p-MODULAR AND LOCALLY ANALYTIC REPRESENTATION THEORY OF p-ADIC GROUPS

FLORIAN HERZIG

Notes taken by Ashwin Iyengar[1] and have not been checked by the speaker. Any errors are due to me.

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1. Talk I

This course will be about mod p and p-adic representations of p-adic reductive groups, and we will focus on $\mathrm{GL}_n(\mathbf{Q}_p)$ to keep things concrete.

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Our motivation are the mod p and p-adic (local) Langlands correspondences, i.e. connections with Galois representations. The first half will roughly be about mod p representations, and the second half will be about p-adic representations (for example, Banach representations, and locally analytic representations).

1.1. **p-adic groups.** For simplicity, assume $G = \mathrm{GL}_n(\mathbf{Q}_p)$. Then G is a topological group with unique maximal compact open $K = \mathrm{GL}_n(\mathbf{Z}_p)$ (up to conjugation). Inside K we have a filtration $K(r) = 1 + p^r \operatorname{Mat}_n(\mathbf{Z}_p) \supseteq K(r+1) \supseteq \cdots$, which form a fundamental system of neighborhoods of $1 \in G$. This implies immediately that G is totally disconnected.

Remark 1.1.1. If H is a topological group, then any open subgroup is closed, and any closed subgroup of finite index is open.

Definition 1.1.2. For us, a profinite group (resp. a pro-p-group) is a compact Hausdorff topological group with a fundamental system of neighborhoods of 1 consisting of open normal subgroups of finite index (resp. of index p).

Example 1.1.3. In our case K is profinite, and $K(r)$ is pro-p: to see this note

$$K(s)/K(s+1) \xrightarrow{\sim} \operatorname{Mat}_n(\mathbf{F}_p)$$

under the map sending $1 + p^s A \mapsto \overline{A}$.

In particular, this implies that G has no $\overline{\mathbf{F}}_p$-valued Haar measure.

Here are some important subgroups:

- The Borel B denotes the upper triangular matrices in G.
- The torus T denotes the diagonal matrices in B.
- The unipotent radical U of B denotes the upper triangular matrices with 1 all along the diagonal.
- More generally if $n = \sum_{i=1}^r n_i$, then P denotes the standard parabolic subgroup of G with r blocks of size $n_1, \ldots, n_r$, and the standard Levi subgroup is the corresponding Levi, isomorphic to $\prod_{i=1}^r \mathrm{GL}_{n_i}$, and N denotes the corresponding unipotent radical.
- Note $B = T \ltimes U$ and more generally $P = M \ltimes N$.
- We let $\overline{P}$ denote the opposite parabolic (this is just the transpose, for GL_n) and $\overline{N}$ its unipotent radical.

Proposition 1.1.4 (Iwasawa Decomposition)**.** *For P any standard parabolic, $G = PK$.*

Proof. Exercise. □

Proposition 1.1.5 (Cartan Decomposition)**.**

$$G = \bigsqcup_{a_1 \geq \cdots \geq a_n} K \operatorname{diag}(p^{a_1}, \ldots, p^{a_n}) K$$

Proof. For GL_n this follows from the theory of elementary divisors, but for more general groups, this is harder, and you need the Bruhat-Tits building. □

1.2. **Smooth representations.** In this section, G is *any* Hausdorff topological group with a fundamental system of neighborhoods consisting of compact open subgroups. Let C be any field (soon we assume that C has characteristic p and that C is algebraically closed).

Suppose π is any representation of G over C.

Lemma 1.2.1. *The following are equivalent:*

(1) *For all $x \in \pi$, $\mathrm{Stab}_G(x)$ contains an open subgroup.*

(2) *$\pi = \bigcup_U \pi^U$ where U runs over compact open subgroups.*

(3) *The action map $G \times \pi \to \pi$ is continuous if π is discrete.*

Definition 1.2.2. If these hold, then π is *smooth.* A map of smooth G-representations is any G-linear map, and this forms an abelian category of smooth representations of G.

Example 1.2.3. If $G = \mathbf{Q}_p^\times$, then a character $\chi : \mathbf{Q}_p^\times \to C^\times$ is smooth if and only if $\ker \chi$ is open. But $\mathbf{Q}_p^\times = \mathbf{Z}_p^\times \times p^{\mathbf{Z}}$, so χ is determined by $\chi(p)$ and a character $\mathbf{Z}_p^\times \to C^\times$ with open kernel. The open subgroups of $\mathbf{Z}_p^\times$ are of the form $1 + p^r\mathbf{Z}_p$ so the character $\mathbf{Z}_p^\times \to C^\times$ must factor via a character

$$\mathbf{Z}_p^\times/(1 + p^r\mathbf{Z}_p) \cong \mathbf{Z}/(p-1)\mathbf{Z} \times \mathbf{Z}/p^{r-1}\mathbf{Z} \to C^\times$$

for some r.

1.3. **Induced representations.** Now suppose $H \leq G$ is some closed subgroup and σ is a smooth H-representation. Then

$$\mathrm{Ind}_H^G(\sigma) := \left\{ f : G \to \sigma \mid \begin{array}{c} f(hg) = hf(g) \text{ for all } h \in H, g \in G, \\ \exists U \text{ compact open where } f(gu) = f(g) \end{array} \right\}$$

This has a G-action:

$$(\gamma \cdot f)(g) = f(g\gamma)$$

and this is forced to be smooth because of the compact open condition in the definition.

Remark 1.3.1. For $f \in \mathrm{Ind}_H^G(\sigma)$, the support $\mathrm{supp}(f) = \{Hg \in H\backslash G \mid f(g) \neq 0\} \subseteq H\backslash G$ is open and closed: if we pick U such that $f(gu) = f(g)$ for all $u \in U$, then the preimage of $\mathrm{supp}(f)$ under $G \to H\backslash G$ is a union of left cosets of U, which is open. Repeat the argument with the complement of $\mathrm{supp}(f)$.

Definition 1.3.2. Let $\text{c-Ind}_H^G \sigma = \left\{ f \in \mathrm{Ind}_H^G \sigma : \mathrm{supp}(f) \text{ compact} \right\}$. This is a subrepresentation of Ind_H^G, called the *compact induction.*

Remark 1.3.3. In the special case where $H\backslash G$ is compact, the two constructions agree. For example, if H is a parabolic subgroup of GL_n, they are the same.

Proposition 1.3.4 (Frobenius Reciprocity)**.** *Say π is a smooth G-representation, and σ is a smooth H-representation. Then*

(1) $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \cong \mathrm{Hom}_H(\pi|_H, \sigma)$, *and this is natural in π, σ.*

(2) *If U is an open subgroup of G, then*

$$\mathrm{Hom}_G(\text{c-Ind}_U^G \sigma, \pi) \cong \mathrm{Hom}_U(\sigma, \pi|_U).$$

Moreover, c-Ind_U^G *is an exact functor.*

Proof Idea. For $\varphi : \pi \to \mathrm{Ind}_H^G \sigma$, we define $\overline{\varphi} : \pi|_H \to \sigma$ sending $x \mapsto \varphi(x)(1)$. Conversely, for $\psi : \pi|_H \to \sigma$, we define $\overline{\psi} : \pi \to \mathrm{Ind}_H^G \sigma$ taking $x \mapsto (g \mapsto \psi(gx))$.

For $g \in G$ and $y \in \sigma$, let $[g, y] \in \text{c-Ind}_U^G \sigma$ denote the function on Ug^{-1} that sends $g^{-1} \mapsto y$, and zero outside. Then $\gamma[g, y] = [\gamma g, y]$ for $\gamma \in G$ and if $u \in U$ we have $[gu, y] = [g, uy]$. Check that $C[G] \otimes_{C[U]} \sigma \cong \text{c-Ind}_U^G \sigma$ sending $g \otimes y \mapsto [g, y]$, and exactness follows from this description because $C[G]$ is free over $C[U]$. □

Proposition 1.3.5. *If $G = \mathrm{GL}_n(\mathbf{Q}_p)$ and P is a standard parabolic, then $\mathrm{Ind}_P^G(-)$ is an exact functor.*

Proof Idea. There exists a continuous section $s : P\backslash G \to G$ to $\pi : G \to P\backslash G$: for example, you can use $G \supseteq P\overline{N}$. Then $\mathrm{Ind}_P^G \sigma \cong \mathscr{C}^\infty(P\backslash G, \sigma)$ as a C-v.s. (locally constant functions): send $f \mapsto (x \mapsto f(s(x)))$. □

Remark 1.3.6. If σ is a smooth representation of M, then first we inflate it to a smooth P-representation via $P \twoheadrightarrow M$, and then induce. By abuse of notation, we notate this $\mathrm{Ind}_P^G(\sigma)$. Furthermore, if $P_1 \subseteq P_2 \subseteq G$ and σ is a smooth representation of M_1, then

$$\mathrm{Ind}_{P_1}^G \sigma \cong \mathrm{Ind}_{P_2}^G \mathrm{Ind}_{P_1}^{P_2}(\sigma \circ (P_1 \twoheadrightarrow M_1)) \cong \mathrm{Ind}_{P_2}^G \mathrm{Ind}_{P_2 \cap M_1}^{M_2} \sigma|_{P_2 \cap M_1}.$$

1.4. **The mod p setting.** From now on, in the mod p setting, we assume that C is an algebraically closed field of characteristic p, and $G = \mathrm{GL}_n(\mathbf{Q}_p)$.

