Homework 6

Introduction to Topology, Spring 2023

Due Mar 9, 2023 at 11:59pm

1. If $f : X \to Y$ is continuous and X is connected, show that $f(X)$ is connected.
2. Show that if $A \subseteq X$ is a connected subspace then its closure $\overline{A}$ is connected.
3. Let $\{X_i\}_{i \in I}$ denote a collection of connected topological spaces (for some arbitrary indexing set I) and let
$$X = \prod_{i \in I} X_i$$
equipped with the product topology. Fix a point $(a_i)_{i \in I} \in X$.
   (a) If $J \subseteq I$ is a finite subset, let $X_J \subseteq X$ denote the subspace of X consisting of all points $(x_i)_{i \in I} \in X$ such that $x_i = a_i$ for all $i \notin J$. Show that X_J is connected.
   (b) Show that the union
$$Y = \bigcup_{J \subseteq I \text{ finite}} X_J$$
is connected.
   (c) Show that the closure of Y in X is equal to X. Conclude that X is connected.
4. If G is a topological group (as defined in the Week 5 group work) show that the connected component of the identity element $e \in G$ is a normal subgroup of G (recall that a subgroup $H \leq G$ is normal if $gHg^{-1} = H$ for all $g \in G$). Hint: use Exercise 1.
5. Here are some examples of how you can use connectedness to distinguish spaces.
   (a) Show that no two of the spaces $(0, 1)$, $(0, 1]$, or $[0, 1]$ are homeomorphic.
   (b) Show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ if $n > 1$.
6. Show that if $A \subseteq \mathbb{R}^2$ is a countable subset, then $\mathbb{R}^2 \setminus A$ is path-connected.