Homework 7

Introduction to Topology, Spring 2023

Due March 13, 2023 at 11:59pm

1. If X is a compact topological space and $Z \subseteq X$ is a closed subset, show that Z is compact.

2. If X is a Hausdorff topological space and $Z \subseteq X$ is a compact subspace (i.e. a subset for which the subspace topology is compact), then Z is closed.

3. (a) Prove the *tube lemma*, which says the following. If X and Y are two topological spaces and Y is compact, then consider the product $X \times Y$ with the product topology. Fix a point $x \in X$. If $N \subseteq X \times Y$ is an open subset containing $\{x\} \times Y$, then show that N contains an open set of the form $U \times Y$ where $U \subseteq X$ is an open set containing x. *Hint: remember that subsets of the form $U \times V$ with $U \subseteq X$ open and $V \subseteq Y$ open form a base of open sets for the product topology!*

   (b) Using the tube lemma, prove that if X and Y are compact, then $X \times Y$ is compact.

   Note that by induction, this implies that any finite product of compact topological spaces is compact. In fact more is true: any (arbitrary, so possibly infinite) product of compact topological spaces is compact; this is called Tichonov's theorem. This is a bit harder to prove though, and is in fact equivalent to the axiom of choice!

4. Show that if X is Hausdorff and $A, B \subseteq X$ are two disjoint compact subsets, then there exist disjoint open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. *In this way, we see that compact sets "behave like points" in some sense.*

5. Show that if Y is compact, then the projection map $\pi_X : X \times Y \to X$ takes closed sets to closed sets.