The following is an extremely important lemma which distinguishes mod p representation theory from the characteristic 0 world.

Lemma 1.4.1 (p-group Lemma)**.** *Any smooth representation $\tau \neq 0$ of a pro-p group H has a fixed vector: $\tau^H \neq 0$.*

Proof. WLOG let $C = \mathbf{F}_p$, and pick some nonzero $x \in \tau$. Since τ is smooth and H is compact, there exists an open normal subgroup $U \leq H$ such that $x \in \tau^U$. Note H/U is a finite p-group (since H is pro-p), and it acts on τ^U. Replacing τ^U by $\mathbf{F}_p[H] \cdot x$, WLOG we can assume $\dim_{\mathbf{F}_p} \tau$ is finite. By picking a basis, we get a map

$$\tau : H \to \mathrm{GL}_n(\mathbf{F}_p),$$

so $\tau(H)$ is a p-group, and thus $\tau(H)$ is contained in the p-Sylow subgroup of $\mathrm{GL}_n(\mathbf{F}_p)$. But the p-Sylow subgroup of $\mathrm{GL}_n(\mathbf{F}_p)$ is conjugate to the unipotent radical of the standard Borel, and thus $\tau(H)$ fixes the first basis vector. □

Corollary 1.4.2. *If $\pi \neq 0$ is a smooth representation of G or K, then $\pi^{K(1)} \neq 0$.*

Proof. The group $K(1)$ is pro-p, so use the lemma. □

Corollary 1.4.3. *Any irreducible smooth K-representation π is trivial on $K(1)$, so we get a bijection*

$$\{\textit{irreducible smooth } K\textit{-representations}\} \overset{\sim}{\longleftrightarrow} \{\textit{irreducible } \mathrm{GL}_n(\mathbf{F}_p)\textit{-representations}\}.$$

Proof. Since $\pi^{K(1)} \neq 0$ and $K(1)$ is normal in K, we see that $0 \neq \pi^{K(1)} \subseteq \pi$ is a subrepresentation. Now we use the fact that π is irreducible to conclude that $\pi^{K(1)} = \pi$ which shows that the action of π factors through $K/K(1) \cong \mathrm{GL}_n(\mathbf{F}_p)$. □

Definition 1.4.4. An irreducible smooth K-representation is called a *weight*.

Corollary 1.4.5. *Any non-zero smooth G-representation π contains a weight V, i.e. $V \subseteq \pi|_K$.*

Proof. Pick a nonzero $x \in \pi^{K(1)}$. Then $C[K] \cdot x = C[K/K(1)] \cdot x$ is finite dimensional, so it contains an irreducible subrepresentation. □

If $n = 1$, then the weights are exactly the irreducible $\mathbf{F}_p^\times$-representations valued in C, which are just parametrized by $\mathbf{F}_p^\times$.

If $n = 2$, the weights are given by $V_{a,b} = \mathrm{Sym}^{a-b}(C^2) \otimes \det^b$ for $(a, b) \in \mathbf{Z}^2$ such that $0 \leq a - b \leq p - 1$ (and $0 \leq b \leq p - 1$).

We can think of $\mathrm{Sym}^{a-b}(C^2)$ as homogeneous polynomials in X, Y of degree $a-b$, and $\mathrm{GL}_2(\mathbf{F}_p)$ acts via

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} f(X,Y) = f(\alpha X + \gamma Y, \beta X + \delta Y)$$

Note $\dim V_{a,b} = a - b + 1$.

Theorem 1.4.6. *If $P = MN$ is the standard parabolic decomposition and V is a weight of G, then the natural map*

$$V^{\overline{N}(\mathbf{F}_p)} \hookrightarrow V \twoheadrightarrow V_{N(\mathbf{F}_p)}$$

is an isomorphism of irreducible $M(\mathbf{F}_p)$-representations (i.e. weights of M). In particular, if $P = B$, $V^{\overline{U}(\mathbf{F}_p)} \cong V_{U(\mathbf{F}_p)}$ is one dimensional.

This theorem allows us to make one parametrization of weights.

Theorem 1.4.7 (Curtis, 1970s...). *There is a bijection*

$$\{\textit{weights of } G\} \cong \left\{(\psi : T(\mathbf{F}_p) \to C^\times, P \textit{ standard parabolic}) \mid \psi \textit{ extends to } P(\mathbf{F}_p)\right\}.$$

The map sends $V \mapsto (\psi_V, P_V)$, which we call the parameter of V, where ψ_V is the action of $T(\mathbf{F}_p)$ on $V^{U(\mathbf{F}_p)}$ and P_V is the largest standard parabolic such that $V_{N(\mathbf{F}_p)}$-coinvariants are still one-dimensional.

Example 1.4.8. Let $n = 2$ again. Then $Y^{a-b} \in V_{a,b}$ is $\overline{U}(\mathbf{F}_p)$-stable and thus generates $V_{a,b}^{U(\mathbf{F}_p)}$, and you can compute that $\psi_{V_{a,b}} : T(\mathbf{F}_p) \to C^\times$ takes $\mathrm{diag}(x,y) \mapsto x^b y^a$. Furthermore $P_{V_{a,b}}$ is G if $a = b$ and B otherwise. Note $\psi_{V_{a,b}}$ extends to $G(\mathbf{F}_p)$ if and only if $a \equiv b \bmod p-1$.

Remark 1.4.9. There is a Steinberg parametrization of weights, which uses algebraic representations of the algebraic group $\mathrm{GL}_{n,\mathbf{F}_p}$.

Remark 1.4.10. The weight $V^{\overline{N}(\mathbf{F}_p)} \cong V_{N(\mathbf{F}_p)}$ of M has parameter $(\psi_V, M \cap P_V)$.

1.5. **Hecke algebras and mod p Satake isomorphism.** If π is an irreducible smooth G-representation, then there exists a weight $V \hookrightarrow \pi|_K$, so by Frobenius reciprocity, we get a map $\text{c-Ind}_K^G V \to \pi$, which is surjective, since π is irreducible and V is nonzero.

Definition 1.5.1. The Hecke algebra of the weight V is $\mathcal{H}_G(V) = \mathrm{End}_G(\text{c-Ind}_K^G V)$.

Lemma 1.5.2. *We have an algebra isomorphism*

$$\mathcal{H}_G(V) \cong \left\{ \varphi : G \to \mathrm{End}_C(V) \mid \begin{array}{c} \mathrm{supp}(\varphi) \textit{ is compact, and} \\ \varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2 \end{array} \right\}$$

where the right hand side has the convolution product

$$(\varphi_1 * \varphi_2)(g) = \sum_{\gamma \in G/K} \varphi_1(g\gamma) \circ \varphi_2(\gamma^{-1})$$

Proof. As vector spaces,

$$\mathcal{H}_G(V) = \mathrm{Hom}_G(\text{c-Ind}_K^G(V), \text{c-Ind}_K^G(V)) = \mathrm{Hom}_K(V, \text{c-Ind}_K^G(V)|_K).$$

In general, we have

$$\mathrm{Maps}(V, \mathrm{Maps}(G, V)) = \mathrm{Maps}(G \times V, V) = \mathrm{Maps}(G, \mathrm{Maps}(V, V)).$$

Then just check that this matches the right hand side in the statement of the lemma. Then just do the computation to check multiplication. ∎

2. Talk II

2.1. **Hecke algebras + mod p Satake isomorphism.** Recall $G = \mathrm{GL}_n(\mathbf{Q}_p)$, and C is an algebraically closed characteristic p coefficient field. We had the congruence subgroups K and $K(r)$, and the standard decomposition of parabolics $P = MN$ for N the unipotent radical and N the Levi.

We defined weights to be irreducible smooth C-valued representations of K, but since we're in characteristic p these are automatically trivial on $K(1)$, and $K/K(1) \cong G(\mathbf{F}_p)$, so these are the same as irreducible representations of $\mathrm{GL}_n(\mathbf{F}_p)$.

Definition 2.1.1. We defined the *Hecke algebra* of V is

$$\mathcal{H}_G(V) = \mathrm{End}_G(\text{c-Ind}_K^G(V)).$$

We had the following lemma:

Lemma 2.1.2. *We have an algebra isomorphism*

$$\mathcal{H}_G(V) \cong \{\varphi : G \to \mathrm{End}_C(V) \mid \mathrm{supp}(\varphi) \text{ compact, and } \varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2\}$$

where the right hand side has the convolution product

$$(\varphi_1 \circ \varphi_2)(g) = \sum_{\gamma \in G/K} \varphi_1(g\gamma) \circ \varphi_2(\gamma^{-1})$$

Proof. Given last time. ∎

Remark 2.1.3. If π is a smooth G-representation, then by Frobenius reciprocity, if V is a weight, then

$$\mathrm{Hom}_K(V, \pi|_K) \cong \mathrm{Hom}_G(\text{c-Ind}_K^G V, \pi),$$

and it's now clear that $\mathcal{H}_G(V)$ acts on the left side of the equality. Explicitly, if $f : V \to \pi|_K$ and $\varphi \in \mathcal{H}_G(V)$, then

$$(f \cdot \varphi)(x) = \sum_{Kg \in K\backslash G} g^{-1} f(\varphi(g)(x)),$$

and now you need to check that this is well-defined.

