL}_n(F))$, and we look at its class in $H^1(G_F, \mathrm{GL}_n(F^{\mathrm{sep}}))$, and Hilbert's theorem 90 exactly says that this is 0. $\square$

Since we're in characteristic p we get the extra structure of Frobenius for free: this is the endomorphism $\varphi : F^{\mathrm{sep}} \to F^{\mathrm{sep}}$ sending $x \mapsto x^p$, which commutes with the G-action. Then starting with a representation (ρ, V), we get an induced map

$$\Phi : D(V) = (V \otimes_{\mathbf{F}_p} F^{\mathrm{sep}})^{G_F} \xrightarrow{\mathrm{id} \otimes \varphi} (V \otimes_{\mathbf{F}_p} F^{\mathrm{sep}})^{G_F}$$

which makes $D(V)$ into a φ-module over F.

Definition 1.2.2. Let A be a ring and let $\varphi : A \to A$ be any endomorphism. Then a **φ-module over** A is a finitely generated A-module D together with a map $\Phi : D \to D$ which is semi-linear with respect to φ, i.e. $\Phi(ad) = \varphi(a)\Phi(d)$ such that

$$\varphi^* D = D \otimes_{A,\varphi} A \to D$$

sending $d \otimes 1 \mapsto \phi(d)$ is an isomorphism.

In the above example, the linearization of Φ is injective because φ is, but $D(V)$ is finite-dimensional over F, so Φ is an isomorphism.

Corollary 1.2.3. *The functor*

$$D : \left\{ \begin{array}{c} \textit{continuous representations of } G_F \\ \textit{on finite dimensional } \mathbf{F}_p\textit{-vector spaces} \end{array} \right\} \to \mathsf{Mod}_{\varphi,F}$$

is fully faithful.

Theorem 1.2.4 (Fontaine)**.** *D is an equivalence of categories with quasi-inverse given by*

$$D \mapsto \mathbf{V}(D) := (D \otimes_F F^{\mathrm{sep}})^{\varphi=1}$$

Proof. Regarding fully faithfulness, we check that $\mathbf{V}$ is a quasi-inverse on the essential image. Given V, we have an isomorphism

$$D(V) \otimes_F F^{\mathrm{sep}} \xrightarrow{\sim} V \otimes_{\mathbf{F}_p} F^{\mathrm{sep}}$$

which is clearly G_F-equivariant, but also φ-invariant, and $(F^{\mathrm{sep}})^{\varphi=1} = \mathbf{F}_p$. Hence

$$(V \otimes_{\mathbf{F}_p} F^{\mathrm{sep}})^{\varphi=1} = V.$$

For essential surjectivity, one reduces to showing that if D a φ-module over F, then $\dim_{\mathbf{F}_p} \mathbf{V}(D) = \dim_F D$. $\square$

Remark 1.2.5. Note it's hard to actually write down representations, so this gives us a way to actually explicitly construct continuous representations of G_F with mod p coefficients, just using linear algebra.

1.3. **Torsion coefficients and $\mathbf{Q}_p$-coefficients.** Suppose F is a field of characteristic p, so that we have access to Frobenius.

Definition 1.3.1. Say $(\mathscr{O}_{\mathcal{E}}, \varphi)$ is a **Cohen ring** for F if $\mathscr{O}_{\mathcal{E}}$ is a complete discrete valuation ring with uniformizer p, residue field F, and a lift φ of Frobenius.

Example 1.3.2. If $F = \mathbf{F}_q((t))$, then we can take $\mathscr{O}_{\mathcal{E}} = W(\mathbf{F}_q)((t))^{\wedge_p}$, and we can set φ to be the Witt vector Frobenius on $W(\mathbf{F}_q)$ and either take $t \mapsto t^p$ or $t \mapsto (1+t)^p - 1$.

Given a Cohen ring $\mathscr{O}_{\mathcal{E}}$ we can set $\mathcal{E} = \mathscr{O}_{\mathcal{E}}[1/p]$. We let $\mathcal{E}^{\mathrm{ur}}$ be the maximal unramified extension of $\mathcal{E}$ with ring of integers $\mathscr{O}_{\mathcal{E}^{\mathrm{ur}}}$, and we can take the p-adic completion $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$, and by inverting p we get $\widehat{\mathcal{E}}^{\mathrm{ur}}$. Then $\mathcal{E}^{\mathrm{ur}}/\mathcal{E}$ is Galois with Galois group G_F, and we have an extension of φ and the G_F-action to $\mathscr{O}_{\mathcal{E}^{\mathrm{ur}}}, \mathcal{E}^{\mathrm{ur}}, \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}, \widehat{\mathcal{E}}^{\mathrm{ur}}$.

Lemma 1.3.3. *The natural map $\mathscr{O}_{\mathcal{E}} \to \mathscr{O}_{\mathcal{E}^{\mathrm{ur}}}$ induces an isomorphism*

$$\mathscr{O}_{\mathcal{E}} \xrightarrow{\sim} (\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{G_F}.$$

Furthermore $\mathbf{Z}_p = (\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{\varphi=1}$, and

$$H^1(G_F, \mathrm{GL}_n(\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})) = \{*\}$$

Proof. Use successive approximation. Filter $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$ (resp. $\mathrm{GL}_n(\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})$) so that the graded pieces look like F^{sep} (resp. $\mathrm{Mat}_n(F^{\mathrm{sep}})$). □

Definition 1.3.4. A φ-module (D, Φ) over $\mathcal{E}$ is called **étale** if there exists a φ module (D', Φ') over $\mathscr{O}_{\mathcal{E}}$ such that

$$(D, \Phi) \cong (D', \Phi') \otimes_{\mathscr{O}_{\mathcal{E}}} \mathcal{E}$$

Not every Φ-module over $\mathcal{E}$ has an integral model, so we need this definition. These form a category $\mathsf{Mod}^{\mathrm{et}}_{\varphi,\mathcal{E}}$.

Corollary 1.3.5.

(1) *Let Λ be a finitely generated $\mathbf{Z}_p$-module with continuous $\rho : G_F \to \mathrm{GL}(\Lambda)$ a continuous representation. Then $D(\Lambda) := (\Lambda \otimes_{\mathbf{Z}_p} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{G_F}$ is a φ-module over $\mathscr{O}_{\mathcal{E}}$ and*

$$D(\Lambda) \otimes_{\mathscr{O}_{\mathcal{E}}} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}} \to \Lambda \otimes_{\mathbf{Z}_p} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$$

is a (G_F, φ)-equivariant isomorphism.

(2) *We get an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{continuous } G_F\textit{-representations} \\ \textit{on finitely generated } \mathbf{Z}_p\textit{-modules} \end{array} \right\} \xrightarrow{D} \mathsf{Mod}_{\varphi,\mathscr{O}_{\mathcal{E}}}$$

with quasi-inverse $\mathbf{V} : D \mapsto (D \otimes_{\mathscr{O}_{\mathcal{E}}} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{\varphi=1}$.

(3) *The functor*

$$\left\{ \begin{array}{c} \textit{continuous } G_F\textit{-representations} \\ \textit{on finite dimensional } \mathbf{Q}_p\textit{-vector spaces} \end{array} \right\} \xrightarrow{D} \mathsf{Mod}_{\varphi,\mathcal{E}}$$

taking $V \mapsto (V \otimes_{\mathbf{Q}_p} \widehat{\mathcal{E}}^{\mathrm{ur}})^{G_F}$ is fully faithful with essential image $\mathsf{Mod}^{\mathrm{et}}_{\varphi,\mathcal{E}}$.

Thus from (3) we see that if $G_F \to \mathrm{GL}(V)$ is a continuous representation, then there exists $\Lambda \subseteq V$ a G_F-stable $\mathbf{Z}_p$-lattice.

1.4. **Local fields in mixed characteristic.** Let $K/\mathbf{Q}_p$ be a finite extension, and let $G_K = \mathrm{Gal}(\overline{K}/K)$. Then we ask the same questions as before: we want to describe continuous representations of G_K on finitely generated

(1) $\mathbf{F}_p$-vector spaces,

(2) $\mathbf{Z}_p$-modules, or

(3) $\mathbf{Q}_p$-vector spaces.

