THE LOCAL LANGLANDS CORRESPONDENCE AND LOCAL-GLOBAL COMPATIBILITY FOR GL_2

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Contents

1. Talk I

This course will give an overview of the Langlands program for GL_n (but we'll say things about how to generalize to arbitrary G here and there).

Here's the rough plan for the five talks:

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(1) Motivation
(2) Automorphic (local)
(3) Galois
(4) Automorphic (global) + Langlands correspondence
(5) Langlands-Kottwitz (for $\mathrm{GL}_{2,\mathbf{Q}}$)

1.1. **Motivation.** Fix F a global field, and let $\mathbf{A}_F = \prod'_v F_v$ denote the usual ring of adèles. Let $\Gamma_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$, and let $\Gamma_v := \mathrm{Gal}(F_v^{\mathrm{sep}}/F_v)$, and fix some embeddings $\Gamma_v \hookrightarrow \Gamma_F$ induced from some $\overline{F} \hookrightarrow \overline{F_v}$.

Conjecture 1.1.1 (Global Langlands, Rough Form)**.** *There is a unique bijection between certain automorphic representations of* $\mathrm{GL}_n(\mathbf{A}_F)$ *and certain Galois representations* $\Gamma_F \to \mathrm{GL}_n(\overline{\mathbf{Q}_\ell})$ *such that* π *corresponds to* ρ *if and only if* π_p *corresponds to* $\rho_v := \rho|_{\Gamma_v}$ *at every finite place* v*, where* $\pi = \otimes'_v \pi_v$*.*

The second part is called "local-global compatibility", and it implicitly refers to a local Langlands correspondence. The goal for the course is to make this precise!

1.2. **Smooth Representations.** We will start in the local setting: let $F/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathscr{O}_F$ and finite residue field k_F with uniformizer ϖ_F. Let $G = \mathrm{GL}_n(F)$ with the p-adic topology. This is a locally compact and locally profinite group with a maximal compact subgroup $\mathrm{GL}_n(\mathscr{O}_F)$, which has a basis of neighborhoods $I_n + \varpi_F^m M_n(\mathscr{O}_F)$ around the identity I_n.

Our coefficient field will be $\mathbf{C}$, with the discrete topology, but most of this also works for $\overline{\mathbf{Q}_\ell}$ (e.g. this is useful when we take étale cohomology). Most of what's to follow works for other reductive groups as well.

Now consider a representation (π, V) of G (note V will usually be infinite dimensional, but there shouldn't really be any topological issues, because the topology on G and the topology on the coefficients, which could either be discrete or ℓ-adic, etc, shouldn't interact anyway: this is what makes local Langlands simpler away from p!).

Definition 1.2.1.

(1) A representation (π, V) is called **smooth** if for every $v \in V$, the stabilizer $\mathrm{Stab}_G(v)$ contains an open subgroup of G. This is equivalent to saying that the map $G \times V \xrightarrow{\pi} V$ is continuous for the discrete topology on V (exercise). (I think this is *not* equivalent, assuming V is infinite dimensional, to saying that $G \to \mathrm{GL}(V)$ is continuous, unless there's some strange topology on $\mathrm{GL}(V)$ that I haven't considered. It certainly shouldn't be true if you put the discrete topology on $\mathrm{GL}(V)$?)
(2) A smooth representation (π, V) is called **admissible** if $\dim V^K < \infty$ (here V^K denotes the invariant vectors in V for the action of K) for all open compact subgroups $K \subseteq G$.

Remark 1.2.1.

(1) As previously mentioned, V is typically infinite dimensional, so admissibility is not a vacuous condition.
(2) An irreducible smooth representation is admissible: this is a non-trivial fact.
(3) From an arbitrary representation (π, V) of G, we can produce a smooth representation $(\pi^{\mathrm{sm}}, V^{\mathrm{sm}})$ by taking the subspace of smooth vectors in V: to see this use the fact that $\mathrm{Stab}_G(v) \cap \mathrm{Stab}_G(v') \subseteq \mathrm{Stab}_G(\alpha v + \beta v')$ for any $\alpha, \beta \in F$ and $v, v' \in V$, and the definition of smooth.

Two questions we can ask: can we classify all representations of one of these types (admissible/smooth)? Can we even construct them? We must answer at least the first question to do local Langlands.

Fact 1.2.1. *If π is irreducible and smooth (hence admissible), then $\mathrm{End}_G(\pi) = \mathbf{C}$: this is a version of Schur's lemma in this context. It follows that when π is irreducible, then there exists a "central character" $\omega_\pi : F^\times = Z(G) \to \mathbf{C}^\times$ such that $\pi(z) = \omega_\pi(z) \cdot \mathrm{id}_V$ for all $z \in Z(G)$.*

We can also define the contragredient (dual) to an irreducible admissible (π, V). It is denoted $(\pi^\vee, V^\vee)$, where $V^\vee = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})^{\mathrm{sm}}$ is the smooth vectors in the usual dual space, and where

$$\pi^\vee(g)(f) : v \mapsto f(\pi(g^{-1}) \cdot v)$$

Exercise 1.2.1.

(1) Show that $(\pi^\vee, V^\vee)$ is admissible if (π, V) is (hint: show that $(\mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})^{\mathrm{sm}})^K = \mathrm{Hom}_{\mathbf{C}}(V^K, \mathbf{C})$).

(2) Show that duality is an involution (this is not completely trivial: note V are typically infinite dimensional! But the fact that we're taking smooth vectors fixes this problem).

(3) Show that

$$\omega_{\pi^\vee} = (\omega_\pi)^{-1}$$

1.3. **Parabolic Induction.** The hope is that we can understand representations of some nice class of subgroups, and then induce them to G. We now try to do this for parabolic subgroups.

Let $\underline{n} = (n_1, \ldots, n_r)$ denote a partition of n. Then let $P_{\underline{n}}$ denote the usual standard parabolic in GL_n of block upper-triangular matrices with block sizes determined by $\underline{n}$. This has a Levi decomposition $P_{\underline{n}} = M_{\underline{n}} N_{\underline{n}}$: here $M_{\underline{n}}$ is block diagonal matrices with blocks in GL_{n_i}, and $N_{\underline{n}}$ is the corresponding unipotent radical.

Let $\delta_{\underline{n}} : M_{\underline{n}} \to \mathbf{C}^\times$ denote the modulus character, given by

$$m \mapsto |\det(\mathrm{ad}(m| \operatorname{Lie} N_{\underline{n}}))|_F$$

Example 1.3.1. For $G = \mathrm{GL}_2(F)$ and $P_{(1,1)} = B$ the Borel, we get that $M = \{\mathrm{diag}(a, d)\}$ and that $\operatorname{Lie} N$ is the set of matrices which are zero outside of the upper right corner, so $\mathrm{ad}(\mathrm{diag}(a, d))$ acts by the character ad^{-1}, and thus $\delta_{(1,1)}(\mathrm{diag}(a, d)) = |ad^{-1}|_F$.

Let $\mathsf{Rep}_{\mathbf{C}}(-)$ denote the category of smooth representations with $\mathbf{C}$-coefficients.