Example 2.1.4. If $V = 1$ (trivial rep) then

$$\mathcal{H}_G(V) = \mathcal{H}_G(1) = \mathcal{C}_c(K\backslash G/K, C).$$

Then $\mathcal{H}_G(V)$ acts on $\mathrm{Hom}_K(1, \pi|_K) = \pi^K$ in the usual double-coset way, i.e.

$$1_{KgK} : \pi^K \to \pi^K, x \mapsto \sum_i g_i^{-1} x$$

where $KgK = \bigsqcup_i K g_i$.

2.2. **Satake isomorphism (mod p).** We want to understand the structure of $\mathcal{H}_G(V)$ now for C-valued representations. We want to embed

$$\mathcal{H}_G(V) \hookrightarrow \mathcal{H}_T(V_{U(\mathbf{F}_p)})$$

and then determine its image. More generally, we want to have

$$\mathcal{H}_G(V) \hookrightarrow \mathcal{H}_M(V_{N(\mathbf{F}_p)})$$

for larger parabolics.

Lemma 2.2.1. *There exists a natural isomorphism*

$$\mathrm{Hom}_G(\text{c-Ind}_K^G(V), \mathrm{Ind}_P^G(-)) \xrightarrow{f \mapsto f_M} \mathrm{Hom}_M(\text{c-Ind}_{M\cap K}^M V_{N(\mathbf{F}_p)}, -)$$

Proof. By Frobenius reciprocity, we get

$$\mathrm{Hom}_G(\text{c-Ind}_K^G(V), \mathrm{Ind}_P^G(-)) \cong \mathrm{Hom}_K(V, \mathrm{Ind}_P^G(-)|_K)$$

But $\mathrm{Ind}_P^G(-)|_K = \mathrm{Ind}_{P\cap K}^K(-|_{P\cap K})$ by the Iwasawa decomposition. Then

$$\mathrm{Hom}_K(V, \mathrm{Ind}_{P\cap K}^K(-|_{P\cap K})) \cong \mathrm{Hom}_{P\cap K}(V|_{P\cap K}, -|_{P\cap K})$$

Now note we started with an M-representation viewed as a P-representation, so this becomes

$$\mathrm{Hom}_{M\cap K}(V_{N\cap K}, -|_{M\cap K})$$

and then finally we use Frobenius reciprocity again to get

$$\mathrm{Hom}_M(\text{c-Ind}_{M\cap K}^M V_{N(\mathbf{F}_p)}, -).$$

□

Observe that any $\varphi \in \mathcal{H}_G(V) = \mathrm{End}(\text{c-Ind}_K^G(V))$ induces a natural transformation of the first functor in the lemma by precomposition, hence also of the second functor. So by Yoneda, there exists a unique $S_M^G(\varphi) \in \mathrm{End}_M(\text{c-Ind}_{M\cap K}^M V_{N(\mathbf{F}_P)}) = \mathcal{H}_M(V_{N(\mathbf{F}_P)})$ such that

$$(f \circ \varphi)_M = f_M \circ S_M^G(\varphi)$$

Exercise 2.2.2. Use this identity to show that $S_M^G : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{N(\mathbf{F}_p)})$ is a C-algebra homomorphism.

Proposition 2.2.3. *Let $p_N : V \to V_{N(\mathbf{F}_p)}$. Explicitly, we have $S_M^G(\varphi) : M \to \mathrm{End}_C(V_{N(\mathbf{F}_p)})$ takes $m \mapsto \sum_{N\cap K\backslash N} p_N \circ \varphi(nm)$.*

Idea of Proof. Find f such that $f_M = \mathrm{id}$ and use the defining formula. □

Definition 2.2.4. Let $T^+ = \{\mathrm{diag}(t_1, \dots, t_n) \in T \mid \mathrm{val}(t_1) \geq \dots \geq \mathrm{val}(t_n)\}$ and

$$\mathcal{H}_T^+(V_{U(\mathbf{F}_p)}) = \left\{\psi : \mathcal{H}_T(V_{U(\mathbf{F}_p)}) : \mathrm{supp}\,\psi \subseteq T^+\right\}$$

Theorem 2.2.5. *The map*

$$S_T^G : \mathcal{H}_G(V) \to \mathcal{H}_T(V_{U(\mathbf{F}_p)})$$

is injective with image $\mathcal{H}_T^+(V_{U(\mathbf{F}_p)})$.

Corollary 2.2.6.

$$\mathcal{H}_G(V) \cong C[\Lambda^+]$$

where $\Lambda^+ = T^+/(T\cap K) = \mathbf{Z}_+^n = \{(\lambda_1, \dots, \lambda_n) \mid \lambda_1 \geq \dots \geq \lambda_n)\}$.

Outline of Proof.

(1) Find nice bases of $\mathcal{H}_G$ and $\mathcal{H}_T$: for $\lambda \in \mathbf{Z}_+^n$ let $t_\lambda := \mathrm{diag}(p^{\lambda_1}, \dots, p^{\lambda_n})$.

Fact: there exists $T_\lambda \in \mathcal{H}_G(V)$ such that $\mathrm{supp}\, T_\lambda = Kt_\lambda K$ and $T_\lambda(t_\lambda) \in \mathrm{End}_C(V)$ is a linear projection. Then if we reduce $K \cap t_\lambda^{-1} K t_\lambda$ to the residue field, we get $P_\lambda(\mathbf{F}_p)$ and note that $T_\lambda(t_\lambda)$ has to factor as

$$V \twoheadrightarrow V_{N_\lambda(\mathbf{F}_p)} \to V^{\overline{N}_\lambda(\mathbf{F}_p)} \hookrightarrow V.$$

and the middle map is $M_\lambda(\mathbf{F}_p)$-linear by Theorem 1.4.6.

The by the Cartan decomposition, deduce that T_λ for all $\lambda \in \mathbf{Z}_+^n$ gives a C-basis of $\mathcal{H}_G$. Similarly, $(\tau_\lambda)_\lambda$ forms a basis for $\mathcal{H}_T$.

(2) To show that S_T^G is injective, prove that

$$S_T^G(T_\lambda) = \tau_\lambda + \sum_{\mu<\lambda} a_\mu \tau_\mu$$

(3) Need to show that $\mathrm{im}(S_T^G) \subseteq \mathcal{H}_T^+$.

(4) Triangular argument of Sug Woo.

□

For the corollary, just note that as T is commutative, $\tau_\lambda \tau_\mu = \tau_{\lambda\mu}$, so we deduce that

$$\mathcal{H}_T^+ \cong C[\mathbf{Z}_+^n] \cong C[x_1, \dots, x_n, x_n^{-1}$$

Proposition 2.2.7. *So we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{H}_G(V) & \xrightarrow{S_M^G} & \mathcal{H}_M(V_{N(\mathbf{F}_p)}) \\ & \searrow^{S_T^G} & \big\downarrow S_T^M \\ & & \mathcal{H}_T(V_{U(\mathbf{F}_p)}), \end{array}$$

hence S_M^G is injective. Moreover, there exists $\varphi \in \mathcal{H}_G(V)$ such that

$$\mathcal{H}_G(V)[\varphi^{-1}] \xrightarrow{\sim} \mathcal{H}_M(V_{N(\mathbf{F}_p)})$$

Proof. The first part is formal.

Note $\mathrm{im}(S_T^G) \cong C[\Lambda^+]$, but

$$\mathrm{im}(S_T^M) \cong C[\Lambda^{+,M}]$$

Note S_M^G is identified with the inclusion

$$C[\Lambda^+] \hookrightarrow C[\Lambda^{+,M},$$

and this is localization at any fixed $\lambda \in \Lambda^+ \cong \mathbf{Z}_+^n$ such that

$$\lambda_1 = \cdots = \lambda_{n_1} > \lambda_{n_1+1} = \cdots = \lambda_{n_1+n_2} > \cdots$$

□

2.3. **Admissible representations and supersingular representations.**

Definition 2.3.1. A smooth G-representation π is admissible if $\dim_C \pi^W < \infty$ for all compact open subgroups W.

Remark 2.3.2. This is stable under taking subrepresentations, less obvious is that it's also stable under taking quotients.

