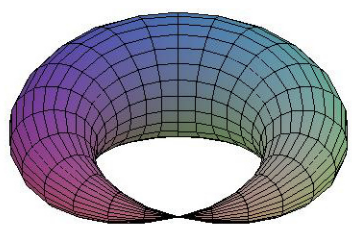
Homework 12

Introduction to Topology, Spring 2023

Due April 24, 2023 at 11:59pm

1. If (X, x) and (Y, y) are two pointed topological spaces (in other words, $x \in X$ and $y \in Y$ are some choice of point) then the *wedge sum* $X \vee Y$ is the quotient of $X \sqcup Y$ where you identify x and y.

   Explain why $S^2 \vee S^1$ (with respect to any choice of points) is homotopy equivalent to the pinched torus (or croissant). This is the shape that you get by taking a torus and then contracting one of its cross-sectional circles to a point.

2. Explain why the one-point compactification of a Möbius strip is homeomorphic to the real projective plane (in other words, the topological space you get by taking the 2-sphere in side $\mathbb{R}^3$ and quotienting by the relation $x \sim -x$).

   Using the fact that

   $$\pi_1(S^2 \vee S^1, *) = \pi_1(S^2, *) * \pi_1(S^1, *) = \pi_1(S^1, *) = \mathbb{Z}$$

   (which follows from the Seifert-van Kampen theorem) and that the fundamental group of the real projective plane is cyclic of order 2 *we can conclude that the open cylinder* $S^1 \times (0, 1)$ *and the "open Möbius strip" (defined by taking* $[0, 1] \times (0, 1)$ *and gluing opposite edges with a twist) are not homeomorphic.*

3. If $E = \{(x, y) \in \mathbb{R}^2 : |y| \leq |x|\}$ and $B = \mathbb{R}$, show that the map $E \to B$ taking $(x, y) \mapsto x$ is a Serre fibration.