Definition 1.3.1 (Normalized Parabolic Induction)**.** There is a functor n-Ind : $\mathsf{Rep}_{\mathbf{C}}(M_{\underline{n}}) \to \mathsf{Rep}_{\mathbf{C}}(G)$ taking

$$\pi_{\underline{n}} \mapsto \mathrm{Ind}_{P_{\underline{n}}}^G(\pi_{\underline{n}} \otimes \delta_{\underline{n}}^{1/2})^{\mathrm{sm}},$$

where G acts by pre-composition on the left

Remark 1.3.1. Note the $\mathrm{Ind}_{P_{\underline{n}}}^G(\pi_{\underline{n}} \otimes \delta_{\underline{n}}^{1/2})^{\mathrm{sm}}$ can be described as the set of locally constant functions $f : G \to P_{\underline{n}}$ such that $f(pg) = \delta_{\underline{n}}^{1/2}(p)\pi_{\underline{n}}(p)f(g)$ for all $p \in P_{\underline{n}}$ and $g \in G$.

Remark 1.3.2.

(1) n-Ind is an exact functor, and has a very explicit left and right adjoint in terms of Jacquet modules.

(2) n-Ind preserves admissibility, and the property of being finite length.

(3) So why do we add this $\delta^{1/2}$? Well, it makes n-Ind preserve the unitary-ness of the representation, and it makes n-Ind "Weyl-symmetric".

In summary, we construct representations of G from representations of a Levi as follows: given $\underline{n} = (n_1, \dots, n_r)$, we take $\pi_i \in \mathrm{Irr}(G_{n_i})$ (here $\mathrm{Irr}(-)$ is the set of irreducible smooth representations), and take

$$\text{n-Ind}_{P_{\underline{n}}}^{G}(\pi_1 \boxtimes \cdots \boxtimes \pi_r),$$

a finite length representation of G (note we're implicitly pulling back along $P_{\underline{n}} \to M_{\underline{n}}$).

1.4. Hierarchy of Representations.

Definition 1.4.1. We define the set of **square-integrable representations**

$$\mathrm{Irr}^2(G) = \left\{(\pi, V) \in \mathrm{Irr}(G) : (g \mapsto f(\pi(g)v)) \in L^2(G) \text{ for all } g \in G, v \in V, f \in V^\vee\right\},$$

where $L^(G)$ is the space of square-integrable functions on G, with respect to the usual Haar measure. We also define the set of **supercuspidal representations**

$$\mathrm{Irr}^{\mathrm{sc}}(G) = \{(\pi, V) \in \mathrm{Irr}(G) : (g \mapsto f(\pi(g)v)) \text{ is compactly supported for all } g \in G, v \in V, f \in V^\vee\}.$$

Then there is a hierarchy

$$\mathrm{Irr}(G) \supseteq \mathrm{Irr}^2(G) \supseteq \mathrm{Irr}^{\mathrm{sc}}(G).$$

Fact 1.4.1. *An irreducible admissible (π, V) is supercuspidal if and only if it's not a subquotient of a parabolic induction.*

1.5. Local Hecke Algebras.

Here we want to give the analog of the group algebra $\mathbf{C}[G]$ in the setting of smooth admissible representations.

Fix a Haar measure on G normalized so that $\mathrm{vol}(\mathrm{GL}_n(\mathscr{O}_F)) = 1$.

Definition 1.5.1. We define the **Hecke algebra** $\mathcal{H}(G) := C_c^\infty(G, \mathbf{C})$, the locally constant compactly supported functions under convolution (this is not commutative!). This is an infinite dimensional $\mathbf{C}$-vector space, and has no unit (the unit wants to be the Dirac δ function, but this isn't locally constant). Another point of view is

$$\mathcal{H}(G) = \bigcup_K C_c^\infty(K\backslash G/K)$$

where K runs over the open compact subgroups. With this description if f, f' are K-bi-invariant, then

$$(f \star f')(g) = \sum_{h, h' \in G/K} f(h) f'(h^{-1}g).$$

Definition 1.5.2. A left-$\mathcal{H}(G)$-module is called **smooth** if $\mathcal{H}(G)V = V$ (you need this because there's no unit in $\mathcal{H}(G)$).

Fact 1.5.1. *There is an equivalence of categories $\mathsf{Rep}_{\mathbf{C}}(G) \xrightarrow{\sim} \mathsf{Mod}^{\mathrm{sm}}_{\mathcal{H}(G)}$ (smooth left-$\mathcal{H}(G)$-modules).*

Sketch of Proof. Given (π, V) and $f \in \mathcal{H}(G)$, then

$$\pi(f) \cdot v = \int_G f(g)(\pi(g) \cdot v) dg.$$

What does this mean? Choose K such that f is K-bi-invariant and K stabilizes v. Then

$$\pi(f) \cdot v = \mathrm{vol}(K) \sum_{g \in G/K} f(g) \pi(g) \cdot v.$$

Since f is compactly supported, the sum is finite. One can check that this doesn't depend on the choice of K, and a simple computation shows that this makes V a left-$\mathcal{H}(G)$-module, which is smooth (take $f = 1_K/\operatorname{vol}(K)$). Conversely, given a left-$\mathcal{H}(G)$-module V, define

$$\pi(g)v = \pi(1_{gK}/\operatorname{vol}(K))v$$

for any open compact $K \subseteq G$ (check this is well-defined). One checks that this is smooth and provides an inverse construction. □

Exercise 1.5.1. For K a compact open subgroup of G, this induces a bijection

$$\left\{\pi \in \operatorname{Irr}(G) : \pi^K \neq 0\right\} \cong \operatorname{Irr}(C_c^\infty(K\backslash G/K))$$

Fix $f \in \mathcal{H}(G)$, and note that f is K-bi-invariant for some open compact $K \subseteq G$. Then a computation shows that $\pi(f)V \subseteq V^K$, so if π is admissible, we can define $\operatorname{tr}\pi(f) \in \mathbf{C}$.

Observe that if π is admissible, then $\pi(f)$ has finite rank ($\pi(f)V \subseteq V^K$), so $\operatorname{tr}\pi(f) \in \mathbf{C}$ is defined.

Let's construct some representations using normalized induction.

Example 1.5.1. If $G = \mathrm{GL}_2(F)$, then note that

$$\text{n-Ind}(\delta^{1/2}) = \operatorname{Ind}_B^G(1)^{\mathrm{sm}} = \left\{f : \mathbf{P}^1(F) \to \mathbf{C} \text{ locally constant}\right\}$$

This has the constant functions as a subspace, which we call 1, so we get a short exact sequence

$$0 \to 1 \to \text{n-Ind}(\delta^{-1/2}) \to \mathrm{St} \to 0.$$

We call the quotient St, the **Steinberg representation**, which it turns out is irreducible and square-integrable.

Here's a construction of the generalized Steinberg representation, for arbitrary n. Let $G_n = \mathrm{GL}_n(F)$.

Take $m \in \mathbf{Z}_{\geq 1}$ and $\pi_0 \in \operatorname{Irr}^{\mathrm{sc}}(G_r)$. Then

$$\text{n-Ind}(\pi_0 \boxtimes \pi_0|\det| \boxtimes \cdots \boxtimes \pi_0|\det|^{m-1})$$

has a unique irreducible quotient called $\mathrm{St}_m(\pi_0)$ which lives in $\operatorname{Irr}^2(G_{rm})$.