The idea is to find a big extension K_∞/K, which should be an infinite and deeply ramified Galois extension, and we write Γ for its Galois group. But for this to be useful, Γ should be as simple as possible, and we want $\mathrm{Gal}(\overline{K}/K_\infty) \cong \mathrm{Gal}(F^{\mathrm{sep}}/F)$ where F is a local field in characteristic p. Then we want the G_{K_∞}-action on $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$ to extend to a continuous G_K-action commuting with φ.

In this case, we get a continuous Γ-action on $(\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{G_{K_\infty}} = \mathscr{O}_{\mathcal{E}}$ commuting with φ.

Definition 1.4.1. A (φ,Γ)**-module** over $\mathscr{O}_{\mathcal{E}}$ (resp. $\mathcal{E}$) is a φ-module over $\mathscr{O}_{\mathcal{E}}$ (resp. $\mathcal{E}$) with a semi-linear Γ-action commuting with φ. A (φ,Γ)-module over $\mathcal{E}$ is called **étale** if its underlying φ-module is étale.

Ok we haven't actually found K_∞ yet, but assume we have this setup.

Theorem 1.4.2.

(1) *There is an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{continuous } G_K\textit{-representations} \\ \textit{on finitely generated } \mathbf{Z}_p\textit{-modules} \end{array} \right\} \to \mathsf{Mod}_{(\varphi,\Gamma),\mathscr{O}_{\mathcal{E}}}$$

sending $\Lambda \mapsto (\Lambda \otimes_{\mathbf{Z}_p} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{G_{K_\infty}}$ *with quasi-inverse given by*

$$D \mapsto (D \otimes_{\mathscr{O}_{\mathcal{E}}} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{\varphi=1}.$$

(2) *There is an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{continuous } G_K\textit{-representations} \\ \textit{on finite dimensional } \mathbf{Q}_p\textit{-vector spaces} \end{array} \right\} \to \mathsf{Mod}^{\mathrm{et}}_{(\varphi,\Gamma),\mathcal{E}}$$

sending $V \mapsto (V \otimes_{\mathbf{Q}_p} \widehat{\mathcal{E}}^{\mathrm{ur}})^{G_{K_\infty}}$ *with quasi-inverse given by*

$$D \mapsto (D \otimes_{\mathcal{E}} \widehat{\mathcal{E}}^{\mathrm{ur}})^{\varphi=1}.$$

So how do we actually define this K_∞? Classically, this was done using the theory of "norm fields", due to Fontaine-Wintenberger. A more general perspective is the tilting equivalence for perfectoid fields, due to Scholze, which we discuss next time.

2. Talk II

2.1. **Perfectoid fields and tilting.**

Definition 2.1.1. A **perfectoid field** K is a complete non-archimedean field of residue characteristic p, which is complete with respect to a valuation v_K (resp. norm $|\cdot|$) such that

(1) v_K is non-discrete, i.e. the corresponding valuation ring $\mathscr{O}_K$ with maximal ideal $\mathfrak{m}_K$ satisfies $\mathfrak{m}_K^2 = \mathfrak{m}_K$.

(2) The map $\mathrm{Frob} : \mathscr{O}_K/p \xrightarrow{x\mapsto x^p} \mathscr{O}_K/p$ is surjective.

Example 2.1.2.

(1) For example, we can look at the field $\mathbf{F}_q((x))$. Unfortunately Frobenius is not surjective here, so instead we take
$$\mathbf{F}_q((x^{1/p^\infty})) = \left(\bigcup_{n\geq 0} \mathbf{F}_q((x^{1/p^n}))\right)^{\wedge_x}.$$
Then this is a perfectoid field: note we had to adjoin all the p-power roots of x so that Frobenius is surjective, and then we need to complete with respect to some norm, and we choose the x-adic norm.

(2) We could also add a lot more and take
$$\overline{\mathbf{F}_q((x))}^{\wedge_x}$$
This is the other main characteristic p example.

(3) We could take $\overline{\mathbf{Q}}_p^{\wedge_p}$, otherwise known as $\mathbf{C}_p$.

(4) Start with $F/\mathbf{Q}_p$ finite and $\pi \in F$ a uniformizer. Then I can take
$$F(\pi^{1/p^\infty})^{\wedge_\pi}$$

(5) Let $F/\mathbf{Q}_p$ is finite, fix ϵ_n a compatible system of p-power roots of 1. Then look at $F(\epsilon_n \mid n \geq 1)^{\wedge_p}$.

Remark 2.1.3. If K is a complete non-archimedean field of characteristic p with a nondiscrete valuation with respect to which K is complete, then perfectoid is the same as perfect.

Definition 2.1.4. Let K be a perfectoid field with ring of integers $\mathscr{O}_K$ and a pseudo-uniformizer $\varpi \in \mathscr{O}_K$, i.e. some element $\varpi \in \mathscr{O}_K$ such that $|p| \leq |\varpi| \leq 1$. Then we define
$$\mathscr{O}_{K^\flat} = \varprojlim_{x \mapsto x^p} \mathscr{O}_K/\varpi.$$
Choose $\varpi^\flat = (\varpi_0^\flat, \varpi_1^\flat, \varpi_2^\flat, \dots) \in \mathscr{O}_{K^\flat}$ such that $\varpi_1^\flat \neq 0$. Then the **tilt** of K is
$$K^\flat = \mathscr{O}_{K^\flat}[1/\varpi^\flat]$$

Lemma 2.1.5.

(1) *$\mathscr{O}_{K^\flat}$ has a valuation defined by*
$$(x_0, x_1, \dots) \mapsto \varinjlim_{n\to\infty} v_K(\widetilde{x}_n^{p^n})$$
for some choice of $\widetilde{x_n} \in \mathscr{O}_K$ lifting x_n.

(2) *$\mathscr{O}_{K^\flat}$ is complete with respect to this valuation, and $\mathscr{O}_{K^\flat}$ does not depend of the choice of ϖ, and the topology defined above does not depend on the valuation on K.*

(3) *$K^\flat$ is a characteristic p perfectoid field: it's complete by the remark above, non-discrete by construction, and we forced Frobenius to be an isomorphism. Note K doesn't depend on $\varpi^\flat$.*

Remark 2.1.6. If K already has characteristic p, then tilting does nothing. In general
$$\mathscr{O}_{K^\flat} = \varprojlim_{x\mapsto x^p} \mathscr{O}_K/\varpi = \varprojlim_{x\mapsto x^p} \mathscr{O}_K$$
is an isomorphism of multiplicative monoids. If we're in characteristic p, this commutes with addition, but in characteristic 0 not quite. In general if $(x^{(n)}), (y^{(n)}) \in \varprojlim_{x\mapsto x^p} \mathscr{O}_K$, then the addition is given by
$$((x^{(n)}) + (y^{(n)}))^{(n)} = \lim_{m\to\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$$
Note that in characteristic p this simplifies to usual addition.

Theorem 2.1.7 (Scholze)**.**

(1) *Let K be a perfectoid field and L/K is a finite extension then L is perfectoid.*

(2) *The map $L \mapsto L^\flat$ defines a degree preserving equivalence of categories between*

$$\{\textit{finite extensions of } K\} \xrightarrow{\sim} \left\{\textit{finite extensions of } K^\flat\right\}$$

Note K is always perfect so this is really an equivalence of finite separable extensions. In particular, we get a canonical isomorphism

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \cong \mathrm{Gal}((K^\flat)^{\mathrm{sep}}/K^\flat)$$

Remark 2.1.8.

(1) It's actually easy to write down a quasi-inverse to this functor. Given K perfectoid there is a canonical ring homomorphism

$$v_K : W(\mathscr{O}_{K^\flat}) \twoheadrightarrow \mathscr{O}_K$$

sending $[(x_0, x_1, \dots)] \mapsto \lim_{n\to\infty} \widetilde{x_n}^{p^n}$ where $\widetilde{x_n} \in \mathscr{O}_K$ lifts x_n. Then given a finite extension $E/K^\flat$ with ring of integers $\mathscr{O}_E$ we let

$$L = W(\mathscr{O}_E) \otimes_{W(\mathscr{O}_{K^\flat}), v_K} K$$

Then L is an untilt of E, i.e. $L^\flat \cong E$.