Lemma 2.3.3. *A smooth rep. π is admissible if and only if there exists $W \leq G$ open and pro-p such that $\dim_C \pi^W < \infty$.*

Proof. One direction is by clear. Let's say $W' \subseteq G$ is any compact open subgroup. Firstly, we can shrink W', so WLOG we can assume that $W' \subseteq W$. Then we see that by Frob Rec.

$$\pi^{W'} = \mathrm{Hom}_{W'}(1, \pi|_{W'}) = \mathrm{Hom}_W(\text{c-Ind}_{W'}^W 1, \pi)$$

Note W' is finite index inside W so $\text{c-Ind}_{W'}^W$ is finite dimensional, so we claim that $\mathrm{Hom}_W(\sigma, \pi)$ is finite dimensional for all finite dimensional smooth σ. We do this by induction. If σ is irreducible and smooth, then $\sigma = 1$ by the p-group lemma. If not, then we have some nontrivial short exact sequence

$$0 \to \sigma' \to \sigma \to \sigma'' \to 0$$

so we get

$$0 \to \mathrm{Hom}_W(\sigma'', \pi) \to \mathrm{Hom}_W(\sigma, \pi) \to \mathrm{Hom}_W(\sigma, \pi').$$

So the middle term is finite dimensional by induction. □

Lemma 2.3.4. *If π is admissible, then it contains an irreducible subrepresentation.*

Proof. Fix any W open pro-p subgroup of G. For all subrepresentations $0 \neq \tau \subseteq \pi$, we have that $0 \neq \tau^W \subseteq \pi^W$ by Lemma 1.4.1. Then choose τ such that $\dim(\tau^W)$ is minimal. Then exercise: $\langle G \cdot \tau^W \rangle$ is irreducible. □

Exercise 2.3.5.

(1) If π is a smooth representation, then π is admissible if and only if all the $\mathrm{Hom}_K(V, \pi)$ are finite dimensional.

(2) If π is irreducible and admissible, then π has a central character.

(3) Show that taking parabolic induction preserves admissibility.

Remark 2.3.6. If π is irreducible and smooth, then it does not have to be admissible!

2.4. **Supersingular representations.** These were first defined by Barthel-Livné for GL_2.

Recall that if π is admissible G-representation and V a weight, then $\mathrm{Hom}_K(V, \pi)$ is finite dimensional, and we have a right action of $\mathcal{H}_G(V)$. So we want to use this to describe a notion of supersingularity.

If $\mathrm{Hom}_K(V, \pi) \neq 0$, then it contains a simultaneous eigenvector for the $\mathcal{H}_G(V)$-action.

Definition 2.4.1. $\mathrm{Eval}_G(V, \pi) := \{\varphi \in \mathrm{Hom}_C(\mathcal{H}_G(V), C) : \varphi \text{ occurs as eigenvalues on } \mathrm{Hom}_K(V, \pi)\}$.

Recall $\mathcal{H}_T^+$ has basis τ_λ for $\lambda \in \mathbf{Z}_+^n$. Note that $\tau_\lambda \in (\mathcal{H}_T^+)^\times$ if and only if $\lambda \in \mathbf{Z}_+^n \cap (-\mathbf{Z}_+^n) = \{(a, \ldots, a) \mid a \in \mathbf{Z}\}$.

Lemma 2.4.2. *If π is an irreducible admissible G-representation and V is a weight, then TFAE:*

(1) *For all $\chi \in \mathrm{Eval}_G(V, \pi)$, $\chi(\tau_\lambda) = 0$ for all $\lambda \in \mathbf{Z}_+^n \backslash \mathbf{Z}_0^+$.*

(2) *For all $\chi \in \mathrm{Eval}_G(V, \pi)$, χ doesn't factor through $S_M^G : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{N(\mathbf{F}_p)})$ for all $M \neq G$.*

Idea. We saw that $\mathcal{H}_G[\tau_\lambda^{-1}] \cong \mathcal{H}_{M_\lambda}$, where M_λ is the centralizer of t_λ. □

Definition 2.4.3. An irreducible admissible G-representation is *supersingular* if it satisfies the equivalent conditions in Lemma 2.4.2 for all weights V.

Theorem 2.4.4 (Breuil)**.** *If $n = 2$ and $\alpha \in C^\times$, then*

$$\text{c-Ind}_K^G(V)/(\tau_{(1,0)}, \tau_{(1,1)} - \alpha)\,\text{c-Ind}_K^G(V)$$

is irreducible admissible supersingular.

But this is very special to $\mathrm{GL}_2(\mathbf{Q}_p)$: not irreducible in general.

3. Talk III

Last time, we talked about Hecke algebras, the mod p Satake transform, and we defined supersingular representations: recall this is defined using the Hecke eigenvalues.

3.1. **Classification in terms of supersingular representations.** If Q is a standard parabolic subgroup, then we define the generalized Steinberg representations

$$\mathrm{St}_Q := \mathrm{Ind}_Q^G(1)/\sum_{Q\subsetneq Q'} \mathrm{Ind}_{Q'}^G(1).$$

The actual Steinberg is when $Q = B$ and trivial if $Q = G$.

Theorem 3.1.1 (Grosse-Klonne, H., T. Ly)**.** *The representations St_Q are irreducible admissible and pairwise non-isomorphic. The irreducible constituents of $\mathrm{Ind}_Q^G(1)$ are the $\mathrm{St}_{Q'}$ for all $Q' \supseteq Q$, each with multiplicity one.*

Proposition 3.1.2. *Suppose σ is an (irreducible/admissible/smooth) M-representation. Then there exists a unique largest parabolic $P(\sigma)$ containing P such that σ considered as a P-representation extends uniquely to $P(\sigma)$, and it carries the same properties as before (irreducible/admissible/smooth).*

Remark 3.1.3. The extension $\widetilde{\sigma}$ is trivial on the unipotent radical on $N(\sigma)$ because $N(\sigma) \subseteq N$.

Example 3.1.4. Say M is the $(2,1)$ Levi inside GL_3. If σ is irreducible admissible, then it's automatically of the form $\tau \boxtimes \chi$, for some GL_2-rep τ and character χ. If $P(\sigma) = G$, then $\widetilde{\sigma}$ is trivial on the normal subgroup generated by

$$N = \begin{pmatrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

which is

$$\mathrm{SL}_3(\mathbf{Q}_p) = \left\langle \begin{pmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\rangle.$$

S since τ is irreducible, it was to be $\chi^{-1} \circ \det$.

Definition 3.1.5. Suppose (P, σ, Q) consists of a standard parabolic P, σ an irreducible admissible supersingular M-representation, and Q a parabolic $P \subseteq Q \subseteq P(\sigma)$. Then Let

$$I(P, \sigma, Q) = \mathrm{Ind}_{P(\sigma)}^G(\widetilde{\sigma} \otimes \mathrm{St}_Q^{P(\sigma)})$$

where $\mathrm{St}_Q^P := \mathrm{Ind}_Q^P(1)/\sum_{Q\subsetneq Q'\subseteq P} \mathrm{Ind}_{Q'}^P(1)$

Remark 3.1.6. As $N \leq P$ and $N \leq Q$, N acts trivially on St_Q^P, and

$$\mathrm{St}_Q^P|_M \cong \mathrm{Ind}_{Q\cap M}^M(1)/\sum_{Q\subsetneq Q'\subseteq P} \mathrm{Ind}_{Q'\cap M}^M$$

In particular, $\mathrm{St}_Q^{P(\sigma)}$ is trivial on $N(\sigma)$, so we can do parabolic induction.

Theorem 3.1.7 (Abe-Henniart-H-Vignéras)**.** *The map from triples (P, σ, Q) as above (up to isomorphism) to irreducible admissible G-representations (up to isomorphism) sending*

$$(P, \sigma, Q) \mapsto I(P, \sigma, Q)$$

is a bijection.

Concretely, P has blocks of size $n_1, \ldots, n_r$, so if

$$\sigma \boxtimes \sigma_1 \boxtimes \cdots \boxtimes \sigma_r$$

for σ_i irreducible admissible supersingular representations of $\mathrm{GL}_n(\mathbf{Q}_p)$, then when you do $P(\sigma)$, you combine the consecutive ones such that

$$n_i = \cdots = n_{i+1} = \cdots = n_j = 1$$

and

$$\sigma_i = \sigma_{i+1} = \cdots = \sigma_j$$

i.e. the consecutive characters.

So any irreducible admissible representation is of the form

$$\mathrm{Ind}_{P'}^{G}(\tau)$$

where $\tau = \tau_1 \boxtimes \cdots \boxtimes \tau_s$ and each τ_i is either supersingular (if $n_i' \geq 2$) or $\tau_i \cong \mathrm{St}_{Q_i}^{\mathrm{GL}_{n_i'}} \otimes (\eta_i \circ \det)$.

Example 3.1.8 ($n = 2$)**.** If $P = B$ and $\sigma = \chi_1 \boxtimes \chi_2$ for $\chi_1 \neq \chi_2$, then since $\chi_1 \neq \chi_2$, we must have $Q = B$, and in this case $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$.

But if $P = B$ and $\sigma = \chi \boxtimes \chi$, then $P(\sigma) = G$, so we can either take $Q = B$ or $Q = G$. If $Q = B$, we get $\mathrm{St} \otimes \chi \circ \det$ and if $Q = G$ we get $\chi \circ \det$.

Finally, if $P = G$, then σ is supersingular, and $Q = G$, so we see the supersingular representations of GL_2.