Fact 1.5.2 ([Zel80], Section 9.3)**.**

$$\operatorname{Irr}^2(G) = \left\{\mathrm{St}_m(\pi_0) : m \mid n, \pi_0 \in \operatorname{Irr}^{\mathrm{sc}}(G_{n/m})\right\}$$

Theorem 1.5.1 ([Zel80], Theorem 6.3)**.** *If $\pi \in \operatorname{Irr}(G)$, then*

$$\pi = \mathrm{St}_{m_1}(\pi_1) \boxplus \cdots \boxplus \mathrm{St}_{m_r}(\pi_r)$$

for unique (m_i, π_i), where $\pi_i \in \operatorname{Irr}(G_{r_i})$ up to permutation, where $\pi_1' \boxplus \cdots \boxplus \pi_r'$ is a certain "distinguished irreducible subquotient" of n-Ind$(\pi_1' \boxtimes \cdots \boxtimes \pi_r')$*, where $\pi_i' \in \operatorname{Irr}^2(G)$.*

So the upshot is that to understand all irreducible representations, you really just need to understand the supercuspidal representations, and you can build the rest of them.

Example 1.5.2. Working through the above definitions, we have the following exact sequence from before:

$$0 \to 1 \to \text{n-Ind}(\delta^{-1/2}) \to \mathrm{St}(|\cdot|^{-1/2}) \to 0$$

Furthermore, $1 = |\cdot|^{-1/2} \boxplus |\cdot|^{1/2}$.

2. Talk II

2.1. **Unramified Hecke Algebra.** Recall $F/\mathbf{Q}_p$ finite extension, and $G = G_n = \mathrm{GL}_n(F)$. We defined the Hecke algebra $\mathcal{H}(G) = C_c^\infty(G, \mathbf{C})$ of locally constant functions on G under convolution. When $K \subseteq G$ is an open compact subgroup, we defined

$$\mathcal{H}(G, K) = C_c^\infty(K\backslash G/K, \mathbf{C}),$$

and we had that

$$\mathrm{Irr}^{\mathrm{ur}}(G) := \left\{\pi \in \mathrm{Irr}\, G : \pi^K \neq 0\right\} \leftrightarrow \sim \mathrm{Irr}(\mathcal{H}(G, K))$$

under the map sending $\pi \mapsto \pi^K$.

Now let's consider $K_0 = \mathrm{GL}_n(\mathscr{O}_F)$. This is a maximal compact subgroup of G.

Definition 2.1.1. If $\pi \in \mathrm{Irr}(G)$ is *unramified* if it has $\pi^{K_0} \neq 0$.

Note for GL_n, this definition doesn't depend on the choice of maximal compact.

When $n = 1$, such π really correspond to unramified characters on the Galois side, under local class field theory. We let

$$\mathcal{H}^{\mathrm{ur}}(G) := \mathcal{H}(G, K_0).$$

Again there's a bijection between the set of irreducible unramified representations of G and $\mathrm{Irr}(\mathcal{H}^{\mathrm{ur}}(G))$, under the map $\pi \mapsto \pi^{K_0}$, so to understand irreducible unramified representations, we can really just study the structure of $\mathcal{H}^{\mathrm{ur}}(G)$. The goal of the Satake isomorphism is to explicitly describe this ring.

2.2. **Satake Isomorphism.** We let $T \subseteq B \subseteq G$, where T is a maximal torus, and B is the standard Borel. In this case, T is the set of diagonal matrices in G, and B is the set of upper triangular matrices in G. In other words, we are in the $\underline{n} = (1, \ldots, 1)$ situation.

We let $\delta : B \twoheadrightarrow T \to \mathbf{R}_{>0}^\times$ where the first map is the natural projection, and the second map is the modulus character

$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto |t_1^{n-1} t_2^{n-3} \cdots t_n^{1-n}|_F.$$

The unramified Hecke algebra of T is by definition

$$\mathcal{H}^{\mathrm{ur}}(T) = C_c^\infty(T(F)/T(\mathscr{O}_F)).$$

But this is relatively easy to understand: note after choosing a uniformizer that there is an obvious (topological, but both sides are discrete) bijection $T(F)/T(\mathscr{O}_F) \xrightarrow{\sim} \mathbf{Z}^n$ and in fact this induces an algebra homomorphism

$$\mathcal{H}^{\mathrm{ur}}(T) \cong \mathbf{C}[t_1^\pm, \ldots, t_n^\pm]$$

One checks easily that convolution of functions is the same as multiplication of polynomials under this identification. Now consider the map

$$\mathcal{H}^{\mathrm{ur}}(G) \xrightarrow{\mathcal{S}} \mathcal{H}^{\mathrm{ur}}(T)$$

defined by

$$f \mapsto \left(t \mapsto \delta^{1/2}(t) \int_N f(tn) dn\right)$$

where dn is the standard Haar measure making $\mathrm{vol}(N) = 1$.

Theorem 2.2.1 (Satake Isomorphism)**.** *The map $\mathcal{S}$ induces an isomorphism*

$$\mathcal{H}^{\mathrm{ur}}(G) \xrightarrow{\sim} \mathbf{C}[t_1^\pm, \ldots, t_n^\pm]^{S_n},$$

where the symmetric group S_n acts by permuting the indices of the x_i.

Idea of Proof. First check that $\mathcal{S}$ is an algebra homomorphism which lands in the S_n-invariants. Then we compare bases: $\mathcal{H}^{\mathrm{ur}}(G)$ has a basis

$$\left\{1_{K_0\,\mathrm{diag}(\omega^{a_1},\ldots,\omega^{a_m})K_0} : a_1 \geq \cdots \geq a_m\right\}$$

(this follows from the Cartan decomposition of G) and $\mathbf{C}[t_1^{\pm},\ldots,t_n^{\pm}]^{S_n}$ has a basis

$$\left\{\sum_{w\in S_n} t_1^{a_{w(1)}}\cdots t_n^{a_{w(n)}} : a_1 \geq \cdots \geq a_m\right\}$$

and one then shows that as you take larger and larger corresponding finite subsets of these bases (under the obvious correspondence), $\mathcal{S}$ remains upper-triangular. ☐

The theorem shows that $\mathcal{H}^{\mathrm{ur}}(G)$ is commutative, so its irreducible modules are one-dimensional. So the map $\mathrm{Irr}^{\mathrm{ur}}(G) \to \mathrm{Irr}(\mathcal{H}^{\mathrm{ur}}(G))$ given by $\pi \mapsto \pi^{K_0}$ lands in 1-dimensional subspaces, with an action of $\mathbf{C}[t_1^{\pm},\ldots,t_n^{\pm}]^{S_n}$. Then if we write

$$p_C(x) = (x-t_1)\cdots(x-t_n) = x^n + C_{n-1}x^{n-1} + \cdots + C_1x + C_0,$$

then the coefficients $C_i \in \mathbf{C}[t_1^{\pm},\ldots,t_n^{\pm}]$ act via nonzero scalars c_i on π^{K_0}, and if we take the roots of the polynomial

$$p_c(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0,$$

we get a set of n nonzero complex numbers which we call the **Satake parameter** $\mathrm{Sat}(\pi)$. Conversely, a set of n nonzero complex numbers determines an action of each C_i, but since $\left\{C_0^{-1}, C_0, \ldots, C_{n-1}\right\}$ generates $\mathbf{C}[t_1^{\pm},\ldots,t_n^{\pm}]^{S_n}$, this uniquely determines an action of $\mathcal{H}^{\mathrm{ur}}(G)$. We have thus proven:

Corollary 2.2.1. *There exists a canonical bijection*

$$\mathrm{Sat} : \mathrm{Irr}^{\mathrm{ur}}(G) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}^{\mathrm{ur}}(G)) \xrightarrow{\sim} (\mathbf{C}^{\times})/S_n$$

You can actually construct an inverse of Sat. Given $\{s_1,\ldots,s_n\} \in (\mathbf{C}^{\times})/S_n$, we can construct an unramified character $\chi_{s_i} : F^{\times} \to \mathbf{C}^{\times}$ taking $a \mapsto s_i^{v(a)}$ (here v is the normalized valuation $F^{\times} \to \mathbf{Z}$). Then

$$\{s_1,\ldots,s_n\} \mapsto \chi_{s_1} \boxplus \cdots \boxplus \chi_{s_n}$$

(the Langlands quotient from Lecture 1) is the inverse.