(2) In fact this works in greater generality. We get a more general tilting equivalence.

$$\{\text{perfectoid } K\text{-algebras}\} \xrightarrow{\sim} \left\{\text{perfectoid } K^\flat\text{-algebras}\right\}.$$

By definition a perfectoid K-algebra is a Banach K-algebra such that $R^\circ \subseteq R$ is open and bounded, where

$$R^\circ = \{x \in R \mid \{x^n\} \text{ is bounded}\}$$

and $R^\circ/\varpi \xrightarrow{x \mapsto x^p} R^\circ/\varpi$ is surjective. Again the equivalence is given by $R \mapsto R^\flat$, and there is still a quasi-inverse given by the construction above. There is an almost purity theorem, which says that given a perfectoid K-algebra R, tilting induces an equivalence

$$\mathrm{FEt}_R \xrightarrow{\sim} \mathrm{FEt}_{R^\flat}$$

between the categories of finite étale R-algebras and finite étale $R^\flat$-algebras.

(3) If you allow all base fields K, tilting does not give an equivalence. For example, take

(a) $K_1 = \mathbf{Q}_p(\epsilon_n \mid n \geq 1)^{\wedge_p}$

(b) $K_2 = \mathbf{Q}_p(p^{1/p^\infty})^{\wedge_p}$

(c) $K_3 = \mathbf{F}_p((x^{1/p^\infty}))$

These are all perfectoid, and $K_1^\flat \cong K_2^\flat \cong K_3^\flat \cong K_3$. Note $K_2^\flat$ has an element

$$\varpi^\flat = (p, p^{1/p}, p^{1/p^2}, \dots).$$

Then $\mathbf{F}_p((x)) \to K_2^\flat$ sending $x \mapsto \varpi^\flat$ induces an isomorphism

$$\mathbf{F}((x^{1/p^\infty})) \cong K_2^\flat.$$

Idea for tilting equivalence. Given $K, \mathscr{O}_K, \varpi$, we have $K^\flat, \mathscr{O}_{K^\flat}, \varpi^\flat$. Note we have $\mathscr{O}_K/\varpi \cong \mathscr{O}_{K^\flat}/\varpi^\flat$ by construction. We then prove that $L \mapsto L^\flat$ is an equivalence as follows: we prove instead that $\mathscr{O}_L \to \mathscr{O}_{L^\flat}$ is an equivalence. But by reducing mod ϖ, we have to prove that given $\mathscr{O}_L$ we can show that $\mathscr{O}_L/\varpi^n$ is the unique flat lift of $\mathscr{O}_L/\varpi$ over $\mathscr{O}_K/\varpi^n$. Note these sorts of lifts are controlled by the cotangent complex, which we show vanishes, so the obstructions vanish, and we get canonical lifts. □

2.2. **Back to (φ, Γ)-modules.** Let $K/\mathbf{Q}_p$ be finite and set $C_1 = \mathbf{C}_p = \widehat{\overline{K}}$. Let $F/\mathbf{F}_p((x))$ be finite and let $C_2 = (F^{\mathrm{sep}})^{\wedge_x}$.

Then the G_K-action on $\overline{K}$ extends to a continuous action of G_K on C_1. Similarly, the G_F-action on F^{sep}) extends to a continuous action of G_F on C_2.

Lemma 2.2.1.

(1) *C_1, C_2 are algebraically closed. In particular, $C_2 = \overline{F}^{\wedge_x}$ (consequence of Krasner's lemma).*

(2) *C_1 and C_2 are perfectoid fields, and $C_2 \cong C_1^\flat$, but this isomorphism is very non-canonical, and depends on the choice of some $\varpi^\flat \in C_1^\flat$.*

(3) *Let $H \subseteq G_K$ (resp $H \subseteq G_F$) be a closed subgroup. Then*
$$C_1^H = (\overline{K}^H)^{\wedge_p} (\textit{resp. } C_2^H = (((F^{\mathrm{sep}})^H)^{\mathrm{perf}})^{\wedge_x})$$
This is a consequence of the Ax-Sen lemma.

Now fix $\epsilon_n \in \overline{K}$ a compatible sequence of p-power roots of 1. Let $K_\infty = K(\epsilon_n \mid n \geq 1)$. Then K_∞/K is a Galois extension, and we let $\Gamma := \mathrm{Gal}(K_\infty/K)$, and we define the cyclotomic character
$$\chi : \Gamma \xrightarrow{\sim} \mathbf{Z}_p^\times, \quad g \cdot \epsilon_n = \epsilon_n^{\chi(g)} \text{ for all } n \geq 1$$
This is an isomorphism is $K = \mathbf{Q}_p$, but in general just lands in an open subgroup of $\mathbf{Z}_p^\times$ (for example if $K = \mathbf{Q}_p(\epsilon_1)$, it should land in $1 + p\mathbf{Z}_p$).

Then $\widehat{K_\infty}$ is a perfectoid field. Then

$$\begin{array}{ccc}
\{\text{finite (sep.) extensions of } K_\infty\} & \xrightarrow{\sim \text{ (comp.)}} & \left\{\text{finite (sep.) extensions of } \widehat{K_\infty}\right\} \\
 & & \Big\downarrow \sim \text{ (tilting)} \\
\left\{\text{finite (sep.) extensions of } \mathbf{F}_q((x))^{\mathrm{perf}}\right\} & \xrightarrow{\sim} & \left\{\text{finite (sep.) extensions of } \widehat{K_\infty}^\flat\right\} \\
\sim \Big\uparrow & & \\
\{\text{finite (sep.) extensions of } \mathbf{F}_q((x))\} & &
\end{array}$$

In conclusion we get an isomorphism $\mathrm{Gal}(\overline{K}/K_\infty) \cong \mathrm{Gal}(F^{\mathrm{sep}}/F)$ (note we're writing $F = \mathbf{F}_q((x))$), which is what we wanted to obtain c.f. the last lecture. Note that G_K acts on $\mathbf{C}_p$ and thus also on $\mathbf{C}_p^\flat \supseteq \widehat{K_\infty}^\flat \supseteq F$. So G_K also acts on $W(\mathbf{C}_p^\flat)$ and $W(\mathscr{O}_{\mathbf{C}_p^\flat})$. Note $\mathbf{C}_p^\flat \supseteq F^{\mathrm{sep}}$. Note that the G_K-action preserves F^{sep}, and the restriction of G_K-action to F^{sep} is just the canonical action of G_F on F^{sep}.

Let $\mathscr{O}_{\mathcal{E}} = W(\mathbf{F}_q)((x))^{\wedge_p} \hookrightarrow W(\mathbf{C}_p^\flat)$ by taking $x \mapsto 1 - [(\epsilon_0, \epsilon_1, \dots)]$. Then $\mathscr{O}_{\mathcal{E}}$ with the restriction of the Witt vector Frobenius on $W(\mathbf{C}_p^\flat)$ is a Cohen ring of F. Then $\widehat{\mathscr{O}_{\mathcal{E}^{\mathrm{ur}}}}$ is the p-adic completion of the maximal unramified extension of $\mathscr{O}_{\mathcal{E}}$ in $W(\mathbf{C}_p^\flat)$, and this is stable under the action of G_K and Frob. The restriction of the G_K-action on $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$ to G_F is the canonical action.

3. Talk III

3.1. **Addendum / Small Correction.** So last time we had $K/\mathbf{Q}_p$ finite, and had constructed $K_\infty = \bigcup_{n\geq 0} K(\epsilon_n)$ by adjoining a compatible system $(\epsilon_n)_{n\geq 0}$ of p^nth roots of unity. We let $\mathbf{F}_q$ denote the residue field of K_∞. In the first lecture, we said that we wanted $F = \mathbf{F}_q((X)) \hookrightarrow \widehat{K_\infty}^\flat$ which has a Γ action via the cyclotomic character. We let $\mathscr{O}_{\mathcal{E}} = W(\mathbf{F}_q)((X))^\vee \hookrightarrow W(\widehat{K}_\infty^\flat)$, which is still acted upon by Γ, and we wanted a Γ-equivariant embedding. But the point is that this is *not automatic*: it definitely depends on the chosen embedding. In this case, $\mathscr{O}_{\mathcal{E}^{\mathrm{unr}}}, \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{unr}}} \hookrightarrow W(\mathbf{C}_p^\flat)$ is automatically G_K-stable.

