Homework 9

Introduction to Topology, Spring 2023

Due April 3, 2023 at 11:59pm

1. Suppose $f, g : I \to X$ are two loops satisfying $f(0) = g(0)$ and $f(1) = g(1)$ and $k : X \to X'$ is another continuous map. If H is a path-homotopy between f and g, show that $k \circ H$ is a path-homotopy between $k \circ f$ and $k \circ g$.

2. Suppose $f, g : X \to Y$ are two continuous maps satisfying $f(x_0) = g(x_0)$. Denote $y_0 := f(x_0)$. A *homotopy from f to g relative to x_0* is a map

$$H : X \times I \to Y$$

such that $H(x_0, t) = y_0$ for all $t \in I$; in this case we say that f and g are *homotopic relative to x_0*.

Show that if f, g are homotopic relative to x_0 then the maps $f_*, g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ are equal.

3. Here we will finish the proof that $\pi_1(X, x_0)$ is a group.

   (a) Show that if $\ell : I \to X$ is a loop based at x_0, then the constant map $c : I \to X$ sending $t \mapsto x_0$ satisfies $[c * f] = [f * c] = [f]$. In other words, c is the *identity element* for the group structure on $\pi_1(X, x_0)$.

   (b) Show that the operation $*$ on $\pi_1(X, x_0)$ is associative.

   (c) If $\ell : I \to X$ is a loop based at x_0, then show that $\ell^{-1}(t) = \ell(1 - t)$ is an inverse to ℓ in $\pi_1(X, x_0)$; in other words, you need to show that $[\ell^{-1} * \ell] = [\ell * \ell^{-1}] = [c]$.

4. This exercise is about varying the basepoint.

   (a) Suppose $x_0, x_1 \in X$ are two points and $\alpha : I \to X$ is a path satisfying $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Then show that the map

$$\begin{aligned} \widehat{\alpha} : \pi_1(X, x_0) &\to \pi_1(X, x_1) \\ [f] &\mapsto [\alpha^{-1} * f * \alpha] \end{aligned}$$

   is well-defined, and is an isomorphism of groups. (You may assume that concatenation of paths is associative; the proof of this is basically same as 3(b)).

   (b) Recall that a group $(G, *)$ is *abelian* if $x * y = y * x$ for all $x, y \in G$.

   Suppose X is path-connected, and fix two points $x_0, x_1 \in X$. Show that $\pi_1(X, x_0)$ is abelian if and only if $\widehat{\alpha} = \widehat{\beta}$ for all pairs of paths α, β from x_0 to x_1.

5. If X, Y are two topological spaces and $x_0 \in X$ and $y_0 \in Y$ are some chosen basepoints then find an isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \xrightarrow{\sim} \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

(recall that if G and H are two groups, then $G \times H$ becomes a group by taking $(g, h) * (g, h') = (gg', hh')$). Hint: think about the universal property of the Cartesian product.