2.3. **Basic Representation Theory.** Let R be a topological ring (e.g. $\overline{\mathbf{Q}}_\ell$, $\overline{\mathbf{Z}}_\ell$, $\overline{\mathbf{F}}_\ell$). Fix $E/\mathbf{Q}_\ell$ a finite extension with ring of integers $\mathscr{O}_E$ and residue field k_E. We can topologize $\mathrm{GL}_n(R)$ with the subspace topology from $M_n(R) \times R$, as usual.

Definition 2.3.1. Say Γ is a topological group and k is a topological field. Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a continuous representation on some finite dimensional k-vector space V. Then there exists some filtration

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_r = V$$

such that V_i/V_{i-1} is irreducible, and define

$$\rho^{\mathrm{ss}} = \bigoplus_{i=1}^{r} V_i/V_{i-1}.$$

We say that ρ is **semisimple** if $\rho \cong \rho^{\mathrm{ss}}$.

Fact 2.3.1 (Brauer-Nesbitt)**.** *If two continuous representations $\rho_1, \rho_2 : \Gamma \to \mathrm{GL}_n(k)$ have the same characteristic polynomial on a dense subset of Γ, then $\rho_1^{\mathrm{ss}} \cong \rho_2^{\mathrm{ss}}$.*

Remark 2.3.1. If k has characteristic 0, then it's enough to impose that the traces are equal in the Brauer-Nesbitt theorem.

From now on, assume Γ is compact, e.g. a profinite group, in applications.

Fact 2.3.2. *If $\rho : \Gamma \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ is continuous, then its image is contained in $\mathrm{GL}_n(E)$ for some finite $E/\mathbf{Q}_\ell$. In fact, ρ has image in $\mathrm{GL}_n(\mathscr{O}_E)$ up to conjugation, i.e. there exists $g \in \mathrm{GL}_n(E)$ such that $g\rho g^{-1}$ lands in $\mathrm{GL}_n(\mathscr{O}_E)$.*

Idea of Proof. The first part uses the Baire category theorem: see Corollary 5 in [Dic01]. It's enough to show that an $\mathscr{O}_E$-sublattice of E^n is stabilized by $\rho(\Gamma)$. But since Γ is compact, we take Γ to be generated by $\rho(\Gamma)(\mathscr{O}_E^n)$. □

2.4. **Construction mod ℓ.** Let $\rho : \Gamma \to \mathrm{GL}_n(E)$ be a continuous rerpesentation. Conjugate it by $g \in \mathrm{GL}_n(E)$ into $\mathscr{O}_E$ and reduce mod ϖ_E and semisimplify, and then call the resulting thing $\overline{\rho}$.

Claim 2.4.1. *The residual representation $\overline{\rho}$ is independent of g.*

Proof. No matter which g you choose, the characteristic polynomial of $g\rho g^{-1}$ will be same after reducing mod $\mathfrak{m}_E$. Now apply Fact 2.3.1. □

Exercise 2.4.1. If Γ is profinite and $\rho : \Gamma \to \mathrm{GL}_n(k)$ is continuous with k discrete or $\mathbf{C}$ with its complex topology, then the image of ρ is finite (hint: think about the kernel of ρ).

2.5. **Galois Groups.** Let F be a number field and v be a place of F with completion F_v. Fix algebraic closures $\overline{F}, \overline{F_v}$ and $i_v : \overline{F} \hookrightarrow \overline{F_v}$, so we get $\Gamma_v \hookrightarrow \Gamma$ induced by the i_v.

The Γ, Γ_v are profinite, hence compact and totally disconnected. If v is nonarchimedean, then we have

$$0 \to I_v \to \Gamma_v \xrightarrow{\alpha} \mathrm{Gal}(\overline{k_v}/k_v) = \widehat{\mathbf{Z}} \to 0$$

(here we use the geometric Frobenius convention). As usual the Weil group W_{F_v} is $\alpha^{-1}(\mathbf{Z})$, and local class field theory gives us

$$F_v^\times \xrightarrow{\sim} W_{F_v}^{\mathrm{ab}}$$

2.6. **Galois Representations.** Start with the local nonarchimedean case. We have a continuous representation $\rho : \Gamma_v \to \mathrm{GL}_n(k)$.

Definition 2.6.1. We say ρ is **unramified** if $\rho(I_v) = 1$ (in particular if ρ is unramified, then there's a well-defined $\rho(\mathrm{Frob}_v)$). In the global context, if $\Gamma \to \mathrm{GL}_n(k)$ is a continuous representation then say ρ is **unramified at** p if $\rho(I_v) = 1$, which is independent of i_v. If S is a finite set of places, we say that ρ is **unramified outside** S (almost everywhere) if $\rho(I_v) \neq 1$ for $v \notin S$.

Fact 2.6.1. *If X/F is a smooth projective variety, then $H^i_{\mathrm{et}}(X_{\overline{F}}, \overline{\mathbf{Q}}_\ell)$ is unramified almost everywhere.*

If ρ is unramified away from some finite set of places S, then ρ factors as $\rho : \Gamma \twoheadrightarrow \Gamma_S \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$, where $\Gamma_S = \mathrm{Gal}(F_S/F)$, where F_S is the maximal algebraic extension of F unramified outside S. Then we can make sense of the (conjugacy classes of) $\{\mathrm{Frob}_v\}_{v \notin S} \subseteq \Gamma_S$. By the Cebotarev density theorem, these Frobenii are dense, so along with Brauer-Nesbitt, this determines ρ^{ss}.

3. Talk III

3.1. **Weil-Deligne Representations.** We want to turn Galois representations into representations "without the topology". This may seem like a strange thing to do, but think about the fact that smooth representations on the automorphic side have the discrete topology, so this might make it easier for us to see a local Langlands correspondence. Furthermore, to formulate a notion of "compatible system" of Galois representations (for varying $\ell \neq p$, such a reinterpretation is helpful because it eliminates the topology on the coefficients.

Let $F/\mathbf{Q}_p$ be a finite extension, and let k be a topological field of characteristic 0. Recall we had the exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_F & \longrightarrow & W_F & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I_F & \longrightarrow & \Gamma_F & \longrightarrow & \widehat{\mathbf{Z}} & \longrightarrow & 0 \end{array}$$

Local class field theory gives us a map $|\cdot|_F : W_F \to W_F^{\mathrm{ab}} \cong F^\times \xrightarrow{|\cdot|_F} \mathbf{Q}_{>0}^\times$.

Definition 3.1.1. A **Weil-Deligne representation** of W_F is a triple (V, r, N), where V is a finite dimensional k vector space and $r : W_F \to \mathrm{GL}_k(V)$ is such that, such that

(1) $r|_{I_p}$ has open kernel: this essentially should say that r is continuous for the discrete topology on k, and
(2) $N \in \mathrm{End}_k(V)$ is an endomorphism such that $r(w)Nr(w)^{-1} = |w|_F \cdot N$ for all $w \in W_F$ (note $|\cdot|_F$ lands in $\mathbf{Q}$ and k has characteristic 0). In particular, N is automatically nilpotent (hint: think about traces).