Lemma 3.1.9. *If σ is an irreducible admissible supersingular representation of M, then $\mathrm{Ind}_P^G(\sigma)$ is of finite length. Its irreducible constituents are precisely the $I(P, \sigma, Q)$ where Q runs through all the possible choices between P and $P(\sigma)$, with multiplicity one.*

Definition 3.1.10. An irreducible admissible G-representation π is *supercuspidal* if it's not a subquotient of $\mathrm{Ind}_P^G(\sigma)$ for $P \neq G$ and for all irreducible admissible σ.

Corollary 3.1.11. *If π is irreducible admissible, then π is supersingular if and only if π is supercuspidal.*

Proof. If π is supercuspidal, then by the Theorem, $\pi = I(P, \sigma, Q)$, so by the lemma, π is a subquotient of $\mathrm{Ind}_P^G(\sigma)$, but by the Theorem, $P = G$, so $\pi = I(G, \sigma, G) = \sigma$, and is therefore supersingular.

In the other direction, if π is supersingular, suppose π occurs in some $\mathrm{Ind}_Q^G(\tau)$ where τ is irreducible admissible and $Q \neq G$. Then the lemma applied to τ and transitivity of parabolic induction implies that π occurs in $\mathrm{Ind}_P^G(\sigma)$, where σ is supersingular, and $P \subseteq Q$. Therefore, by the lemma again,

$$\pi \cong I(P, \sigma, Q')$$

for some Q', but on the other hand, $\pi \cong I(G, \pi, G)$, so $P = G$ and $Q = G$. ☐

Idea of proof of theorem::: If we want to show that $\mathrm{Ind}_P^Q(\sigma)$ is irreducible, then take $\tau \subseteq \mathrm{Ind}_P^G(\sigma)$ nonzero and then take $V \hookrightarrow \tau|_K$. Then

$$\text{c-Ind}_K^G V \to \tau \hookrightarrow \mathrm{Ind}_P^G(\sigma)$$

Then this factors through $C \otimes_{\mathcal{H}_G(V),\chi} \text{c-Ind}_K^G(V)$ for some $\chi : \mathcal{H}_G(V) \to C$. Then if $P_V \subseteq P$, then

$$C \otimes_{\mathcal{H}_G(V),\chi} \text{c-Ind}_K^G(V) \cong \mathrm{Ind}_P^G(C \otimes_{\mathcal{H}_M} \text{c-Ind}\, V_{N(\mathbf{F}_p)})$$

3.2. **p-adic representations.** Compared to the mod p representation theory, we need to go through some basic stuff to even access the basic objects and definitions. There is a lot more topology and analysis involved, because the topologies are now very compatible.

To avoid confusion, let $E/\mathbf{Q}_p$ be a finite extension of $\mathbf{Q}_p$: this will be our new coefficient field. $\mathscr{O} = \mathscr{O}_E$ will be the ring of integers, and V will now be an E-vector space.

3.3. **Some functional analysis.** The basic reference is Schneider's book. We can either consider seminorms or lattices.

Definition 3.3.1.

(1) A non-archimedean *seminorm* is a function

$$|\cdot| : V \to \mathbf{R}_{\geq 0}$$

such that

- $|x+y| \leq \max(|x|, |y|)$,
- $|\lambda x| = |\lambda|_E |x|$ for all $\lambda \in E$ and $x \in V$

and we say that it's a norm if

- $|x| = 0$ if and only if $x = 0$.

A *lattice* in V is an $\mathscr{O}$-submodule $\Lambda \subseteq V$ that spans V as an E-vector space.

(2) A *locally convex vector space (lcv)* is a vector space V equipped with a topology defined by seminorms $\{|\cdot|_i\}_{i\in I}$, where the basic opens are given by

$$x_0 + \{|x|_{i_1} \leq \epsilon, \dots |x|_{i_n} \leq \epsilon \text{ for some } i_j \in I, \epsilon > 0\}$$

with this definition, it's easy to see that V is a topological vector space (CHECK THIS). Equivalently, its topology is defined by lattices in the sense that the basic opens are of the form $x_0 + \Lambda_j$ for some $j \in J$ where the Λ_j are a family of lattices such that

(a) For all $\alpha \in E^\times$ and $j \in J$, there exists some other $k \in J$ such that $\alpha\Lambda_j \supseteq \Lambda_k$ (we want to make sure that when we scale a lattice it's still open).

(b) For $i, j \in J$, $\Lambda_i \cap \Lambda_j \supseteq \Lambda_k$.

(3) The dictionary is as follows: if $|\cdot|$ is a seminorm, then $\{|x| \leq \epsilon\}$ is a lattice. If Λ is a lattice, then $|x|_\Lambda := \inf_{x\in\lambda\Lambda} |\lambda|_E$.

By convention all lcv will be Hausdorff, i.e. $\bigcap \Lambda = \{0\}$, where Λ runs over open lattices.

Exercise 3.3.2. If V is a lcv and $W \subseteq V$ the subspace topology on W and quotient topology on V/W are lcv.

Remark 3.3.3. We usually consider $W \subseteq V$ closed, because then V/W is Hausdorff.

Exercise 3.3.4. If $(V_i)_{i\in I}$ is a family of lcv, then so is $\prod_{i\in I} V_i$ for the product topology.

Similarly, we can put a topology on $\varprojlim_i V_i$ and it's lcv.

On $V := \bigoplus_{i\in I} V_i$ take the finest locally convex topology such that each $V_i \to V$ is continuous, this should be lcv.

Similarly, we could take $\varinjlim_i V_i$, it should be lcv.

Exercise 3.3.5. If V is a lcv, so is its strong dual

$$V_b' := \mathrm{Hom}_E^{\mathrm{cts}}(V, E)$$

with the topology defined by the lattices

$$\{f \mid |f(B)| \leq \epsilon\}$$

for all B bounded subsets of V and all $\epsilon > 0$ (uniform converges in each bounded subset). Here, $B \subseteq V$ is bounded if for all $\Lambda \subseteq V$ open lattice, there exists $\alpha \in E$ such that $B \subseteq \alpha\Lambda$.

Definition 3.3.6. A lcv V is Banach (Fréchet) if its topology can be defined by a single (a countable family of) (semi)norm(s), and for which it's complete with respect to the topology (i.e. Cauchy sequences converge).

Clearly a Banach lcv V is Fréchet. A Fréchet space is metrizable.

Remark 3.3.7. A Banach space does *not* carry a fixed norm, but sometimes it can be useful to fix one.

Proposition 3.3.8. *A finite dimensional vector space carries a unique Hausdorff lcv topology. If $V = E^n$, we can define it in this non-archimedean world using the supremum norm:*

$$||\underline{a}|| := \max_{1 \leq i \leq n} |a_i|.$$

This is clearly a Banach topology, complete because E is complete.

Example 3.3.9. If I is a set, consider

$$\ell^\infty(I) := \{\text{bounded functions } I \to E \text{ with the sup. norm}\}$$

Inside, we have $c_0(I) = \{f \in \ell^\infty(I) \mid \text{ for all } \epsilon > 0, |\{|f| > \epsilon\}| < \infty\}$. Think of $I = \mathbf{N}$.

If X is a compact topological space, then we have

$$\mathscr{C}^0(X, E)$$

with the sup norm, and this is Banach.

Remark 3.3.10. For Fréchet spaces, we have the Open Mapping Theorem, and the Closed Graph Theorem.

4. Talk IV

4.1. **Recollections.** Recall that we take a finite extension $E/\mathbf{Q}_p$, and $\mathscr{O} = \mathscr{O}_E$ denotes its ring of integers, and V is an E-vector space. Recall that we are interested in locally convex (lcv) vector spaces V, i.e. those where a fundamental neighborhood basis of 0 is given by a family of lattices or semi-norms.

By convention, we assumed that V is always Hausdorff.

Definition 4.1.1. A map $f : V \to W$ of Banach spaces is **compact** if $\overline{f(V^\circ)}$ is relatively compact for any/some unit ball $V^\circ \subseteq V$.

Definition 4.1.2. A locally convex V is of **compact type** if

$$V \cong \varinjlim_{n \geq 1} V_n$$

where V_n is Banach and $V_n \to V_{n+1}$ are injective and compact.

Example 4.1.3. If $\dim_E V$ is countable, then we can equip it with the finest locally convex topology. Then $V = \bigcup_{n \geq 1} V_n$, where $V_1 \subseteq V_2 \subseteq \cdots$ which are all finite dimensional, so V is clearly of compact type.

Fact 4.1.4.

(1) *If V is of compact type and $W \subseteq V$ is a closed subspace, then both W (with the subspace topology) and V/W (with the quotient topology) are of compact type.*

(2) *The strong dual induces equivalences of categories:*

$$\{\textit{compact type spaces}\} \overset{\sim}{\longleftrightarrow} \{\textit{“nuclear” Fréchet spaces}\}$$

taking

$$\varinjlim_n V_n \mapsto \varprojlim_n (V_n)_b^\vee$$

where $\vee$ denotes the continuous linear dual, and b denotes the strong topology.

4.2. **Locally analytic manifolds.** First let's discuss manifolds.

