Homework 4

Introduction to Topology, Spring 2023

Due February 20, 2023 at 11:59pm

1. Let $X = \{a, b, c\}$. The powerset is $\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$. Given a topology on X, we can get another topology by permuting the elements of X. For instance, if I take the topology $\mathcal{T} = \{\varnothing, \{a, b\}, X\}$ and act by the permutation $a \mapsto b, b \mapsto c, c \mapsto a$, then I get $\mathcal{T} = \{\varnothing, \{b, c\}, X\}$.

   Say two topologies are *equivalent* if one is obtained from the other via permutation. This defines an equivalence relation, and it is a fact that there are 9 equivalence classes. Write down a representative of each equivalence class. (you don't need to rigorously prove that they are all distinct).

2. (a) If X is a set, let
   $$\mathcal{T}_{\text{cof}} = \{U \in \mathcal{P}(X) : U^c \text{ is finite}\} \cup \{\varnothing\}$$
   Show that $\mathcal{T}_{\text{cof}}$ is a topology; this is called the "cofinite topology".

   (b) Which sets are closed in the cofinite topology?

3. (a) Show that the intersection of two topologies is a topology.

   (b) If $\mathcal{B}$ is a base of open sets, show that the topology it generates (consisting of $\varnothing$ and unions of sets in $\mathcal{B}$) is equal to the intersection of all topologies containing $\mathcal{B}$.

   (c) A subset $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a *sub-base of open sets* if it satisfies the property that for every $x \in X$ there exists $S \in \mathcal{S}$ such that $x \in S$. Let $\mathcal{B}_{\mathcal{S}}$ denote the subset of $\mathcal{P}(X)$ consisting of *finite intersections* of elements of $\mathcal{S}$. Show that $\mathcal{B}_{\mathcal{S}}$ is a base of open sets.

4. If X is a topological space, $A \subseteq X$ is a subset and $x \in X$ is a point, we say that x is a *limit point*[1] *for* A if every open neighborhood U of x contains a point of A which is different from x. In a metric space, a limit point for A is the same as the limit of a convergent sequence in A, but you don't need to check this.

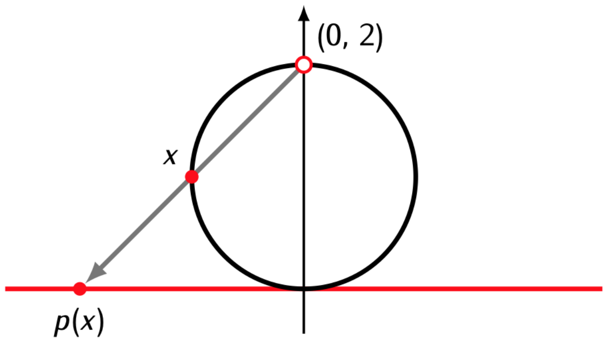
   (a) If $\mathbb{N}$ has the cofinite topology (see Problem 2), prove that a subset $Z \subseteq \mathbb{N}$ contains all of its limit points if and only if $Z = \mathbb{N}$ or Z is finite. (do this directly; don't use the next exercise!)

   (b) Now if X is any topological space, show that a subset $Z \subseteq X$ is closed if and only if it contains all of its limit points.

5. Let $S = \{0, 1\}$ be the Sierpinski space from class, whose topology is $\{\varnothing, \{1\}, S\}$. If $(X, \mathcal{T})$ is another topological space, write down a natural bijection
   $$C(X, S) \cong \mathcal{T}$$
   where $C(X, S)$ denotes the set of continuous functions $X \to S$.

[1] You may also see this referred to as a *cluster point* or an *accumulation point* in some references.

6. Let S^1 denote the circle of radius 1 around the point $(0, 1)$. Define a map $p : S^1 - \{(0, 2)\} \to \mathbb{R}$ by taking a point x to the intersection of the line connecting $(0, 2)$ to x with the x-axis:



Show that p is a homeomorphism. In this way, we see that topologically, you can construct a circle by taking the real line and "adding a point at $\pm\infty$"