Definition 3.1.2. We say that $\sigma = (V, r, N)$ is **Frobenius semisimple** if $r(w)$ is semisimple for all $w \in W_F$. This is equivalent to saying that $r(w)$ is semisimple for some w such that $|w| \neq 1$. Say σ is **semisimple** if it's Frobenius-semisimple and $N = 0$. Say σ is **unramified** if $r(I_F) = 1$ and $N = 0$.

Example 3.1.1. (V, r, N) is Frob-semisimple and unramified if and only if $N = 0$ and $r = \bigoplus_{i \in I} \chi_i$ for some unramified characters of W_F, i.e. a character $\chi_i : W_F \to k^\times$ which factors through the quotient $W_F/I_F \cong \mathbf{Z}$. So basically an n-dimensional Frob-ss unramified Weil-Deligne representation is the same as an element of $(k^\times)^n/S_n$, by sending $r \mapsto \{\chi_i(\mathrm{Frob})\}_{i=1}^n$.

There exists a Frobenius semisimplification $(V, r, N) \mapsto (V, r^{\mathrm{ss}}, N)$ by taking the semisimple part of r.

Example 3.1.2. Consider $r : W_F \to W_F/I_F \cong \mathbf{Z} \to \mathrm{GL}_2(k)$ taking

$$1 \mapsto \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

Then this is not Frobenius semisimple, but on the other hand, r^{ss} takes 1 to $\mathrm{diag}(a, a)$.

There is a hierarchy

$$\mathsf{Rep}_k(\mathrm{WD}_F)^{\mathrm{Frob-ss}} \supseteq \mathrm{Indec}(\mathrm{WD}_F)^{\mathrm{Frob-ss}} \supseteq \mathrm{Irr}(\mathrm{WD}_F)$$

If the definition of $\mathrm{Indec}(\mathrm{WD}_F)^{\mathrm{Frob-ss}}$ seems confusing, remember that these are supposed to be indecomposable as *Weil-Deligne*-representations, but that the actually underlying representation of the Weil group is always semisimple: if N doesn't vanish, then you may not be able to decompose the Weil-Deligne representation, but ρ might not be irreducible.

Under local Langlands, these are supposed to match up with the hierarchy given on the automorphic side:

$$\mathrm{Irr}(G) \supseteq \mathrm{Irr}^2(G) \supseteq \mathrm{Irr}^{\mathrm{sc}}(G)$$

Furthermore, just like on the automorphic side, we will show that you can build all of $\mathsf{Rep}_k(\mathrm{WD}_F)^{\mathrm{Frob-ss}}$ out of $\mathrm{Irr}(\mathrm{WD}_F)$, and the construction process looks very similar. In fact, we leave this as a multi-part exercise.

Remark 3.1.1. If σ is an irreducible Weil-Deligne representation, then $N = 0$: note N is nilpotent, so if it were nonzero, then it would map V onto a proper nonzero subrepresentation of V.

Let $\sigma = (V, r, N)$ be an irreducible Weil-Deligne representation (so in particular $N = 0$). Now we define, for $m \geq 1$,

$$\mathrm{Sp}_m(\sigma) = (V^{\oplus m}, r \oplus r|\cdot|_F \oplus \cdots \oplus r|\cdot|_F^{m-i}, N_m),$$

where $N_m(v_1, \ldots, v_m) = (0, v_1, \ldots, v_{m-1})$.

Exercise 3.1.1.

(1) Show that $\mathrm{Sp}_m(\sigma)$ is a Weil-Deligne representation.

(2) Show that there is a bijection between $\mathrm{Indec}(\mathrm{WD}_F)^{\mathrm{Frob-ss}}$ and $\{(m, \sigma_0) : m \geq 1, \sigma_0 \in \mathrm{Irr}(\mathrm{WD}_F)\}$, taking $(m, \sigma_0) \mapsto \mathrm{Sp}_m(\sigma_0)$.

(3) Finally, show that there is a bijection between $\mathsf{Rep}(\mathrm{WD}_F)^{\mathrm{Frob-ss}}$ and

$$\{(m, \sigma_i)\}\ /\text{permutation}$$

where $(m, \sigma_i) \mapsto \bigoplus_i \mathrm{Sp}_{m_i}(\sigma_i)$.

Hint for (2) and (3): decompose ρ into irreducible factors, and think about how N has to act on each factor.

3.2. **Local Galois Representations vs Weil-Deligne Representations.** Again let $F/\mathbf{Q}_p$ be a finite extension. Then there are two cases: $\ell \neq p$ and $\ell = p$. Now let's use $k = \overline{\mathbf{Q}}_\ell$-coefficients.

Proposition 3.2.1. *In the $\ell \neq p$ case, there is a fully faithful functor*

$$\mathsf{Rep}_{\overline{\mathbf{Q}}_\ell}(\Gamma_F) \to \mathsf{Rep}_{\overline{\mathbf{Q}}_\ell}(\mathrm{WD}_F)$$

taking $(V, \rho) \mapsto (V, r, n)$.

A special case of this is when it happens that $\rho(I_F)$ is finite: then $r = \rho|_{W_F}$ and $N = 0$. In general, N remembers the infinite unipotent action of the ℓ-part of tame inertia.

One Word Proof. Grothendieck's ℓ-adic monodromy theorem. □

Note however, that this is not an equivalence of categories.

In the $\ell = p$ case, something deeper is happening. In this case, we need p-adic Hodge theory, and we'll just say something vague about this for now:

Proposition 3.2.2. *There is a functor (not fully faithful) $\mathsf{Rep}^{\mathrm{dR}}_{\overline{\mathbf{Q}}_p}(\Gamma_F) \to \mathsf{Rep}_{\overline{\mathbf{Q}}_p}(\mathrm{WD}_F)$ produced by p-adic Hodge theory.*

3.3. **Local Langlands for** GL_n**.** First we explain the $n=1$ case. An admissible smooth representation of $F^\times$ is just a character $F^\times \to \mathbf{C}^\times$ which vanishes on $1=\mathfrak{m}_F^r$ for some $r>0$, and a 1-dimensional Weil-Deligne representation is just a map $W_F^{\mathrm{ab}} \to \mathbf{C}^\times$ whose restriction to I_F^{ab} factors through a finite quotient. But local class field theory gives us an isomorphism $F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ which takes $\mathscr{O}_F^\times$ to I_F^{ab}, so we really have the same objects on either side.

Now we can state local Langlands in general.