Definition 4.2.1. If $a \in \mathbf{Q}_p^d$ and $r > 0$, we define the **closed ball**

$$B_r(a) = \left\{ x \in \mathbf{Q}_p^d \mid ||x - a|| \le r \right\}.$$

These are actually compact and open as well.

Definition 4.2.2. A **($\mathbf{Q}_p$-)locally analytic manifold** of dimension d is a paracompact Hausdorff topological space M along with a maximal atlas of charts (U, φ_U) where $U \subseteq M$ is open which cover M, and $\varphi_U : U \xrightarrow{\sim} B_U \subseteq \mathbf{Q}_p^d$ where B_U is a closed ball such that $\varphi_{U_i} \circ \varphi_{U_j}^{-1} : \varphi_{U'}(U \cap U') \xrightarrow{\sim} \varphi_U(U \cap U')$ is locally analytic, i.e. locally given by a convergent power series.

We get a category of locally analytic manifolds.

Definition 4.2.3. A **locally analytic group** (or **p-adic Lie group**) is a group object in the category of locally analytic manifolds.

Example 4.2.4. Examples are $\mathrm{GL}_n(\mathbf{Q}_p)$, $\mathrm{GL}_n(K)$, $K/\mathbf{Q}_p$ a finite extension.

Remark 4.2.5. Any locally analytic manifold is strictly paracompact, meaning that you can refine an open cover by a locally finite cover consisting of *disjoint* open sets.

4.3. **Locally analytic functions.**

Definition 4.3.1. If $B = B_r(a) \subseteq \mathbf{Q}_p^d$ and V is a Banach space, with some fixed norm $||\cdot||$, let

$$\mathcal{C}^{\mathrm{rig}}(B, V) := \left\{ f = \sum_{i \in \mathbf{N}^d} v_i (x_1 - a_1)^{i_1} \cdots (x_d - a_d)^{i_d} \mid \lim_{|i| \to \infty} ||v_i|| r^{|i|} = 0 \right\}.$$

Furthermore, let $||f||_B := \max_i ||v_i|| r^{|i|} \in \mathbf{R}_{\ge 0}$.

Lemma 4.3.2.

(1) $||\cdot||_B$ *is independent of the choice of a, because we are in the non-archimedean world.*

(2) $(\mathcal{C}^{\mathrm{rig}}(B, V), ||\cdot||_B)$ *is complete, i.e. Banach.*

Proof. ??? □

Remark 4.3.3. We have a continuous injective evaluation map

$$\mathcal{C}^{\mathrm{rig}}(B, V) \to \mathcal{C}^0(B, V).$$

Definition 4.3.4. If $B_1, B_2 = B_r(a)$ are closed balls in $\mathbf{Q}_p^d$, then let

$$\mathcal{C}^{\mathrm{rig}}(B_1, B_2) := \left\{ f + a \mid f \in \mathcal{C}^{\mathrm{rig}}(B_1, \mathbf{Q}_p^d) \mid ||f||_{B_1} \le r \right\}$$

and this is independent of the choice of a, and composition is well-defined.

Definition 4.3.5. Suppose M is a locally analytic manifold and V a Banach space. Then we define

$$\mathcal{C}^{\mathrm{an}}(M, V) := \varinjlim_{M = \bigsqcup_{i \in I} U_i \text{ charts } \varphi : U_i \xrightarrow{\sim} B_i \text{ ball}} \prod_{i \in I} \mathcal{C}^{\mathrm{rig}}(B_i, V)$$

In this limit, transition maps are refinements: say $(U_i, \varphi_i)_{i \in I} \le (W_j, \psi_j)_{j \in J}$ if for all $j \in J$ there exists a unique $i(j) \in I$ such that $W_j \subseteq U_{i(j)}$ such that the map

$$B_j \xrightarrow{\psi_j^{-1}} W_j \subseteq U_{i(j)} \xrightarrow{\varphi_{i(j)}} B_{i(j)}$$

lives in the image of $\mathcal{C}^{\mathrm{rig}}(B_j, B_{i(j)}) \to \mathcal{C}^0(B_j, B_{i(j)})$. Then we get transition maps

$$\mathcal{C}^{\mathrm{rig}}(U_{i(j)}, V) \to \mathcal{C}^{\mathrm{rig}}(W_j, V),$$

which induces

$$\prod_{i\in I} \mathcal{C}^{\mathrm{rig}}(U_i, V) \to \prod_{j\in J} \mathcal{C}^{\mathrm{rig}}(W_j, V),$$

which is continuous and injective.

Remark 4.3.6. The transition maps are compatible with compositions, and any two indices admit a common refinement, which implies that $\mathcal{C}^{\mathrm{an}}(M, V)$ is locally convex and we have a continuous evaluation map

$$\mathcal{C}^{\mathrm{an}}(M, V) \to \mathcal{C}^0(M, V).$$

Exercise 4.3.7. If $M = \mathbf{Z}_p \subseteq \mathbf{Q}_p$ then the set $\{(a + p^n\mathbf{Z}_p, \mathrm{id}) \mid a \in \mathbf{Z}/p^n\}_{n\geq 0}$ is cofinal among all indices. So

$$\mathcal{C}^{\mathrm{an}}(\mathbf{Z}_p, V) = \varinjlim_{n\geq 0} \prod_{a\in\mathbf{Z}/p^n} \mathcal{C}^{\mathrm{rig}}(a + p^n\mathbf{Z}_p, V).$$

The transition maps are compact, which implies that $\mathcal{C}^{\mathrm{an}}(\mathbf{Z}_p, E)$ is of compact type. The fact that the transitions maps are compact comes down to the fact that

$$\mathcal{C}^{\mathrm{rig}}(\mathbf{Z}_p, E) \to \mathcal{C}^{\mathrm{rig}}(p\mathbf{Z}_p, E)$$

is compact.

Proposition 4.3.8. *If M is compact and $V = E$ (or more generally V is of compact type) then $\mathcal{C}^{\mathrm{an}}(M, V)$ is of compact type.*

More generally, if V is locally convex, we define

$$\mathcal{C}^{\mathrm{an}}(M, V) := \varinjlim_{(U_i,\varphi_i,V_i)} \prod_{i\in I} \mathcal{C}^{\mathrm{rig}}(U_i, V_i)$$

over V_i Banach with a continuous injection $V_i \hookrightarrow V$, and where (U, φ_i) are before.

Proposition 4.3.9. *If $M = \bigsqcup_{i\in I} M_i$, then*

$$\mathcal{C}^{\mathrm{an}}(M, V) \cong \prod_{i\in I} (M_i, V).$$

4.4. **Locally analytic and Banach space representations.** Now G is a locally analytic group.

Definition 4.4.1. A **Banach space representation** of G is a Banach space V and a continuous linear action $G \times V \to V$. It is unitary if there exists a G-invariant norm defining the topology on V.

Remark 4.4.2. Continuous is equivalent to separately continuous. Closed subrepresentations and quotients are still Banach.

Example 4.4.3.

(1) A finite dimensional continuous representation (with its unique Hausdorff topology) is a Banach space representation.

(2) If $H \leq G$ is a closed subgroup such that $H\backslash G$ is compact and W is any Banach representation of H, then

$$(\mathrm{Ind}_H^G W)^{C_0} = \left\{ f : G \xrightarrow{\mathrm{cts}} W \mid f(hg) = hf(g) \right\}.$$

There always exists a section of $s : H\backslash G \to G$, and we can use this to get an isomorphism

$$(\mathrm{Ind}_H^G W)^{C_0} \cong \mathcal{C}^0(H\backslash G, W),$$

which is again a Banach space, using the supremum norm. For example, this works when P is a parabolic, or we get $\mathcal{C}^0(G, E)$ if G is compact and $H = 1$.

(3) If G is compact, then any Banach space representation is unitary (via averaging, as usual).

Definition 4.4.4. A **locally analytic representation** of G is a compact type space V and a continuous linear action $G \times V \to V$ such that orbit maps $o_v : G \to V$ sending $g \mapsto gv$ are locally analytic, i.e. $o_v \in \mathcal{C}^{\mathrm{an}}(G, V)$ for all $v \in V$.

Example 4.4.5.

(1) Finite dimensional representations are locally analytic: the point is that any continuous homomorphism $G \to \mathrm{GL}_n(E)$ is locally analytic).

(2) If $H \leq G$ is closed with compact quotient, we let
$$(\mathrm{Ind}_H^G W)^{\mathrm{an}} := \left\{ f : G \xrightarrow{\text{locally analytic}} W \mid f(hg) = hf(g) \right\} \cong \mathcal{C}^{\mathrm{an}}(H\backslash G, W),$$
which is of compact type because $H\backslash G$ is compact and W is of compact type.

(3) Say V^{sm} is a **smooth representation** of countable dimension (here o_v is locally constant!)

(4) If $G = \mathbf{G}(\mathbf{Q}_p)$, where $\mathbf{G}$ is an algebraic group and V_{alg} is a (finite dimensional) algebraic representation of G, then V_{alg} is locally analytic, and things of the form $V_{\mathrm{alg}} \otimes V_{\mathrm{sm}}$ are called "locally algebraic". (Warning: these are not abelian categories!!)

