Homework 8

Introduction to Topology, Spring 2023

Due March 27, 2023 at 11:59pm

1. We will finish the proof of Proposition 8.1.4 in the notes, as follows.

   (a) Prove the *pasting lemma*: if X is a topological space and $Z_1, \ldots, Z_n \subseteq X$ is a finite list of closed subsets satisfying $Z_1 \cup \cdots \cup Z_n = X$, show that a map $f : X \to Y$ is continuous if and only if the restriction
   $$f|_{Z_i} : Z_i \to Y$$
   is continuous for all $i = 1, \ldots, n$.

   (b) Use this to conclude that the map H_3 in Proposition 8.1.4 of the notes is continuous.

   (c) Show by finding an explicit counterexample that the pasting lemma is not true if you replace "finite list of closed subsets $Z_1, \ldots, Z_n \subseteq X$" with "infinite list of closed subsets $Z_i \subseteq X$".

2. Show that the map H from Example 8.1.2 is continuous.

3. Show that if $f, f' : X \to Y$ are homotopic and $g, g' : Y \to Z$ are homotopic, then $g \circ f \sim g' \circ f'$.

4. A space is called *contractible* if the identity map is null-homotopic.

   (a) Show that a contractible space is path-connected.

   (b) Show that if Y is contractible any two maps $f, g : X \to Y$ are homotopic, where X is any topological space.

5. Show that two finite topological spaces, both with the discrete topology, are homotopy equivalent if and only if they have the same number of points.

6. Show that $\mathbb{R}^3 - \{(0,0,z) : z \in \mathbb{R}\}$ is homotopy equivalent to a circle. (Hint: draw a picture. you already know that the circle and $\mathbb{R}^2 - \{(0,0)\}$ are homotopy equivalent, so you can just find a homotopy equivalence to $\mathbb{R}^2 - \{(0,0)\}$).