Theorem 3.3.1 (Local Langlands: Harris-Taylor, Henniart, Scholze)**.** *There is a unique bijection*

$$\mathrm{Irr}_{\mathbf{C}}(\mathrm{GL}_n(F)) \xrightarrow{\mathrm{LL}_n} \mathsf{Rep}_{\mathbf{C}}^{n-\dim}(\mathrm{WD}_F)^{\mathrm{Frob-ss}}$$

such that

(1) *When $n=1$, LL is given by local class field theory, as described above.*

(2) *If π corresponds to ρ, then ω_π corresponds to* $\det(\sigma)$, *$\pi^\vee$ corresponds to $\sigma^\vee$, and for any smooth $\chi: F^\times \to \mathbf{C}$, one has that $\pi \otimes (\chi \circ \det)$ corresponds to $\sigma \otimes \chi$.*

(3) $\mathrm{Irr}^2(\mathrm{GL}_n(F))$ *corresponds to* $\mathrm{Indec}(\mathrm{WD}_F)^{\mathrm{Frob-ss}}$, $\mathrm{Irr}^{\mathrm{sc}}(\mathrm{GL}_n(F))$ *corresponds to* $\mathrm{Irr}(\mathrm{WD}_F)$, *and unramified things match up, i.e.* $\boxplus_i \mathrm{St}_{m_i}(\pi_i)$ *matches* $\bigoplus_i \mathrm{Sp}_{m_i}(\sigma_i)$ *if each π_i matches with σ_i.*

(4) *If π_i corresponds to σ_i for $i=1,2$ then*

$$(L/\epsilon)(s,\pi_1\times\pi_2) = (L/\epsilon)(s,\sigma_1\otimes\sigma_2)$$

Remark 3.3.1. There exist operations you can do on the Weil-Deligne side like base change to a finite extension, induction from a finite extension, tensor product.

Henniart proved (1) and (4) for ϵ-factors are enough to pin down the local Langlands correspondence uniquely.

4. Talk IV

4.1. **Automorphic Representations.** Now we're in the global setting, so from now on F will be a finite extension of $\mathbf{Q}$. Let $G=\mathrm{GL}_{n,F}$ now be the *algebraic* group defined over F. We consider the center $Z=\mathrm{GL}_1=Z(\mathrm{GL}_n)$, and $\mathbf{A}_F=\mathbf{A}\otimes_{\mathbf{Q}} F$, and we fix a continuous **central character**

$$\omega : Z(\mathbf{A}_F)/Z(F) = \mathbf{A}_F^\times/F^\times \to \mathbf{C}^\times,$$

Consider $L^2(G(F)\backslash G(\mathbf{A}_F),\omega)$: these are square integrable functions (for the Haar measure on $G(\mathbf{A}_F)$) such that $f(zg)=\omega(z)f(g)$ for all $z\in Z(\mathbf{A}_F)$ and $g\in G(\mathbf{A}_F)$. If $|\omega|=1$ then $f\in L^2$ means

$$\int_{G(F)Z(F)\backslash G(\mathbf{A}_F)} |f(g)|^2 dg < \infty$$

Since ω is unitary, then the quotient by the center in the integral is fine. On the other hand, if ω is not unitary, you need to scale this somehow for it to be well-defined.

Definition 4.1.1. Say $f\in L^2$ is **cuspidal** if

$$\int_{N_{\underline{n}}(F)\backslash N_{\underline{n}}(\mathbf{A}_F)} f(ng)dn = 0$$

for all nontrivial partitions $\underline{n}$ for all $g\in G(\mathbf{A}_F)$. We call $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A}_F),\omega)$ is the Hilbert space of **cuspidal functions** for the usual inner product. A fancier way of saying this is that "for each parabolic, the constant term map is zero on f".

This may look strange, but it captures the notion of "vanishing at the cusps" from modular forms for the case $G=\mathrm{GL}_{2,\mathbf{Q}}$.

Definition 4.1.2. An irreducible representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ is **cuspidal automorphic** if π is a closed $G(\mathbf{A}_F)$-submodule (for the topology coming from the Hilbert space structure) of $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A}_F), \omega)$.

4.2. **Flath Decomposition.** By definition, $G(\mathbf{A}_F) = \prod'_v G(F_v)$: this is the restricted product at all places, even archimedean ones (for a general reductive group, you really need to use an integral model in order to define a restricted product, i.e. you need some way of picking a maximal compact at all places for the definition to make sense).

If π is cuspidal automorphic (or irreducible admissible, which we haven't defined yet) decomposes as

$$\pi = \widehat{\bigotimes}' \pi_v,$$

where now π_v is an irreducible representation of $G(F_v)$:

(1) If v is a finite place, π_v^{sm} is dense in π_v, and π_v^{sm} is irreducible smooth. But for the purposes of our lecture, let's just assume π_v is smooth: we don't actually lose anything by doing this because if you take the unitary completion you get the actual π_v.

(2) If v is an infinite place, then π_v is a representation of $G(\mathbf{R})$ or $G(\mathbf{C})$, and we get an analogous dense $\pi_v^{\mathrm{sm}} \subseteq \pi_v$, which consists of smooth K_v-finite vectors, where K_v is the maximal compact in $G(F_v)$, except this time π_v^{sm} is only a $(\mathfrak{g} = \mathrm{Lie}\, G(F_v), K_v)$-module, so it's not actually defined over the whole $G(F_v)$.

Fact 4.2.1 (Strong multiplicity one for GL_n)**.** *If π, π' are cuspidal automorphic representations and $\pi_v \cong \pi'_v$ for all but finitely many v, then $\pi = \pi'$ as subrepresentations of L^2_{cusp}.*

Note this means that there is really at most *one subrepresentation* of $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A}_F), \omega)$ with fixed local components at all but finitely many places.

4.3. **Infinitesimal Characters.** Say $v \nmid \infty$ is an infinite place. Then we attach the *infinitesimal character*

$$\mathrm{inf} : \mathrm{Irr}(\mathrm{GL}_n(F_v)) \to (\mathbf{C}^n/S_n)^{[F_v:\mathbf{R}]}$$

Roughly the way this works: π gets an action of $Z(U(\mathfrak{g}_v))$, the center of the universal enveloping algebra of $\mathfrak{g}_v = \mathrm{Lie}\, G(F_v)$, and it acts by a character. But the Harish-Chandra isomorphism tells us that

$$Z(U(\mathfrak{g}_v)) \cong ((\mathbf{C}[t_1, \dots, t_n])^{S_n})^{\otimes [F_v:\mathbf{R}]}$$

Then $\mathrm{inf}(\pi)$ should be thought of a Satake parameter in the archimedean case.

Definition 4.3.1 (Buzzard-Gee)**.** A cuspidal automorphic representation π of $G(\mathbf{A}_F)$ is *L-algebraic* if $\mathrm{inf}(\pi) \in (\mathbf{Z}^n/S_n)^{[F_v:\mathbf{R}]}$, and *$C$-algebraic* if $\mathrm{inf}(\pi) \in ((\mathbf{Z} + \frac{n-1}{2})^n/S_n)^{[F_v:\mathbf{R}]}$.

Exercise 4.3.1. If $G = \mathrm{GL}_{2,\mathbf{Q}}$, then a classical cusp form of weight $k \geq 2$ gives rise to a cuspidal automorphic representation π_f. Our normalization is $\mathrm{inf}(\pi_f) = (k - \frac{3}{2}, -\frac{1}{2})$. Then $|\det|_{\mathbf{A}_F} : \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F) \to \mathbf{R}^\times_{>0}$, so

$$\pi_f \otimes |\det|^a \xrightarrow{\mathrm{inf}} \left(\frac{k-1}{2} - a, -\frac{1}{2} - a\right).$$

4.4. **Global Langlands Correspondence for GL_n.** This is based on work of Langlands himself, but also work of Clozel, Fontaine-Mazur. Fix a prime ℓ and $\iota : \overline{\mathbf{Q}}_\ell \xrightarrow{\sim} \mathbf{C}$.