(1) If $K/\mathbf{Q}_p$ unramified, then $K = W(k)[1/p]$, so $k = \mathbf{F}_q$ in this case (essentially K_∞ is totally ramified). Furthermore, $\pi_n = 1 - \epsilon_n \in K_n = K(\epsilon_n)$ is a uniformizer, since the ϵ_n are the only source of ramification. Now choose the embedding

$$F = k((X)) \hookrightarrow k((X^{1/p^\infty})) \hookrightarrow \widehat{K}_\infty^\flat$$

sending $x \mapsto \underline{\epsilon} - 1 = (\epsilon_1, \epsilon_2, \dots) - 1$. Also choose

$$W(k)((X))^\vee \hookrightarrow W(\widehat{K}_\infty^\flat)$$

sending $X \mapsto [\underline{\epsilon}] - 1$. Then F and $\mathscr{O}_{\mathcal{E}}$ are φ and Γ-stable, and then you need to check that the embeddings preserve φ and Γ.

Moreover, $k((X^{1/p^\infty})) \hookrightarrow \widehat{K}_\infty^\flat$ is an isomorphism.

Proof. It's enough to show that $\bigcup_{n\geq 0} k[[X^{1/p^n}]] \subseteq \mathscr{O}_{\widehat{K}_\infty^\flat}$ is dense. Recall

$$\mathscr{O}_{\widehat{K_\infty}^\flat} = \varprojlim_{x\mapsto x^p} \mathscr{O}_{K_\infty}/p \xrightarrow{\mathrm{pr}_m} \mathscr{O}_{K_\infty}/p = \bigcup_{n\geq 0} W(k)[\pi_n]/p$$

So it's enough to show that $\overline{\pi_n} = \overline{1 - \epsilon_n}$ is always in the image of the projection. But in fact take

$$\mathrm{pr}_m(X^{p^{m-n}}) = \overline{\pi_n}$$

□

(2) Now suppose $K/\mathbf{Q}_p$ is arbitrary. Let L be the maximal unramified subextension. Then the previous step says that $F' = k((X)) \hookrightarrow \widehat{L}_\infty^\flat$ and $\mathscr{O}_{\mathcal{E}'} \hookrightarrow W(\widehat{L}_\infty^\flat)$.

But then $K_\infty = KL_\infty$, and $\widehat{K_\infty}$ is a perfectoid field extension of $\widehat{L_\infty}$. Then let F be the finite separable extension of F' inside $\mathbf{C}_p^\flat$ corresponds to $\widehat{K_\infty}^\flat/\widehat{L_\infty}$. This is Galois stable, and the unique unramified lift $\mathscr{O}_{\mathcal{E}}/\mathscr{O}_{\mathcal{E}'}$ of F'/F inside $W(\mathbf{C}_p^\flat)$, is stable under G_K.

So in summary,

$$\begin{aligned}\{\text{continuous reps of } G_K \text{ on f.g. } \mathbf{Z}_p\text{-modules}\} &\xrightarrow{\sim} \{(\varphi,\Gamma)\text{-modules over } \mathscr{O}_{\mathcal{E}}\} \\ &\xrightarrow{\sim} \left\{(\varphi, G_K)\text{-modules over } \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{unr}}}\right\} \\ &= \left\{\varphi\text{-mods over } \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{unr}}} \text{ with semilinear } G_K\text{-action, comm. with } \varphi\right\}\end{aligned}$$

3.2. **φ-modules and the Fargues-Fontaine curve.** Let F be a perfectoid field of characteristic p, (for example $\mathbf{C}_p^\flat$, or $\widehat{K}_\infty^\flat$). We define

$$A_{\mathrm{inf}} = A_{\mathrm{inf}}(F) = W(\mathscr{O}_F),$$

which has a φ-action via Frobenius. We choose a pseudo-uniformizer $\varpi \in \mathscr{O}_F$ and define

$$Y_F = \mathrm{Spa}(A_{\mathrm{inf}}(F), A_{\mathrm{inf}}(F)) \setminus V(p[\varpi]).$$

This gives you an adic space, independent of the choice of ϖ, which is equipped with an automorphism φ, which is induced by the φ-action on $A_{\mathrm{inf}}(F)$, which has no fixed points and which acts freely, i.e. $\varphi^{\mathbf{Z}}$ acts totally discontinuously and freely. In this case, we always know how to form the quotient in the category of locally ringed spaces.

Here's a bit of an explanation of what we just did. As a set,

$$|\,\mathrm{Spa}(A_{\mathrm{inf}}(F), A_{\mathrm{inf}}(F))| = \{v : A_{\mathrm{inf}}(F) \to \Gamma_v \sqcup \{0\} \text{ continuous valuation such that } v(A_{\mathrm{inf}}(F)) \leq 1\}_{/\cong}.$$

This Γ_v is supposed to be a totally ordered abelian group, and then we extend this by demanding that 0 is smaller than everything. Continuity means that for all $\gamma \in \Gamma_v$ the set $\{f \in A_{\inf}(F) \mid v(f) \leq \gamma\} \subseteq A_{\inf}(F)$ is open for the $(p, [\varpi])$-adic topology on $A_{\inf}(F)$ (this is really a mixture of the topology on the residue field F via ϖ and the Witt vector bit via p). Then

$$V(p[\varpi]) = \{v \in \mathrm{Spa}(A_{\inf}(F), A_{\inf}(F)) \mid v(p) = 0 \text{ and } v([\varpi]) = 0\}.$$

In some sense this is kind of huge.

We make Y_F into a locally ringed space as follows. First of all,

$$B^b = A_{\inf}(F)\left[\frac{1}{p[\varpi]}\right] = \left\{\sum_{n>>-\infty} [x_n]p^n \in W(F)[1/p] \mid x_n \in F \text{ bounded}\right\}$$

which is not quite the ring of functions on Y_F. Then for $0 \leq \rho \leq 1$, we define a norm $|\cdot|_\rho$ on B^b by

$$\left|\sum [x_n]p^n\right|_\rho := \max_n |x_n|\rho^n \in \mathbf{R}$$

For a closed interval $I \subseteq [0,1]$ define $B_I = B_{F,I}$ as the completion of B^b with respect to the family of norms $\{|\cdot|_\rho \mid \rho \in I\}$.

Theorem 3.2.1 (Fargues-Fontaine)**.** *If $1 \notin I$, and the endpoints of I are in $|F^\times|$ then B_I is a principal ideal domain.*

So we define $Y_{F,I} = \mathrm{Spa}(B_I, B_I^\circ) = \{v \in Y_F \mid v \text{ extends to a continuous valuation on } B_I\}$. Then

$$Y_{F,I_1} \cap Y_{F,I_2} = Y_{F,I_1 \cap I_2},$$

and

$$Y_F = \bigcup_{I \subseteq (0,1)} Y_{F,I}$$

and we give it the topology such that the $Y_{F,I}$ are open. To form a basis for the topology, you need all rational open subsets.

Then we make Y_F into a topologically locally ringed space with structure sheaf $\mathscr{O}_{Y_F}$ with

$$\Gamma(Y_{F_I}, \mathscr{O}_{Y_F}) = B_I.$$

Of course one needs to check that this is a sheaf, and that the stalks are local rings.

Furthermore, for $f \in B^b$, we have $|\varphi(f)|_{\rho^p} = |f|_\rho^p$. So φ extends to $B_{[a,b]} \cong B_{[a^p,b^p]}$, so

$$\varphi : Y_{F,[a^p,b^p]} \xrightarrow{\sim} Y_{F,[a,b]}$$

and if $[a,b]$ is small enough then it will be disjoint from $[a^p, b^p]$, so the map

$$\varphi : Y_F \xrightarrow{\sim} Y_F$$

is totally disconnected and free.

Definition 3.2.2. The **(adic) Fargues-Fontaine curve** is the locally ringed space $X_F = Y_F/\varphi^{\mathbf{Z}}$.

