Homework 2

Introduction to Topology, Spring 2023

Due February 6, 2023 at 11:59pm

This week's exercises are meant to get you to play around with/think about metric spaces.

1. Show that if a metric space M has the discrete metric, then every closed ball can be written as an open ball, and every open ball can be written as a closed ball.

2. If (M_1, d_1) and (M_2, d_2) are two metric spaces, show that the following functions on $M_1 \times M_2$ are metrics:

   (a) $d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$

   (b) $d((x_1, x_2), (y_1, y_2)) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$

   (c) $d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$.

   Now suppose M_1 and M_2 are both $\mathbb{R}$ with the Euclidean metric. For each metric d above, draw $B_1((0, 0))$, the unit ball of radius 1 around zero.

3. Fix a set M, and fix two metrics d_1 and d_2. Let $B_r(x, d_i)$ denote the open ball of radius $r > 0$ around the point $x \in M$ with respect to d_i, for $i = 1, 2$.

   We say that d_1 and d_2 are *equivalent* if for every $x \in M$ and every $r > 0$ there exists $r', r'' > 0$ such that

   $$B_{r'}(x; d_1) \subseteq B_r(x; d_2) \text{ and } B_{r''}(x; d_2) \subseteq B_r(x; d_1).$$

   In other words, the open balls for d_1 and d_2 "nest".

   (a) Let d denote the usual Euclidean metric on $\mathbb{R}^n$. Let

   $$d_{\max}((x_i)_{i=1,\ldots,n}, (y_i)_{i=1,\ldots,n}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

   Prove that d and $d_{\max}$ are equivalent.

   (b) Show that if M is a finite set, then any two metrics on M are equivalent.

   (c) Find an example of a set M and two metrics on it which are *not* equivalent (and prove it!)

4. If (M, d) is a metric space and $x_1, x_2, \ldots$ is a sequence which converges to both $x \in M$ and $y \in M$, show (using the definition of convergence given in class) that $x = y$.

5. Fix (M_1, d_1) and (M_2, d_2) two metric spaces.

   (a) Show that the definition of a continuous map presented in class (also see Definition 2.3.1 in the notes) is equivalent[1] to the following definition: $f : M_1 \to M_2$ is continuous if whenever $x_1, x_2, \ldots$ converges to $x \in M$, the sequence $f(x_1), f(x_2), \ldots$ converges to $f(x) \in M$.

   (b) Now suppose $M_2 = \mathbb{R}$ and d_2 is the Euclidean metric. Show that if $f, g : M_1 \to \mathbb{R}$ are two continuous functions then $f + g$ and $f * g$ are continuous functions (here we add and multiply pointwise).

[1] We will later see that this equivalence does *not* hold in general topological spaces.

6. Let $C([a,b],\mathbb{R})$ denote the space of continuous functions $f : [a,b] \to \mathbb{R}$. Here $[a,b]$ has the Euclidean metric.

   (a) Show that $(C([a,b],\mathbb{R}), d_{\max})$ is a metric space, where

   $$d_{\max}(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|.$$

   (you can use the fact that a continuous function on a closed interval attains its maximum and minimum)

   (b) If $x \in [a,b]$, show that the *evaluation map at* x

   $$\begin{aligned} \mathrm{ev}_x : C([a,b],\mathbb{R}) &\to \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

   is continuous with respect to the metric given in part (a).

   (c) Show that

   $$d(f,g) = \int_a^b |f(x) - g(x)| \mathrm{d}x$$

   is also a metric on $C([a,b],\mathbb{R})$.

   (d) If $x \in [a,b]$, show by example that the evaluation map at x

   $$\begin{aligned} \mathrm{ev}_x : C([a,b],\mathbb{R}) &\to \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

   is *not* continuous with respect to the metric given in part (c).