Conjecture 4.4.1. *There exists a unique bijection*

$$\left\{ \begin{array}{c} \textit{cuspidal automorphic } C\textit{-algebraic} \\ \textit{representations } \pi \textit{ of } \mathrm{GL}_n(\mathbf{A}_F) \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \textit{irreducible continuous representations} \\ \rho : \Gamma_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell) \\ \textit{unramified almost everywhere, such that} \\ \rho_v \textit{ is de Rham for all } v \nmid \ell \end{array} \right\}$$

such that $\pi \leftrightarrow \rho$ if and only if $\pi_v \leftrightarrow \iota\,\mathrm{WD}(\pi_v)^{\mathrm{Frob-ss}} \otimes |\cdot|^{\frac{1-n}{2}}$ for all $v \nmid \infty$ under the local Langlands correspondence.

In the above theorem cuspidal should corresponds to irreducible and C-algebraic should corresponds to de Rham.

Note that compatibility of the system of Galois representations is implicit in the statement: this suggests that these should really come from something motivic, so maybe it should be the ℓ-adic realization of some kind of motive underlying the correspondence: see the Fontaine-Mazur conjecture for more details on this.

Remark 4.4.1. Here is a summary of some of the results known about this. The automorphic to Galois direction is almost always done using the ℓ-adic cohomology of Shimura varieties, which is where you find Galois representations that should be compatible with your original automorphic representation. In the other direction, the Taylor-Wiles-Kisin or Calegari-Geraghty methods are the two main methods people use to prove automorphy lifting theorems.

The automorphic to Galois direction is done if F is CM or totally real, and π is *regular*, which means that $\inf(\pi)$ consists of n distinct numbers: this is due to Harris-Lan-Taylor-Thorne, and by Scholze using a different method.

Now we will state our target theorem for this course. Let ℓ and ι be as before, but now $G = \mathrm{GL}_{2,\mathbf{Q}}$ and π is a cuspidal, automorphic, regular, and C-algebraic representation, unramified outside some finite set of places S: this is the same as saying that π comes from a cuspidal eigenform of weight $k \geq 2$.

Theorem 4.4.1 (Eichler-Shimura, Deligne)**.** *There exists a Galois representation*

$$\rho_\pi : \Gamma_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

such that local-global compatibility holds at all places $v \notin S \cup \{\ell\}$. Explicitly, this means that $p^{1/2}\,\mathrm{Sat}(\pi)$ is the set of eigenvalues of $\rho(\mathrm{Frob}_v)$ via ι.

Remark 4.4.2. We will follow the Langlands-Kottwitz method. Then local-global compatibility at S for $v \neq \ell$ is done by Carayol, and at $v = \ell$ was done by Saito.

Idea of Proof. Realize this Galois representation as the π-part in the cohomology of modular curves. □

5. Talk V

References for this talk are [Tay04], [Kud94], [Wed08], [Sch11], [Kot92].

Let ℓ be a prime and $\iota : \overline{\mathbf{Q}}_\ell \to \mathbf{C}$ be an isomorphism. Let π be a cuspidal automorphic regular C-algebraic representation of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ unramified outside a finite set of places S (recall these come from cusp forms of weight $k \geq 2$). Then our goal is to prove the following theorem:

Theorem 5.0.1. *There exists a unique representation*

$$\rho_\pi : \Gamma \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

unramified outside $S \cup \{\ell\}$ such that local-global compatibility holds, i.e. the elements of $p^{1/2}\,\mathrm{Sat}(\pi_p)$ are the eigenvalues of $\rho_\pi(\mathrm{Frob}_p)$ (via ι) for all $p \notin S \cup \{\ell\}$.

We will prove this via the cohomology of modular curves.

5.1. **Modular Curves.** Fix $N \geq 5$.

Definition 5.1.1. The modular curve (of full level N) M_N is the scheme defined over $\mathbf{Z}[1/N]$ representing the functor $\mathsf{Sch}_{\mathbf{Z}[1/N]} \to \mathsf{Set}$ given by

$$S \mapsto \left\{(E/S, \alpha_N) : E \text{ elliptic curve over } S \text{ and } \alpha_N : (\mathbf{Z}/N\mathbf{Z})^2_S \xrightarrow{\sim} E[N]\right\} / \sim$$

Remark 5.1.1. Evaluating at $\mathbf{C}$-points, we have as usual

$$\mathcal{M}_N(\mathbf{C}) \cong \bigsqcup \Gamma(N)\backslash\mathcal{H}$$

where $\Gamma(N)$ is the usual subgroup $\ker(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}))$ and $\mathcal{H}$ is the complex upper half plane.

If $N|N'$, then there is a map $\mathcal{M}_{N'} \to \mathcal{M}_N$ taking $(E, \alpha_{N'}) \mapsto (E, \alpha_{N'}|_{(\mathbf{Z}/N\mathbf{Z})^2})$ so we can take the limit

$$\mathcal{M} = \varprojlim_N \mathcal{M}_N.$$

This gets a Hecke action of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^\infty)$ from the Hecke action at each level N (the transition maps are Hecke-equivariant). We have a universal elliptic curve $\mathcal{E}_N^{\mathrm{univ}} \xrightarrow{p} \mathcal{M}_N$ living over the moduli problem, and we let

$$\mathcal{L}_{N,k} := \mathrm{Sym}^{k-2}(R^1 p_* \mathcal{E}_N^{\mathrm{univ}}).$$

(recall that $\inf(\pi_\infty) = (k - \frac{3}{2}, -\frac{1}{2})$. In fact we will assume for simplicity that $k = 2$ so that $\mathcal{L}_{N,k} = \overline{\mathbf{Q}}_\ell$. These live over each $\mathcal{M}_N$, and we look at the compactly supported étale cohomology

$$H^1_{c,k}(\mathcal{M}) := \varinjlim_N H^1_c(\mathcal{M}_{N,\overline{\mathbf{Q}}}, \mathcal{L}_{N,k}).$$

The transition maps commute with the Hecke action, and we have a Galois action by construction of étale cohomology, so we get an action of $\Gamma_{\mathbf{Q}} \times \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^\infty)$.

Now we are in a nice setup, because now we have both a Hecke and a Galois action. The way this should work then is that

$$\rho_\pi := \mathrm{Hom}_{G(\mathbf{A}_{\mathbf{Q}}^\infty)}((\pi^\infty)^\vee, H^1_{c,k}(\mathcal{M})).$$

In other words, the Galois representation comes from the piece of cohomology which has the same Hecke eigensystem as our original cuspidal automorphic representation π. Note one can show that $\dim \rho_\pi = 2$ via Eichler-Shimura or a "Matsushima-type formula".

But our main problem is to prove local-global compatibility: we need to relate π_p with $\rho_\pi(\mathrm{Frob}_p)$. Consider the following variant

$$\mathcal{M}^p := \varprojlim_{(N,p)=1} \mathcal{M}_N,$$

which is now defined over $\mathbf{Z}_{(p)}$ so that we can look mod p. Let

$$H^1_{c,k}(\mathcal{M}^p) := \varinjlim_{(N,p)=1} H^1_c(\mathcal{M}_{N,\overline{\mathbf{Q}}}, \mathcal{L}_{N,k}).$$

Then one can still check that

$$\rho_\pi \cong \mathrm{Hom}_{G(\mathbf{A}_{\mathbf{Q}}^{\infty,p})}((\pi^{\infty,p})^\vee, H^1_{c,k}(\mathcal{M}^p))$$

To see this, use the fact that $\dim \pi_p^{G(\mathbf{Z}_p)} = 1$ along with strong multiplicity one for GL_2 (note that the $G(\mathbf{Z}_p)$-invariants of $H^1_{c,k}(\mathcal{M})$ gives you $H^1_{c,k}(\mathcal{M}^p)$).