Remark 3.2.3. So how do we think about the universal cover Y_F? First assume F is algebraically closed (e.g. $\mathbf{C}_p^\flat$). There is a bijection

$$\{\text{“classical points” of } Y_F\} \xrightarrow{\sim} \{\text{ideals } (p-[a]) \subseteq A_{\inf}(F), 0 \neq a \in \mathscr{O}_F, 0 < |a| < 1\}$$
$$\xrightarrow{\sim} \left\{\text{perfectoid fields } E \text{ in characteristic 0 such that } E^\flat \cong F\right\}.$$

Here a classical point is a valuation which factors through a maximal ideal of B_I, and the second map takes $\langle p-[a]\rangle$ to $(W(\mathscr{O}_F)/(p-[a]))[1/p]$, which is algebraically closed perfectoid, and has characteristic 0.

Because of this description, one can think of Y_F as a "punctured open unit disk", and the coordinate function on Y_F is p, except that different a could give you the same ideal, and it's not clear which ones (to me at least, right now).

Now F is arbitrary again. Let's write $B = \Gamma(Y_F, \mathscr{O}_{Y_F}) = \varprojlim_I B_I$, which is the same as the Fréchet completion of B^b with respect to all the norms.

Theorem 3.2.4 (Fargues-Fontaine)**.**

$$\Gamma(X_F, \mathscr{O}_{F_X}) = B^{\varphi=\mathrm{id}} = \mathbf{Q}_p$$

Note that a vector bundle $\mathcal{V}$ on X_F is the same as a a vector bundle $\mathcal{V}$ on Y_F and an isomorphism $\varphi^*\mathcal{V} \xrightarrow{\sim} \mathcal{V}$. Note a φ-module over B which is finite projective gives rise to a vector bundle on X_F. In fact, every vector bundle arises in this way.

3.3. **Classification of Vector Bundles.**

Now suppose F is algebraically closed. We say that a **divisor**, as usual, is a formal sum $D = \sum n_i x_i$, where each x_i is a classical point, as described earlier.

Proposition 3.3.1. *Every line bundle $\mathcal{L}$ on X_F is associated to a divisor $D = \sum n_i x_i$, e.g.*

$$\mathscr{O}(-x) = \text{ ideal sheaf of functions vanishing at } x.$$

This gives us a well-defined notion of the degree of a line bundle:

$$\mathcal{L} \cong \mathscr{O}\left(\sum n_i x_i\right) \mapsto \sum n_i.$$

More generally, if $\mathcal{V}$ be a vector bundle on X_F we define

$$\deg \mathcal{V} = \deg(\Lambda^{\operatorname{rank} V})$$

and we define the **slope** of $\mathcal{V}$ to be

$$\mu(\mathcal{V}) = \deg(\mathcal{V}) / \operatorname{rank}(\mathcal{V}).$$

Definition 3.3.2. A vector bundle $\mathcal{V}$ is called **semistable of slope** μ if $\mu(\mathcal{V}) = \mu$ and the slope of any sub-vector-bundle of $\mathcal{V}$ is $\leq \mu$.

Theorem 3.3.3 (Fargues-Fontaine)**.**

(1) *Every vector bundle on X_F decomposes as a direct sum of semistable vector bundles.*

(2) *For every $\mu \in \mathbf{Q}$, there exists a unique indecomposable semistable vector bundle of slope μ.*

3.4. **Construction of $\mathcal{V}_\mu$.**

Let $\mu = r/s$ where r, s coprime, and say $s \geq 1$. Let

$$D_\mu = (W(\overline{\mathbf{F}_p})[1/p])^s$$

and then we make D_μ into a φ-module over $W(\overline{\mathbf{F}_p})[1/p]$ by requiring that the matrix of φ is

$$\begin{pmatrix} 0 & 0 & \cdots & p^r \\ 1 & 0 & \cdots & \\ 0 & 1 & \ddots & \\ 0 & 0 & \ddots & \end{pmatrix}$$

Then the Dieudonné-Manin classification tell you that these are the unique indecomposable φ-modules over this field.

Let D_μ be the pullback (as a φ-module) of D_μ along the map in the top row:

$$\begin{array}{ccccccc}
\Gamma(Y_F, \mathscr{O}_{Y_F}) & \longleftarrow & A_{\inf}(F)\left[\frac{1}{p[\varpi]}\right] & \longleftarrow & A_{\inf}(F)\left[\frac{1}{p}\right] & \longleftarrow & W(\overline{\mathbf{F}_p})\left[\frac{1}{p}\right] \\
 & & & & \uparrow & & \uparrow \\
 & & & & A_{\inf}(F) & \longleftarrow & W(\overline{\mathbf{F}_p}) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathscr{O}_F & \longleftarrow & \overline{\mathbf{F}_p}
\end{array}$$

This defines a vector bundle D_μ on Y_F with $\varphi^* D_\mu \cong D_\mu$, which descends to a vector bundle on X_F.

But you have to be careful: this construction is not full, it's just faithful and essentially surjective. There are some endomorphisms which appear when you pass to vector bundles.

4. Talk IV

4.1. **Correction.** Yesterday, for F a characteristic p perfectoid field, we defined Y_F and had a covering $\bigcup_{I\subseteq(0,1)} Y_{F,I}$. But we said yesterday that $Y_{F,I}$ forms a basis for the topology, but this isn't true!

In fact, a basis for the topology on $Y_{F,I} = \mathrm{Spa}(B_I, B_I^\circ)$ is given by taking $f_1, \dots, f_n, g \in B_I$ such that $(f_1, \dots, f_n, g) = (1)$. Then we define the **rational open subset**

$$U\left(\frac{f_1,\dots,f_n}{g}\right) := \{x \in Y_{F,I} \mid |f_i(x)| \le |g(x)| \text{ for all } i\},$$

where we write $|f(x)|$ in place of $v(f)$, where $x = v$ is the valuation.

One can then really check that if $I' \subseteq I$ with endpoints in $|F^\times|$, then $Y_{F,I'} \subseteq Y_{F,I}$ is a rational open.

Example 4.1.1. So as we said yesterday if F is algebraically closed and $x \in Y_F$ is a classical point, then it corresponds to the ideal $\langle p - [a]\rangle$ for some $a \in \mathscr{O}_F$ with $0 < |a| < 1$. We also said that this defines an untilt E of F by taking

$$\mathscr{O}_E := A_{\inf}(F)/(p - [a]) \subseteq E$$

and inverting p. Then $E^\flat = F$ because by definition,

$$\mathscr{O}_E/p = \mathscr{O}_F/a,$$

and then just take the inverse limit over Frobenius. For $f \in A_{\inf}(F)$, let $\overline{f}$ be its image in $\mathscr{O}_E/p = \mathscr{O}_F/a$. Let $\overline{f}' \in \mathscr{O}_F$ denote a lift. Then

$$|f(x)| = |\overline{f}'|$$

which is independent of the lift. The classification from last time implies

Corollary 4.1.2.

$$\{\textit{semi-stable slope 0 vector bundles on } X_F\} \xrightarrow{\sim} \{\textit{finite dimensional } \mathbf{Q}_p\textit{-vector spaces}\}$$

where we take $\mathcal{V} \mapsto \Gamma(X_F, \mathcal{V})$ *in one direction and* $V \mapsto V \otimes_{\mathbf{Q}_p} \mathscr{O}_{X_F}$ *in the other.*

Now let F be arbitrary again. Then curve X_F has an algebraic variant

$$X_F^{\mathrm{alg}} := \mathrm{Proj} \bigoplus_{d\ge 0} B^{\varphi = p^d}$$

where $B = \Gamma(Y_F, \mathscr{O}_{Y_F})$ is the Fréchet completion of B^b as before.

Theorem 4.1.3 (Fargues-Fontaine)**.**

(1) X_F^{alg} *is a 1-dimensional regular Noetherian scheme.*

(2) *There is a morphism of locally ringed spaces*

$$X_F \to X_F^{\mathrm{alg}}$$

which induces a bijection

$$\{\textit{classical points of } X_F\} \xrightarrow{\sim} \left\{\textit{closed points of } X_F^{\mathrm{alg}}\right\},$$

(we didn't define classical points for non-algebraically closed fields, but there is a definition) inducing isomorphisms

$$\widehat{\mathscr{O}}_{X_F^{\mathrm{alg}},x} \xrightarrow{\sim} \widehat{\mathscr{O}}_{X_F,x}$$

for x a classical point.

