# Homework 1

Introduction to Topology, Spring 2023

*Due January 30, 2023 at 11:59pm*

This week's homework is meant to be a review of preliminaries, to make sure that you are familiar with some set theory and related material.

1. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is:
   (a) injective but not surjective
   (b) surjective but not injective
   (c) neither injective nor surjective
   (d) bijective

   Your answer should be a precisely defined function on all of $\mathbb{R}$, not just a picture. In parts (a) and (b) and (c), find nonempty subsets of the domain and the codomain where the function is bijective.

2. Prove *de Morgan's laws*: in other words, show that if $X \subset Z$ and $Y \subset Z$ then
$$(X \cap Y)^c = X^c \cup Y^c$$
   and
$$(X \cup Y)^c = X^c \cap Y^c$$
   Also show that
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

3. If $X, Y, Z$ are three sets, construct natural maps
$$\mathrm{Fun}(X, Y \times Z) \to \mathrm{Fun}(X, Y) \times \mathrm{Fun}(X, Z)$$
$$\mathrm{Fun}(X \sqcup Y, Z) \to \mathrm{Fun}(X, Z) \times \mathrm{Fun}(Y, Z)$$
   and show that they are bijections.

4. If $X$ is a finite set, show that $|X| < |\mathcal{P}(X)|$ *without computing* $|\mathcal{P}(X)|$ *in terms of* $|X|$. Then compute the size of $|\mathcal{P}(X)|$. [1]

5. Let $f : A \to B$ and $g : B \to C$ be functions and let $A' \subset A$ and $B' \subset B$ and $C' \subset C$ be subsets.
   (a) Show that $A' \subset f^{-1}(f(A'))$ and that equality holds if $f$ is injective.
   (b) Show that $f(f^{-1}(B')) \subset B'$ and that equality holds if $f$ is surjective.
   (c) Show that $(g \circ f)^{-1}(C') = f^{-1}(g^{-1}(C'))$.

6. Let $f : X \to Y$ be a function.
   (a) Define a relation on $X$ by setting
$$x_0 \sim x_1 \text{ if } f(x_0) = f(x_1)$$
   Show that $\sim$ defines an equivalence relation on $X$.

[1] Bonus question, not for credit: if $X$ is infinite and countable, is $\mathcal{P}(X)$ countable or uncountable? Can you prove it?

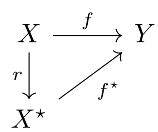
(b) Let $X^\star$ denote the set of equivalence classes under $\sim$ and let $r : X \to X^\star$ denote the map sending $x \in X$ to its equivalence class under $\sim$. Show that $r$ is surjective.

(c) Show that there is an injective map

$$f^\star : X^\star \to Y$$

such that $f^\star \circ r = f$. If $f$ is surjective, then show that $f^\star$ is bijective.

We can express the fact that $f^\star \circ r = f$ by drawing a *diagram*




and saying that it *commutes*; this means that if you start at $X$ and go down and then up-right, that's the same as going right.