Homework 5

Introduction to Topology, Spring 2023

Due February 27, 2023 at 11:59pm

1. If X and Y are two topological spaces, let $C(X, Y)$ denote the set of continuous functions from X to Y. Recall from Homework 1 that you constructed a bijection

$$\mathrm{Fun}(X, Y \times Z) \xrightarrow{\sim} \mathrm{Fun}(X, Y) \times \mathrm{Fun}(X, Z).$$

Show that this bijection naturally gives rise to a bijection

$$C(X, Y \times Z) \xrightarrow{\sim} C(X, Y) \times C(X, Z)$$

If you want, prove the analogous statement for the disjoint union (you don't need to turn this in.)

2. Suppose Y and Z are topological spaces, and suppose P is a topological space equipped with two continuous maps $\pi_Y : P \to Y$ and $\pi_Z : P \to Z$. We say that the triple (P, π_Y, π_Z) **satisfies property** $\Pi(Y, Z)$ if for any topological space X and any two continuous maps $f : X \to Y$ and $g : X \to Z$ there exists a unique continuous map $h : X \to P$ such that $\pi_Y \circ h = f$ and $\pi_Z \circ h = g$. Another way of phrasing these two equalities is to say that the following diagram *commutes*[1]:

$$\begin{tikzcd}
X \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& P \arrow[r, "\pi_Y"] \arrow[d, "\pi_Z"] & Y \\
& Z &
\end{tikzcd}$$

The previous exercise implies that $Y \times Z$ (with the usual projection maps) satisfies property $\Pi(Y, Z)$. Show that if (P, π_Y, π_Z) satisfies property $\Pi(Y, Z)$ then P is homeomorphic to $Y \times Z$.

This is called the "universal property of the product". You have now shown that up to homeomorphism, there's only one thing satisfying the universal property of the product.

3. Suppose $X_1, X_2, \dots,$ is an infinite collection of topological spaces and let $X = \prod_{i=1}^{\infty} X_i$.

   (a) Show that the collection of subsets

$$\left\{ \prod_{i=1}^{\infty} U_i : U_i \subseteq X_i \text{ is open for all } i \right\} \subseteq \mathcal{P}(X)$$

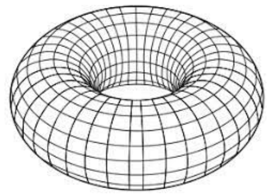
   satisfies the axioms of a "base of open sets" as defined in class.

   (b) The topology it generates is called the *box topology*. Give an example of a collection of X_i for which the box topology is not the same as the product topology; you should find an open set in the box topology which is not open in the product topology.

(as a side note, the box topology for a finite product is the same as the product topology).

[1] To commute means that if you start at X and follow the arrows to Y or Z, it doesn't matter which path you take, you always get the same function.

4. The *torus* T is the surface of a donut and looks like



The torus lives in $\mathbb{R}^3$, and the topology on T is the subspace topology from $\mathbb{R}^3$. Show that T is homeomorphic to $S^1 \times S^1$ with the product topology, where S^1 denotes the circle with its usual metric topology. (hint: you first need to find a description of circle and of the torus, but you can pick one convenient enough to use elementary trig functions, which you are allowed to assume are continuous)

5. Explain using words and pictures (i.e. you don't need to rigorously prove this) why the torus T from the previous exercise is homeomorphic to the quotient space $\mathbb{R}^2/\sim$ where $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 - x_2 \in \mathbb{Z}$ and $y_1 - y_2 \in \mathbb{Z}$.