Theorem 4.1.4 (GAGA of Kedlaya-Liu). *If F is algebraically closed, then pullback along $X_F \to X_F^{\mathrm{alg}}$ induces an equivalence of categories*

$$\left\{\textit{vector bundles on } X_F^{\mathrm{alg}}\right\} \xrightarrow{\sim} \{\textit{vector bundles on } X_F\}$$

Remark 4.1.5. Here, being algebraically closed might not be necessary, but we couldn't find a reference.

Now let $\infty \in X_F^{\mathrm{alg}}$ be any closed point, and define

$$B_e := \Gamma(X_F^{\mathrm{alg}} \setminus \{\infty\}, \mathscr{O}_{X_F^{\mathrm{alg}}}) = B[\frac{1}{t}]^{\varphi=\mathrm{id}}$$

for some $t \in B$ such that $V(t) = \mathrm{pr}^{-1}(\infty) \subseteq Y_F$. Then

$$\left\{\text{vector bundles on } X_F^{\mathrm{alg}}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{finite projective } B_e\text{-modules } M \text{ and} \\ \Lambda \subseteq M \otimes_{B_e[1/t]} \widehat{\mathscr{O}}_{X_F^{\mathrm{alg}},\infty}[1/t] \text{ an } \widehat{\mathscr{O}}_{X_F^{\mathrm{alg}},\infty}\text{-lattice}\end{array}\right\}.$$

This follows from the usual Beauville-Lazslo gluing lemma.

4.2. Equivariant Vector Bundle.

Now say $K/\mathbf{Q}_p$ is finite and let $\mathbf{C}_p = \widehat{\overline{K}}$, which has a G_K-action. Then G_K acts isometrically on $\mathbf{C}_p^\flat$ by functoriality, and on $W(\mathscr{O}_{\mathbf{C}_p^\flat})$ as well. So G_K acts on $Y_{\mathbf{C}_p^\flat}$ and $X_{\mathbf{C}_p^\flat}$.

Corollary 4.2.1. *The map*

$$\left\{\begin{array}{c}\textit{continuous } G_K\textit{-representations} \\ \textit{on finite dimensional } \mathbf{Q}_p\textit{-vector spaces}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}G_K\textit{-equivariant vector bundles} \\ \textit{on } X_{\mathbf{C}_p^\flat}\textit{, semistable of slope } 0\end{array}\right\}$$

taking $V \mapsto V \otimes_{\mathbf{Q}_p} \mathscr{O}_{X_{\mathbf{C}_p^\flat}}$ (with the diagonal G_K-action) is an equivalence of categories.

Remark 4.2.2. If $\mathcal{V}$ is an equivariant vector bundle and $U \subseteq X_{\mathbf{C}_p^\flat}$ is open and $H \subseteq G_K$ is the stabilizer of U, then H is asked to act continuously on $\Gamma(U, \mathcal{V})$: this should be part of the definition.

Corollary 4.2.3. *Let $\theta : W(\mathscr{O}_{\mathbf{C}_p^\flat}) \to \mathscr{O}_{\mathbf{C}_p^\flat}$ be the canonical surjection. Then θ defines a classical point $x_0 \in Y_{\mathbf{C}_p^\flat}$, so let $\infty \in X_{\mathbf{C}_p^\flat}$ denote its image. Then ∞ is stabilized by G_K, and we get a G_K-action on $B_e = \Gamma(X_{\mathbf{C}_p^\flat}^{\mathrm{alg}}, \mathscr{O}_{X_{\mathbf{C}_p^\flat}^{\mathrm{alg}}})$. Then*

$$\left\{G_K\textit{-equivariant vector bundles on } X_{\mathbf{C}_p^\flat}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}(M,\Lambda),\ M \textit{ a finite projective } B_e\textit{-module with semilinear } G_K\textit{-action} \\ \textit{and } \Lambda \subseteq M \otimes_{B_e} \widehat{\mathscr{O}}_{X_{\mathbf{C}_p^\flat},\infty}[1/t] \textit{ a } G_K\textit{-stable } \widehat{\mathscr{O}}_{X_{\mathbf{C}_p^\flat},\infty}\textit{-lattice}\end{array}\right\}$$

Definition 4.2.4. We write

$$B_{\mathrm{dR}}^+ := \widehat{\mathscr{O}}_{X_{\mathbf{C}_p^\flat},\infty} = \widehat{\mathscr{O}}_{Y_{\mathbf{C}_p^\flat},x_0},$$

which is the completion of $W(\mathscr{O}_{\mathbf{C}_p^\flat})[1/p]$ with respect to $\ker(W(\mathscr{O}_{\mathbf{C}_p^\flat}) \to \mathbf{C}_p)$.

Example 4.2.5. Let $\underline{\epsilon} = (\epsilon_n)_{n\geq 0} \in \mathscr{O}_{\mathbf{C}_p^\flat}$ and write

$$t = \log([\underline{\epsilon}]) := \sum_{n\geq 1} (-1)^{n+1} \frac{([\underline{\epsilon}]-1)^n}{n}.$$

This does not converge in A_{inf} or B^b, but in all the B_I for $I \subseteq (0,1)$ it does converge. Then we compute that $\varphi(t) = pt$ and if $g \in G_K$, then $g \cdot t = \chi(g)t$, where χ is the cyclotomic character (for this, one shows that $\log([\underline{\epsilon}]^a) = a\log([\underline{\epsilon}])$). Then

$$V(t) = \text{pr}^{-1}(\infty) \subseteq Y_F$$

An equivariant vector bundle on $X_{\mathbf{C}_p^\flat}$ is a φ-module over B is a φ-module over $W(\overline{\mathbf{F}_p})[1/p]$. $\mathscr{O}_{X_{\mathbf{C}_p^\flat}}$ is B with $\varphi : 1 \mapsto 1$ is $brev\mathbf{Q}_p$ sending $1 \mapsto 1$. $\mathscr{O}_{X_{\mathbf{C}_p^\flat}}(\infty)$ is $t^{-1}B$ (i.e. B with $\varphi(1) = p^{-1}$) is $brev(\mathbf{Q}_p)$ $1 \mapsto p^{-1}$.

$\mathscr{O}_{X_{\mathbf{C}_p^\flat}}$ is trivially semistable of slope 0, and it has trivial action, so it corresponds to the trivial G_K-representation of $G_{\mathbf{Q}_p}$. However, $\mathscr{O}_{X^\flat_{\mathbf{C}_p}}(\infty)$ is not semistable of slope 0, and

$$\Gamma(X_{\mathbf{C}_p^\flat}, \mathscr{O}_{X_{\mathbf{C}_p^\flat}}(\infty)) = (t^{-1}B)^{\varphi=\text{id}} = B^{\varphi=p}$$

which is an infinite dimensional $\mathbf{Q}_p$-vector space! Note

$$\text{Hom}_\varphi(D_0, D_{-1}) = 0$$

and

$$\text{Hom}_\varphi(\mathscr{O}_{X_{\mathbf{C}_p^\flat}}, \mathscr{O}_{X_{\mathbf{C}_p^\flat}}(\infty)) = B^{\varphi=\text{id}}$$

which is huge.

Let $\mathcal{L}$ be the line bundle on $X_{\mathbf{C}_p^\flat}$ which corresponds to $t^{-1}B$ with $\phi = p\varphi$. Then $\mathcal{L}$ becomes semistable of slope 0, and the corresponding Galois representation is $t^{-1}\mathbf{Q}_p$, which is $(t^{-1}B)^{\varphi=\text{id}}$, and G_K acts via χ^{-1}.

4.3. **Galois descent and decompletion/deperfection.** Let F be a perfectoid field of characteristic p. Write $C = \widehat{\overline{F}} = \widehat{F^{\text{sep}}}$, has a G_F-action. Then by functoriality the inclusion $F \to C$ induces

$$X_C \to X_F \text{ and } X_C^{\text{alg}} \to X_F^{\text{alg}}$$

These morphisms are G_F-equivariant.