4.5. **Duality and admissibility: mod p.** For this section, G is a compact locally analytic group (e.g. $\mathrm{GL}_n(\mathbf{Z}_p)$).

First we discuss the mod p case. Let $C/\mathbf{F}_p$ be a finite field and let
$$D^\infty(G) := \mathcal{C}^{\mathrm{an}}(G, C)^\vee = (\varinjlim_{U \leq G \text{ open normal}} \mathcal{C}(G/U, \mathbf{C}))^\vee = \varprojlim_{U \leq G \text{ open normal}} C[G/U] = C[[G]].$$
This is Noetherian (Lazard). If V is a smooth representation of C, then $V = \varinjlim_{U \leq G \text{ open normal}} V^U$, which has an action of $C[[G]]$ in the limit, and thus $V^\vee$ does as well.

Then duality in this case gives a map
$$\{\text{smooth } G\text{-reps}\} \xrightarrow{\sim} \{D^\infty(G)\text{-modules with profinite top. such that action is cts.}\}$$
sending $V \mapsto V^\vee$ ($\varinjlim W \mapsto \varprojlim W^\vee$).

Remark 4.5.1.

(1) By (a version of) Nakayama's lemma, V is admissible if and only if $V^\vee$ is a finitely generated module over $D^\infty(G)$.

(2) Any finitely generated $D^\infty(G)$-module carries a unique profinite topology such that the action is continuous, so the above duality restricts to
$$\{\text{admissible } G\text{-reps}\} \xrightarrow{\sim} \{\text{finitely generated } D^\infty(G)\text{-modules}\}.$$
Now the right hand side is an abelian category, because $D^\infty(G)$ is Noetherian.

Corollary 4.5.2. *The LHS is closed under quotients.*

4.6. **Duality and admissibility: Banach case.** Now let's go back to the p-adic case. Let
$$D^c(G) := \mathcal{C}^0(G, E)' \cong \mathscr{O}[[G]][1/p]$$
where $\mathscr{O}[[G]] = \varprojlim_{n, U \leq G \text{ open normal}} (\mathscr{O}/\varpi^n)[G/U]$. This is a profinite ring, Noetherian by Lazard.

If V is a Banach representation, then it is unitary (recall G is compact). In particular, there exists a G-invariant lattice $V^\circ \subseteq V$. By definition
$$V^\circ = \varprojlim_{n \geq 0} V^\circ/\varpi^n V^\circ,$$

each of which carries an action of $(\mathscr{O}/\varpi^n)[[G]]$, so we get an action of $\mathscr{O}[[G]]$ in the limit. So V, V' become $D^c(G)$-modules.

Definition 4.6.1. Say V is **admissible** if $V^\vee$ is finitely generated as a $D^c(G)$-module.

Theorem 4.6.2 (Schneider-Teitelbaum)**.** *There is a bijection*

$$\{\text{admissible Banach representations of } G\} \xrightarrow{\sim} \{\text{finitely generated } D^c(G)\text{-modules}\}$$

sending $V \mapsto V^\vee$.

Example 4.6.3. The dual of $\mathcal{C}^0(G, E)$ is $D^c(G)$.

Corollary 4.6.4.

(1) *Any map $f : V \to W$ of admissible Banach space representations is strict (i.e. $V/\ker f \cong \operatorname{im} f$ is a topological isomorphism).*

(2) *Any closed subspace W and quotient V/W are again admissible if V is admissible.*

(3) *Have usual kernel/cokernel with the induced topology.*

5. Talk V

Let G be a locally analytic group.Recall that a Banach representation is a continuous map

$$G \times V \to V$$

where V is a Banach space. A locally analytic representation is a continuous map

$$G \times V \to V$$

where V is of compact type and the orbit map $o_v : G \to V$ is locally analytic for all $v \in V$.

Assume G is compact. Recall that in the Banach case, we defined

$$D^c(G) := \mathcal{C}^0(G, E)' \cong \mathscr{O}[[G]][1/p]$$

and so V and V' become finitely generated modules over $D^c(G)$, and this is an equivalence (cf. Theorem 4.6.2), and therefore we get an abelian category.

5.1. Duality and admissibility: locally analytic case.

Now we can still define the distribution algebra analogously:

$$D^{\mathrm{an}}(G) := \mathcal{C}^{\mathrm{an}}(G, E)'_b$$

which is a nuclear Fréchet space. We have Dirac distributions δ_g for $g \in G$, which span a dense subspace of the analytic distributions.

Theorem 5.1.1 (de Lacroix)**.** *There is a unique continuous multiplication $*$ on $D^{\mathrm{an}}(G)$ such that*

$$\delta_g * \delta_h = \delta_{gh}.$$

Concretely, if $\delta_1, \delta_2 \in D^{\mathrm{an}}(G)$, we can compute

$$(\delta_1 * \delta_2)(f) = \delta_1(g_1 \mapsto \delta_2(g_2 \mapsto f(g_1 g_2)))$$

If V is a locally analytic representation then there's a unique separately continuous action of $D^{\mathrm{an}}(G) \times V \to V$ such that $\delta_g v = gv$, and same for V'.

But now $D^{\mathrm{an}}(G)$ is not Noetherian in general.

Theorem 5.1.2 (Schneider-Teitelbaum)**.**

$$\left\{ \begin{array}{c} \textit{locally analytic representations} \\ \textit{on compact type spaces} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{separately continuous } D^{\mathrm{an}}(G)\textit{-modules} \\ \textit{on nuclear Fréchet spaces} \end{array} \right\}$$

taking $V \mapsto V_b'$.

Remark 5.1.3. If $\mathfrak{g} = \mathrm{Lie}(G)$, then we get a map $\mathfrak{g} \to D^{\mathrm{an}}(G)$ via

$$X \mapsto (f \mapsto \frac{d}{dt}|_{t=0} f(e^{tX})).$$

Note the exponential map $\mathfrak{g} \to G$ is defined near the identity.

Remark 5.1.4. We have a subring $D^c(G) \hookrightarrow D^{\mathrm{an}}(G)$, but again, $D^{\mathrm{an}}(G)$ is not necessarily Noetherian.

Example 5.1.5. Take $G = \mathbf{Z}_p$. Mahler showed that

$$\mathcal{C}^0(\mathbf{Z}_p, E) = \left\{ \sum_{n \geq 0} a_n \binom{x}{n} \mid a_n \in E, a_n \to 0 \right\}$$

So

$$\mathcal{C}^{\mathrm{an}}(\mathbf{Z}_p, E) = \left\{ \sum_{n \geq 0} a_n \binom{x}{n} \mid |a_n| r^n \to 0 \text{ for some } r > 1 \right\}$$

Then we have the Amice transform $D^{\mathrm{an}}(\mathbf{Z}_p) \xrightarrow{\sim} \{\text{rigid analytic functions on the open unit disc}\} =: \mathcal{C}^{\mathrm{rig}}(X_{<1})$, which is an algebra isomorphism sending

$$\delta \mapsto \delta((1+T)^x) = \sum_{n \geq 0} \delta\left(\binom{x}{n}\right) T^n$$

But note $\mathcal{C}^{\mathrm{rig}}(X_{<1}) \cong \varprojlim_{r<1, r \in p^{\mathbf{Q}}} \mathcal{C}^{\mathrm{rig}}(X_{\leq r})$, and note that $\mathcal{C}^{\mathrm{rig}}(X_{\leq r})$ is a noetherian PID.

In general, Schneider-Teitelbaum showed that $D^{\mathrm{an}}(G)$ is a Fréchet-Stein algebra.

Definition 5.1.6. A Fréchet algebra A is Fréchet-Stein if there exist seminorms $q_1 \leq q_2 \leq \cdots$ defining the topology on A such that

(1) The multiplication $A \times A \to A$ is continuous with respect to q_n for all n (which implies that $A \cong \varprojlim_{n \geq 1} A_{q_n}$).

(2) The completion A_{q_n} is left Noetherian.

(3) A_{q_n} is flat as a right $A_{q_{n+1}}$-module.

Definition 5.1.7. If A is Fréchet-Stein, then an A-module M is coadmissible if

(1) $M_n := A_{q_n} \otimes_A M$ is finitely generated for all n, and

(2) $M \to \varprojlim_n M_n$ is a bijection.

This mimics the definition of the definition of a coherent sheaf on a non-affinoid which has an exhaustive decreasing cover by affinoid things. It doesn't depend on the choice of q_n.

Fact 5.1.8.

(1) *In the above definition, coadmissible modules M are the same as a compatible sequence M_n, each finitely generated A_{q_n}-modules*

(2) *The category of coadmissible modules is an abelian subcategory of the category of A-modules.*

(3) *Any finitely presented A-module is coadmissible.*

Remark 5.1.9. Any coadmissible M carries a canonical topology: first M_n carries a unique Banach topology by finite generation, then we take the inverse limit topology from $M \xrightarrow{\sim} \varprojlim_n M_n$. Then any map between coadmissible modules is continuous and strict.

