Homework 3

Introduction to Topology, Spring 2023

Due February 13, 2023 at 11:59pm

1. In this exercise we will give a cleaner statement of continuity for a metric space.
   (a) First suppose A and B are two sets and $f : A \to B$ is any function.
      i. Show that $U \subseteq V \subseteq B$ implies $f^{-1}(U) \subseteq f^{-1}(V)$.
      ii. Show that if $(U_i)_{i\in I}$ is a collection of subsets of B and $U = \bigcup_{i\in I} U_i$ then
      $$f^{-1}(U) = \bigcup_{i\in I} f^{-1}(U_i).$$
      iii. Prove Part ii. for intersections instead of unions.
   (b) Suppose (M_1, d_1) and (M_2, d_2) are metric spaces and $f : M_1 \to M_2$ is a function. Show that f is continuous according to any of the definitions in the notes if and only if
   $$U \subseteq M_2 \text{ is open } \implies f^{-1}(U) \subseteq M_1 \text{ is open.}$$
   (c) Using (b) show that a function f is continuous if and only if
   $$Z \subseteq M_2 \text{ is closed } \implies f^{-1}(Z) \subseteq M_1 \text{ is closed.}$$
   (d) Finally, show that a subset $Z \subset M$ in a metric space (M, d) is closed if and only if whenever a sequence $x_1, x_2, \dots \in Z$ converges to $x \in M$, we actually have $x \in Z$.
2. A map $f : M_1 \to M_2$ is called *open* if
   $$U \subseteq M_1 \text{ is open } \implies f(U) \subseteq M_2 \text{ is open.}$$
   This looks similar to the definition of continuity, but it's very different! Find an example of a function f which is
   (a) continuous and open
   (b) continuous and not open
   (c) open and not continuous
   (d) neither open nor continuous
3. Show that if M is a sequentially compact metric space and $N \subset M$ is a closed subset, then N is also sequentially compact.
4. Fix (M, d) a metric space and fix a decreasing sequence of subsets
   $$M \supseteq C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$$
   Show that if each C_n is nonempty and sequentially compact, then $\bigcap_{n\in\mathbb{N}} C_n$ is not empty.

5. If (M, d) is a metric space and $\{C_i\}_{i \in I}$ is an arbitrary collection of sequentially compact subsets, then $\bigcap_{i \in I} C_i$ is sequentially compact. Under the same setup, show that if I is finite, then $\bigcup_{i \in I} C_i$ is sequentially compact as well.

6. Consider the metric space $C([a, b], \mathbb{R})$ from Homework 2; recall this is the set of continuous functions $f : [a, b] \to \mathbb{R}$, and the metric is

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

   (a) Show that $C([a, b], \mathbb{R})$ is not bounded.

   (b) Show that a sequentially compact metric space is bounded, and conclude that $C([a, b], \mathbb{R})$ is not sequentially compact.