Theorem 4.3.1 (Algebraic Galois descent)**.** *The natural map*

$$\left\{\textit{vector bundles on } X_F^{\text{alg}}\right\} \xrightarrow{\sim} \left\{G_F\textit{-equivariant vector bundle on } X_C^{\text{alg}}\right\}.$$

is an equivalence of categories.

Now let $K/\mathbf{Q}_p$ be a finite extension and let K_∞/K be a Galois extension with Galois group $\Gamma = \text{Gal}(K_\infty/K)$ such that $\widehat{K_\infty}$ is perfectoid.

Corollary 4.3.2.

$$\left\{\Gamma\textit{-equivariant vector bundles on } X_{\widehat{K_\infty}^\flat}\right\} \xrightarrow{\sim} \left\{G_K\textit{-equivariant vector bundles on } X_{\mathbf{C}_p^\flat}^{\text{alg}}\right\}.$$

This should really work for any K_∞, but in the cyclotomic case $K_\infty = K((\epsilon_n)_{n\geq 0})$ we can say more.

We already know that (here $B = \Gamma(Y, \mathscr{O}_Y)$, and we abbreviate $Y = Y_{\widehat{K_\infty}^\flat}$).

$$\begin{aligned} \left\{\text{vector bundles on } X_{\widehat{K_\infty}^\flat}\right\} &\xrightarrow{\sim} \{\varphi\text{-modules over } B\} \\ &\xrightarrow{\sim} \left\{\varphi\text{-modules over } B^{(0,r]}\right\} \\ &\left(= \left\{B^{(0,r]}\text{-modules with } \varphi^* M \xrightarrow{\sim} M \otimes_{B^{(0,r]}} B^{(0,r^p])}\right\}\right) \\ &\xrightarrow{\sim} \left\{\varphi\text{-modules over } \lim_{r\to 0} B^{(0,r]} \left(= \widetilde{B}^\dagger_{\mathrm{rig},K} \text{ of Berger}\right)\right\} \end{aligned}$$

The same remark applies to (φ, Γ)-modules. Now let us write Δ^* to mean the punctured open unit disk over $W(\mathbf{F}_q)[1/p]$, where $\mathbf{F}_q$ is the residue field of K_∞. For simplicity assume $K/\mathbf{Q}_p$ is unramified. Then we get a map $\Gamma(\Delta^*, \mathscr{O}_{\Delta^*}) \hookrightarrow B$ sending $X \mapsto [\underline{\epsilon}] - 1$. Then we let

$$\mathcal{R} = \lim_{\rho\to 1} \Gamma(\{\rho \le |X| < 1\}, \mathscr{O}_{\Delta^*}) \subseteq \widetilde{B}^\dagger_{\mathrm{rig},K} = \lim_{r\to 0} B^{(0,r]}$$

is stable under (φ, Γ), and is called the **Robba ring**, which is $B^\dagger_{\mathrm{rig},K}$ for Berger.

Note φ is not an isomorphism on $\mathcal{R}$, only injective. So $\widetilde{B}^\dagger_{\mathrm{rig},K}$ is a Fréchet completion of $\bigcup_{n\ge 0} \varphi^{-n}(\mathcal{R})$.

Theorem 4.3.3 (Decompletion and deperfection)**.**

$$\{(\varphi,\Gamma)\text{-}\textit{modules over } \mathcal{R}\} \xrightarrow{\sim} \left\{(\varphi,\Gamma)\text{-}\textit{modules over } \widetilde{B}^\dagger_{\mathrm{rig},K}\right\}.$$

5. Talk V

5.1. **Crystalline representations and Fontaine's period rings.** We resume with the usual setup. Let $K/\mathbf{Q}_p$ be a finite extension and let $\mathbf{C}_p = \widehat{\overline{K}}$ with its G_K-action. We have the curves $Y_{\mathbf{C}_p^\flat}$ and $X_{\mathbf{C}_p^\flat}$, both with G_K-actions, and a G_K-equivariant quotient map $Y_{\mathbf{C}_p^\flat} \xrightarrow{\mathrm{pr}} X_{\mathbf{C}_p^\flat}$. We have a fixed point $\infty \in X$ corresponding to the untilt $\mathbf{C}_p$, and this corresponds to a the canonical surjection $W(\mathscr{O}_{\mathbf{C}_p^\flat}) \to \mathscr{O}_{\mathbf{C}_p}$. We let

$$B^+_{\mathrm{dR}} = \widehat{\mathscr{O}}_{X,\infty} = \widehat{\mathscr{O}}_{Y,x_0}.$$

Note $\mathrm{pr}^{-1}(\infty) = V(t) \subseteq Y$, where $t = \log([\underline{\epsilon}]) \in B$, and we can also view t as an element of $\widehat{\mathscr{O}}_{Y,x_0} = B^+_{\mathrm{dR}}$ and it is a uniformizer there.

We have seen that we have an equivalence of categories

$$\left\{\begin{array}{c}\text{continuous } G_F\text{-representations} \\ \text{on finite dimensional } \mathbf{Q}_p\text{-vector spaces}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c} G_K\text{-equivariant vector bundles} \\ \text{on } X_{\mathbf{C}_p^\flat}\text{, semistable of slope } 0\end{array}\right\}$$

via the functors of global sections, and base change. So since the whole point of this correspondence is to construct Galois representations, we now want to describe certain interesting vector bundles whose global sections give us Galois representations.

Well, we have also seen that we have a functor

$$\left\{\varphi\text{-modules over } W(\overline{\mathbf{F}_p})[1/p]\right\} \xrightarrow{\sim} \left\{\text{vector bundles on } X_{\mathbf{C}_p^\flat}\right\}.$$

By functoriality, we can enrich this over G_K. Note $(\breve{\mathbf{Q}}_p)^{G_K} = W(k)[1/p] = K_0$, so in summary we get a diagram

$$\begin{array}{ccc} \left\{G_K\text{-equivariant } \varphi\text{-modules over } W(\overline{\mathbf{F}_p})[1/p]\right\} & \xrightarrow{\ \sim\ } & \left\{G_K\text{-equivariant vector bundles on } X_{\mathbf{C}_p^\flat}\right\} \\ \big\uparrow & \nearrow & \\ \{\varphi\text{-modules over } K_0\} & & \end{array}$$

Note that given an arbitrary φ-module (D,ϕ), the image $\mathcal{V}(D,\phi)$ is usually not usually semistable of slope 0. To get around this, we take a B_{dR}^+-lattice $\Lambda \subseteq \mathcal{V}(D,\phi)\otimes \widehat{\mathscr{O}}_{X,\infty}[1/t]$ and modify our vector bundle (recall that Beauville-Lazslo gluing works well in our situation) by Λ to get a vector bundle $\mathcal{V}(D,\phi,\Lambda)$. Note $V(D,\phi)\otimes\widehat{\mathscr{O}}_{X,\infty}[1/t]$ has a G_K-action, and if Λ is G_K-stable, then the modification $\mathcal{V}(D,\phi,\Lambda)$ will again be G_K-equivariant. Then if $\mathcal{V}(D,\phi,\Lambda)$ is semistable of slope 0, then

$$\Gamma(X,\mathcal{V}(D,\phi,\Lambda))$$

is a continuous G_K-representation on a finite dimensional $\mathbf{Q}_p$-vector space, whose rank is $\dim_{K_0} D$.

Definition 5.1.1. Representations arising this way are called **crystalline**.

Definition 5.1.2. Let A_{cris} denote the p-adic completion of the divided power envelope of $A_{\mathrm{inf}}(\mathbf{C}_p^\flat)$ with respect to $\ker(A_{\mathrm{inf}} \xrightarrow{\theta} \mathscr{O}_{\mathbf{C}_p})$. Actually, $\ker\theta$ is principal and generated by one element ξ, and

$$A_{\mathrm{cris}} := A_{\mathrm{inf}}[\xi^n/n!, n\geq 1]^{\wedge_p}$$

We let

$$B_{\mathrm{cris}}^+ := A_{\mathrm{cris}}[1/p]$$

and one can check that $t=\log([\underline{\epsilon}])$ converges in A_{cris}, so we define

$$B_{\mathrm{cris}} := B_{\mathrm{cris}}^+[1/t] = A_{\mathrm{cris}}[1/t].$$

Remark 5.1.3.