Now the new main problem is to "compute" the action of $\Gamma_{\mathbf{Q}} \times \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$ on $H^1_{c,k}(\mathcal{M}^p)$ at unramified primes, but since the action is unramified at p, we are really studying the action of $\langle \mathrm{Frob}_p \rangle \times \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$.

5.2. **Outline of the argument.** Now we give an outline of the main argument.

(1) Describe the action of $\langle \mathrm{Frob}_p \rangle \times \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$ on $\mathcal{M}^p(\overline{\mathbf{F}}_p)$ in terms of "linear algebraic" data.
(2) Obtain a trace formula computing the action of $\langle \mathrm{Frob}_p \rangle \times \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$ on $H^*_{c,k}(\mathcal{M}^p)$ by applying a suitable fixed point formula.
(3) Massage the formula and make it look like the Selberg trace formula.
(4) Compare the outcome of Step 3 with the Selberg trace formula. Basically, they look really similar, but the difference is that the Selberg trace formula knows nothing about the action of Frob_p, but it does know about the full $G(\mathbf{A}_{\mathbf{Q}})$.
(5) If all goes well, we get an equality, for $j \geq 1$

$$\mathrm{tr}(f_p^{(j)} | \pi_p) = \mathrm{tr}(\mathrm{Frob}_p^j | \rho_\pi)$$

where $f_p^{(j)} \in \mathcal{H}^{\mathrm{ur}}(G(\mathbf{Q}_p))$ are some explicit Hecke operators. Morally what's going on is that we need to relate the actions of Frob_p and $G(\mathbf{Q}_p)$ on both sides of the trace formula, as mentioned in the previous step.

So there are too many things to go through: for us, we'll describe step 1.

5.3. **$\overline{\mathbf{F}}_p$-points of $\mathcal{M}^p$.** This is in the spirit of Langlands-Rapoport for modular curves.

Let $\widehat{\mathbf{Z}}^p := \varprojlim_{(N,p)=1} (\mathbf{Z}/N\mathbf{Z})$ and we let

$$\mathcal{E}^0 := \left\{ E \xrightarrow{\text{elliptic curve}} \overline{\mathbf{F}}_p \right\} / \text{isogeny}$$

Then given $E \to \overline{\mathbf{F}}_p$, let

$$T^p(E) := \varprojlim_{(N,p)=1} E[N]$$

and let

$$V^p(E) := T^p(E) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Note $T^p(E)$ is a free $\widehat{\mathbf{Z}}^p$-module of rank 2, and is a $\widehat{\mathbf{Z}}^p$-lattice in $V^p(E)$. We also need $\check{T}_p(E)$, which is the covariant Dieudonné module of $E[p^\infty]$. This is free of rank 2 over $\check{\mathbf{Z}}_p := W(\overline{\mathbf{F}}_p)$ and there are F^{-1}, V^{-1} actions such that $F^{-1}V^{-1} = V^{-1}F^{-1} = p$. Then $T_p'(E) \subseteq V_p'(E)$ is an F^{-1}, V^{-1}-invariant lattice. The Frobenius on $\check{\mathbf{Q}}_p = W(\overline{\mathbf{F}}_p) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is σ-semilinear.

So away from p we care about $T^p(E)$ and $\check{T}_p(E)$. Let's describe

$$\mathcal{M}^p(\overline{\mathbf{F}}_p) = \varprojlim_{(N,p)=1} \mathcal{M}_N(\overline{\mathbf{F}}_p) = \left\{ (E \xrightarrow{\text{elliptic curve}} \overline{\mathbf{F}}_p, \alpha : (\widehat{\mathbf{Z}}^p)^2 \xrightarrow{\sim} T^p(E)) \right\} / \sim .$$

But now we partition this into isogeny classes:

$$\bigsqcup_{E_0 \in \mathcal{E}^0} \left\{ (E, \alpha) \in \mathcal{M}^p(\overline{\mathbf{F}}_p) : \text{ there is an isogeny } E \to E_0 \right\} =$$

$$\bigsqcup_{E_0 \in \mathcal{E}^0} \{ (L^p, \phi^p, L_p) \} / (\sim, \mathrm{Aut}^0(E_0))$$

where $L^p \subseteq V^p E_0$ is a $\widehat{\mathbf{Z}}^p$-lattice, $\phi^p : (\widehat{\mathbf{Z}}^p)^2 \xrightarrow{\sim} L^p$ is a trivialization, and $L_p \subseteq \check{V}_p E_0$ is a F^{-1}, V^{-1}-invariant $\check{\mathbf{Z}}_p$-lattice.

How do we get this characterization? Given (E, α) and an isogeny $f : E \to E_0$ take $L^p = f(T^p(E))$, $\varphi^p : (\widehat{\mathbf{Z}}^p)^2 \xrightarrow{\alpha} T^p(E) \xrightarrow{f} f(T_p(E)) = L^p$. Then $L_p = f(\check{T}_p(E))$. Then we quotient by $\mathrm{Aut}^0(E_0)$ to forget the choice of f.

Exercise 5.3.1. Check that this really gives you a bijection.

So we let

$$X^p(E_0) := \{(L^p, \phi^p)\}$$

and this has an action of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$ (if $g \in G(\widehat{\mathbf{Z}}^p)$, then the action is given by $\phi^p \mapsto \phi^p \cdot g$). At p, we let

$$X_p(E_0) := \{L_p\}$$

parametrizing F^{-1}, V^{-1}-invariant lattices in $\check{V}_p(E_0)$, which has a Frob_p-action by F. Write

$$I(E) = \mathrm{Aut}^0(E_0)$$

the self-quasi-isogenies. Then the summary is that

$$\mathcal{M}^p(\overline{\mathbf{F}}_p) = \bigsqcup_{E_0 \in \mathcal{E}^0} I(E_0)\backslash X^p(E_0) \times X_p(E_0)$$

and this map is an $\langle \mathrm{Frob}_p \rangle \times \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$-equivariant identification.

Now let's get a more group-theoretic description of $X^p(E_0)$ and $X_p(E_0)$.

Exercise 5.3.2. $X^p(E_0)$ is a $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$-torsor.

At p, fix $\varphi_{0,p} : \widehat{\check{\mathbf{Z}}_p}^2 \xrightarrow{\sim} \check{T}_p E_0$. Then $\check{X}_p(E_0)$ consists of $L_p \subseteq V_p E_0 \subseteq \check{\mathbf{Q}}_p^2$ which are F^{-1}, V^{-1}-invariant. Equivalently $L_p \subseteq F(L_p) \subseteq p^{-1} L_p$, and the $\overline{\mathbf{F}}_p$-dimension of the quotient is 2 ((1,1)).

Writing $F = b\sigma$ for $b \in G(\check{\mathbf{Q}}_p)$ and $L_p = g_p(\check{\mathbf{Z}}_p^2)$ for $g \in G(\check{\mathbf{Q}}_p)/G(\check{\mathbf{Z}}_p)$, then the condition becomes

$$g_p^{-1} b\sigma(g_p) \in \check{K}^p \,\mathrm{diag}(1, p^{-1}) \check{K}_p.$$

with $\check{K}_p = G(\check{\mathbf{Z}}_p)$.

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