The idea of the proof that $D^{\mathrm{an}}(G)$ are Fréchet-Stein is that we pass to a small compact open subgroup that is "uniform pro-p". One consequence of this is that topologically, we have a homeomorphism

$$\mathbf{Z}_p^d \xrightarrow{\sim} G$$

Then they use some results of Lazard on Mahler expansions, etc.

Definition 5.1.10. A locally analytic representation V is **admissible** if V_b' is isomorphic to a coadmissible module with its canonical topology.

As before, we get

$$\{\text{admissible locally analytic representations}\} \xrightarrow{\sim} \{\text{coadmissible modules over } D^{\mathrm{an}}(G)\}$$

and thus we get an abelian category.

Corollary 5.1.11.

(1) *Any map of admissible G-representations is strict with closed image.*

(2) *Closed subrepresentations/Hausdorff quotients are admissible.*

(3) *We have the usual kernel and cokernel.*

Example 5.1.12. If V is admissible and smooth, then it's admissible locally analytic. If V is an admissible Banach representation of G, then let

$$V_{\mathrm{an}} := \{v \in V \mid o_v \in \mathcal{C}^{\mathrm{an}}(G, V)\}$$

which takes the subspace topology from $\mathcal{C}^{\mathrm{an}}(G, V)$.

Theorem 5.1.13 (Schneider-Teitelbaum)**.**

(1) *The V_{an} are compact type and dense in V.*

(2) *V_{an} form an admissible locally analytic representation and $(V_{\mathrm{an}})' \cong D^{\mathrm{an}}(G) \otimes_{D^c(G)} V'$.*

(3) *$V \mapsto V_{\mathrm{an}}$ is exact.*

5.2. **Orlik-Strauch Representations.** Now let $G = \mathrm{GL}_n(\mathbf{Q}_p)$. We write $P = MN$ the usual parabolic decomposition for some P. Let $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{p} = \mathrm{Lie}(P)$, etc. We have a universal enveloping algebra $U(\mathfrak{g})$, etc, by which we really mean $U(\mathfrak{g}) \otimes_{\mathbf{Q}_p} E$.

Definition 5.2.1. A finite dimensional irreducible representation of $\mathfrak{m}$ over E is **algebraic** if it integrates to a finite dimensional algebraic representation of the Levi M.

Example 5.2.2. If $P = B$, then $M = T$, and $\mathfrak{t} = \mathbf{Q}_p^d \to E$ is some map $x \mapsto \sum \lambda_i x_i$, and this is algebraic if and only if $\lambda_i \in \mathbf{Z}$, and in this case, this integrates to the character sending

$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto \prod t_i^{\lambda_i}.$$

Definition 5.2.3. The objects of the category $\mathcal{O}_{\mathfrak{p}}^{\mathrm{alg}}$ are finitely generated $U(\mathfrak{g})$-modules L such that $L|_{\mathfrak{m}}$ is a direct sum of irreducible algebraic representations of $\mathfrak{m}$, and such that for all $x \in L$ we have that $U(\mathfrak{m}) \cdot x$ is finite dimensional.

Morphisms are $U(\mathfrak{g})$-linear maps.

Example 5.2.4.

(1) Note $\mathcal{O}_{\mathfrak{g}}^{\mathrm{alg}}$ is the category of algebraic representations of $\mathfrak{g}$.

(2) In general, if W is an irreducible algebraic $\mathfrak{m}$-representation, consider it as a module over $U(\mathfrak{p})$ via the projection $U(\mathfrak{p}) \twoheadrightarrow U(\mathfrak{m})$. Then the (generalized) Verma module is

$$M(W) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$$

lives in $\mathcal{O}_{\mathfrak{p}}^{\mathrm{alg}}$. (exercise: use that $M(W) = U(\overline{\mathfrak{n}}) \otimes_E W$ by PBW).

Fact 5.2.5. *Here are some facts about $\mathcal{O}_{\mathfrak{p}}^{\mathrm{alg}}$.*

(1) *It's abelian.*

(2) *It's closed under sub/quotient/$\oplus$.*

(3) *Every object has finite length.*

(4) *If $P \subseteq Q$ then $\mathcal{O}_{\mathfrak{q}}^{\mathrm{alg}} \subseteq \mathcal{O}_{\mathfrak{p}}^{\mathrm{alg}}$.*

Now fix $L \in \mathcal{O}_{\mathfrak{p}}^{\mathrm{alg}}$, and π_M an admissible smooth M-representation. Then there exists some $W \subseteq L$ finite dimensional, stable under $\mathfrak{p}$ such that W generates L.

$$0 \to \partial \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \to L \to 0$$

Note the $\mathfrak{p}$-action on W integrates to an algebraic P-action: the idea is that it's clearly true for the M-action by axiom (2), and for the $\mathfrak{n}$-action use part (3) and the exponential map $\mathfrak{n} \xrightarrow{\sim} N$.

Now consider

$$\mathcal{C}^{\mathrm{an}}(G, W' \otimes \pi_M),$$

which carries two G-actions, by both left/right-translation. Differentiate the left one and get an action of $\mathfrak{g}$ on $\mathcal{C}^{\mathrm{an}}(G, W' \otimes \pi_M)$, i.e. $X \cdot f = \frac{d}{dt}|_{t=0} f(e^{tX}(-))$.

Then we get a pairing

$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W) \times \mathrm{Ind}_P^G(W' \otimes \pi_M)^{\mathrm{an}} \to \mathcal{C}^{\mathrm{an}}(G, \pi_M)$$

sending

$$(X \otimes w, f) \mapsto (g \mapsto \langle (X \cdot f)(g), w \rangle)$$

Definition 5.2.6. Note ∂ acts on $\mathrm{Ind}_P^G(W' \otimes \pi_M)^{\mathrm{an}}$ via $U(G) \otimes_{U(\mathfrak{p})} W$ and the above pairing.

$$\mathcal{F}_P^G(L, \pi_M) := [\mathrm{Ind}_P^G(W' \otimes \pi_M)^{\mathrm{an}}]^{\partial = 0}$$

which is a closed G-subrepresentation of $\mathrm{Ind}_P^G(W' \otimes \pi_M)^{\mathrm{an}}$.

Example 5.2.7. Note $\mathcal{F}_P^G(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W, \pi_M) = \mathrm{Ind}_P^G(W' \otimes \pi_M)^{\mathrm{an}}$: in this case $\partial = 0$.

Theorem 5.2.8 (Orlik-S?)**.**

(1) *$\mathcal{F}_P^G$ is independent of the choice of W.*

(2) *$\mathcal{F}_P^G(L, \pi_M)$ is admissible, and this is functorial and exact in both L and π_M.*

(3) *If $Q \supseteq P$ and $L \in \mathcal{O}_{\mathfrak{q}}^{\mathrm{alg}}$*

$$\mathcal{F}_P^G(L, \pi_M) \cong \mathcal{F}_Q^G(L, (\mathrm{Ind}_{P \cap M_Q}^M \pi_M)^{\infty})$$

(4) *If L and π_M are irreducible and P is maximal for L (i.e. $L \notin \mathcal{O}_{\mathfrak{q}}^{\mathrm{alg}}$ for $Q \supsetneq P$) then $\mathcal{F}_P^G(L, \pi_M)$ is topologically irreducible.*

Corollary 5.2.9. *If π_M is of finite length then $\mathcal{F}_P^G(L, \pi_M)$ is topologically of finite length.*

5.3. $n = 2$**.** Now take $\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z}^2 \subseteq \mathfrak{t}'$ with $\lambda_1 \geq \lambda_2$. Then we get the following sequence in $\mathcal{O}$. Note $L(\lambda)$ is the unique irreducible quotient of the Verma module.

$$0 \to L(\lambda') \to M(\lambda) \to L(\lambda) \to 0$$

where $\lambda' = (12) \circ \lambda = (\lambda_2 - 1, \lambda_1 + 1)$. Note $L(\lambda)$ lies in $\mathcal{O}_{\mathfrak{g}}^{\mathrm{alg}}$, but $L(\lambda')$ is infinite dimensional.

Let $\chi = \chi_1 \otimes \chi_2$ be a smooth character $T \to E^\times$. By Orlik-Strauch, we get

$$0 \to \mathcal{F}_B^G(L(\lambda), \chi) \to \mathcal{F}_B^G(M(\lambda), \chi) \to \mathcal{F}_B^G(L(\lambda'), \chi) \to 0$$

Note $\mathcal{F}_B^G(M(\lambda), \chi) = \mathrm{Ind}_B^G(\chi_\lambda^{-1} \otimes \chi)^{\mathrm{an}}$. Furthermore, the quotient $\mathcal{F}_B^G(L(\lambda'), \chi)$ is irreducible. Also

$$\mathcal{F}_B^G(L(\lambda), \chi) \cong \mathcal{F}_G^G(L(\lambda), (\mathrm{Ind}_B^G(\chi))^\infty)$$

is irreducible if and only if $\chi_1\chi_2^{-1} \neq 1, |\cdot|^2$.

lastly, the quotient is $M(\lambda')$, so it's a principal series for λ'.

References