(1) The G_K-action on A_{inf} extends to A_{cris}, B_{cris} and B_{cris}^+.

(2) The (Witt vector) Frobenius $\varphi: A_{\mathrm{inf}}\to A_{\mathrm{inf}}$ extends to an injective map $\varphi: A_{\mathrm{cris}}\to A_{\mathrm{cris}}$. These further extends to $\varphi: B_{\mathrm{cris}}^{(+)} \to B_{\mathrm{cris}}^{(+)}$ commuting with G_K.

(3) One checks that we get a canonical map $B_{\mathrm{cris}}^+ \hookrightarrow B_{\mathrm{dR}}^+$. Warning: this is just a ring map. The canonical topologies on both rings are not really compatible: B_{cris}^+ has the p-adic topology and B_{dR}^+ has the t-adic topology, so this is not a topological embedding (although it is continuous). But on the other hand, the map is at least G_K-equivariant.

We can use this embedding to define a filtration on B_{cris} by

$$\mathrm{Fil}^i B_{\mathrm{cris}} := B_{\mathrm{cris}} \cap t^i B_{\mathrm{dR}}^+$$

Note that this is not the same filtration as $t^i B_{\mathrm{cris}}^+$.

Proposition 5.1.4. *The rings B_{cris} and B_{dR} are G_K-regular. We have $B_{\mathrm{cris}}^{G_K} = K_0$ and $B_{\mathrm{dR}}^{G_K} = K$. Furthermore, $(B_{\mathrm{cris}}^+)^{\varphi=\mathrm{id}} = \mathbf{Q}_p$.*

Definition 5.1.5. Let V be a continuous G_K-representation on a finite dimensional $\mathbf{Q}_p$-vector space. Then V is called **de Rham** if V is B_{dR}-admissible and V is called **crystalline** if it is B_{cris}-admissible.

Moreover, recall that

$$D_{\mathrm{dR}}(V) = (V\otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})^{G_K}$$

is a finite dimensional K-vector space and

$$D_{\mathrm{cris}}(V) = (V\otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})^{G_K}$$

is a finite dimensional K_0-vector space.

Remark 5.1.6. Crystalline representations are de Rham, because $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ is G_K-equivariant. Furthermore

$$\begin{aligned} D_{\mathrm{cris}}(V) \otimes_{K_0} K &= D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{dR}}^{G_K} \\ &= (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{dR}})^{G_K} \\ &= D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \otimes_{B_{\mathrm{cris}}} B_{\mathrm{dR}})^{G_K} \\ &\cong (V \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}} \otimes_{B_{\mathrm{cris}}} B_{\mathrm{dR}})^{G_K} \\ &= D_{\mathrm{dR}}(V). \end{aligned}$$

Since $\varphi : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$ is G_K-equivariant and $t^i B_{\mathrm{dR}}^+ \subseteq B_{\mathrm{dR}}$ is G_K-stable, we get extensions

$$\phi : D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V)$$

which are φ-linear isomorphism (because injective) and we get an exhaustive and separated **Z**-filtration

$$\mathrm{Fil}^i D_{\mathrm{dR}}(V) := (V \otimes_{\mathbf{Q}_p} t^i B_{\mathrm{dR}}^+)^{G_K}$$

by K-vector spaces.

Definition 5.1.7. A **filtered φ-module for** K is a finite dimensional K_0-vector space D along with $\phi : D \to D$ a φ-linear isomorphism plus an exhaustive and separated filtration $\mathrm{Fil}^i D_K \subseteq D_K = D \otimes_{K_0} K$ by sub-K-vector spaces.

In particular we have contructed a functor

$$D_{\mathrm{cris}} : \{\text{crystalline representations}\} \to \{\text{filtered } \varphi\text{-modules}\}.$$

Theorem 5.1.8. *A filtered φ-module (D, ϕ, Fil) is* ***weakly admissible*** *if it is semistable of slope 0 for the slope theory defined by the slope*

$$\mu(D, \phi, \mathrm{Fil}) := v_p(\det \phi) - \sum_{i \in \mathbf{Z}} i \dim_K \mathrm{gr}_i D_K$$

Theorem 5.1.9 (Fontaine, Colmez-Fontaine, Berger, Kedlaya, Kisin)**.**

(1) *D_{cris} is fully faithful.*

(2) *The essential image consists of the* ***weakly admissible*** *objects, and a quasi-inverse is given by*

$$V_{\mathrm{cris}}(D, \phi, \mathrm{Fil}) := \mathrm{Fil}^0(D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi = \mathrm{id}} = (D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi = \mathrm{id}} \cap \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}}),$$

where we put the $\otimes$-product filtration on $D_K \otimes_K B_{\mathrm{dR}}$.

Theorem 5.1.10 (Faltings, Tsuji, ...)**.** *Let X/K be a proper smooth algebraic variety. Then*

$$V = H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbf{Q}_p)$$

is a finite dimensional continuous G_K-representation. Furthermore, V is always de Rham, and there is a canonical G_K-equivariant isomorphism

$$H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}}$$

which thus induces

$$D_{\mathrm{dR}}(H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbf{Q}_p)) = H^i_{\mathrm{dR}}(X/K)$$

In fact, this identifies the filtrations on both sides. If X has good reduction and $\mathcal{X}/\mathscr{O}_K$ is a smooth proper model, then $H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbf{Q}_p)$ is crystalline, and we have a (canonical) (G_K, φ)-equivariant isomorphism

$$H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}} \xrightarrow{\sim} H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k)) \otimes_{W(k)} B_{\mathrm{cris}}$$

In particular $D_{\mathrm{cris}}(V) \cong H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$ as φ-modules over K_0.

5.2. **Relation with the Fargues-Fontaine curve.**

Proposition 5.2.1.

(1) $(B_{\mathrm{cris}}^+[1/t])^{\varphi=\mathrm{id}} = B_{\mathrm{cris}}^{\varphi=\mathrm{id}} = B_e = (B[1/t])^{\varphi=\mathrm{id}}$.

(2) $(B_{\mathrm{cris}}^+)^{\varphi=p^d} = B^{\varphi=p^d}$.

In particular,

$$X_{\mathbf{C}_p^\flat}^{\mathrm{alg}} := \mathrm{Proj} \bigoplus_d B^{\varphi=p^d} = \mathrm{Proj} \bigoplus_d (B_{\mathrm{cris}}^+)^{\varphi=p^d}$$

and we get a functor

$$\{\varphi\text{-modules over } K_0\} \to \{\varphi\text{-modules over } B_{\mathrm{cris}}^+\} \to \left\{\text{graded} \bigoplus_d (B_{\mathrm{cris}}^+)^{\varphi=p^d}\text{-modules}\right\} \to \left\{\text{vector bundles on } X_{\mathbf{C}_p^\flat}^{\mathrm{alg}}\right\}$$

and one checks that $(D, \phi) \mapsto \mathcal{V}(D, \phi)$. One computes

$$\Gamma(X^{\mathrm{alg}} \setminus \{\infty\}, \mathcal{V}(D, \phi)) = (D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=\mathrm{id}}.$$

Lemma 5.2.2. *Given an exhaustive separated filtration of D_K, we can produce a G_K-stable B_{dR}^+-lattice $\Lambda \subseteq \mathcal{V}(D, \phi) \otimes B_{\mathrm{dR}} = D_K \otimes_K B_{\mathrm{dR}}$, by sending*

$$(\mathrm{Fil}^i D_K)_i \mapsto \sum \mathrm{Fil}^i D_K \otimes t^{-i} B_{\mathrm{dR}}^+.$$

This is bijective.

$$\Gamma(X^{\mathrm{alg}}, \mathcal{V}(D, \phi, \Lambda)) = \Lambda(X^{\mathrm{alg}} \setminus \{\infty\}, \mathcal{V}(D, \phi)) \cap \Lambda \subseteq \mathcal{V}(D, \phi) \otimes B_{\mathrm{dR}} = V_{\mathrm{cris}}(D, \phi, \mathrm{Fil}).$$

Proposition 5.2.3. *$\mathcal{V}(D, \phi, \Lambda)$ is semistable of slope 0 if and only if $(D, \phi, \mathrm{Fil}^\bullet)$ is weakly admissible.*